Complete Solutions Manual
to Accompany

Probability and Statistics
for Engineering and the Sciences

NINTH EDITION

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CHAPTER 1

Section 1.1

1.

- **a.** *Los Angeles Times, Oberlin Tribune, Gainesville Sun, Washington Post*
- **b.** Duke Energy, Clorox, Seagate, Neiman Marcus
- **c.** Vince Correa, Catherine Miller, Michael Cutler, Ken Lee
- **d.** 2.97, 3.56, 2.20, 2.97

2.

- **a.** 29.1 yd, 28.3 yd, 24.7 yd, 31.0 yd
- **b.** 432 pp, 196 pp, 184 pp, 321 pp
- **c.** 2.1, 4.0, 3.2, 6.3
- **d.** 0.07 g, 1.58 g, 7.1 g, 27.2 g

3.

- **a.** How likely is it that more than half of the sampled computers will need or have needed warranty service? What is the expected number among the 100 that need warranty service? How likely is it that the number needing warranty service will exceed the expected number by more than 10?
- **b.** Suppose that 15 of the 100 sampled needed warranty service. How confident can we be that the proportion of *all* such computers needing warranty service is between .08 and .22? Does the sample provide compelling evidence for concluding that more than 10% of all such computers need warranty service?

4.

a. Concrete populations: all living U.S. Citizens, all mutual funds marketed in the U.S., all books published in 1980
Hypothetical populations: all grade point averages for University of California undergraduates during the next academic year, page lengths for all books published during the next calendar year, batting averages for all major league players during the next baseball season

b. (Concrete) Probability: In a sample of 5 mutual funds, what is the chance that all 5 have rates of return which exceeded 10% last year?
Statistics: If previous year rates-of-return for 5 mutual funds were 9.6, 14.5, 8.3, 9.9 and 10.2, can we conclude that the average rate for all funds was below 10%?
(Hypothetical) Probability: In a sample of 10 books to be published next year, how likely is it that the average number of pages for the 10 is between 200 and 250?
Statistics: If the sample average number of pages for 10 books is 227, can we be highly confident that the average for all books is between 200 and 245?

5.

a. No. All students taking a large statistics course who participate in an SI program of this sort.

b. The advantage to randomly allocating students to the two groups is that the two groups should then be fairly comparable before the study. If the two groups perform differently in the class, we might attribute this to the treatments (SI and control). If it were left to students to choose, stronger or more dedicated students might gravitate toward SI, confounding the results.

c. If all students were put in the treatment group, there would be no firm basis for assessing the effectiveness of SI (nothing to which the SI scores could reasonably be compared).

6. One could take a simple random sample of students from all students in the California State University system and ask each student in the sample to report the distance form their hometown to campus. Alternatively, the sample could be generated by taking a stratified random sample by taking a simple random sample from each of the 23 campuses and again asking each student in the sample to report the distance from their hometown to campus. Certain problems might arise with self reporting of distances, such as recording error or poor recall. This study is enumerative because there exists a finite, identifiable population of objects from which to sample.

7. One could generate a simple random sample of all single-family homes in the city, or a stratified random sample by taking a simple random sample from each of the 10 district neighborhoods. From each of the selected homes, values of all desired variables would be determined. This would be an enumerative study because there exists a finite, identifiable population of objects from which to sample.

8.

a. Number observations equal 2 x 2 x 2 = 8

b. This could be called an analytic study because the data would be collected on an existing process. There is no sampling frame.

9.

a. There could be several explanations for the variability of the measurements. Among them could be measurement error (due to mechanical or technical changes across measurements), recording error, differences in weather conditions at time of measurements, etc.

b. No, because there is no sampling frame.

Section 1.2

10.

a.

5	9	
6	33588	
7	00234677889	
8	127	
9	077	stem: ones
10	7	leaf: tenths
11	368	

A representative strength for these beams is around 7.8 MPa, but there is a reasonably large amount of variation around that representative value.

(What constitutes large or small variation usually depends on context, but variation is usually considered large when the range of the data – the difference between the largest and smallest value – is comparable to a representative value. Here, the range is 11.8 – 5.9 = 5.9 MPa, which is similar in size to the representative value of 7.8 MPa. So, most researchers would call this a large amount of variation.)

b. The data display is not perfectly symmetric around some middle/representative value. There is some positive skewness in this data.

c. Outliers are data points that appear to be *very* different from the pack. Looking at the stem-and-leaf display in part (a), there appear to be no outliers in this data. (A later section gives a more precise definition of what constitutes an outlier.)

d. From the stem-and-leaf display in part (a), there are 4 values greater than 10. Therefore, the proportion of data values that exceed 10 is 4/27 = .148, or, about 15%.

11.

3L	1
3H	56678
4L	000112222234
4H	5667888
5L	144
5H	58
6L	2
6H	6678
7L	
7H	5

stem: tenths
leaf : hundredths

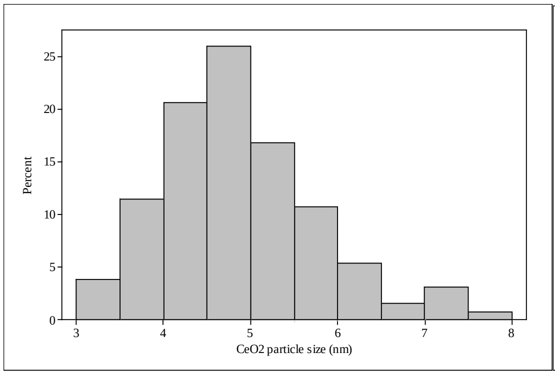
The stem-and-leaf display shows that .45 is a good representative value for the data. In addition, the display is not symmetric and appears to be positively skewed. The range of the data is .75 – .31 = .44, which is comparable to the typical value of .45. This constitutes a reasonably large amount of variation in the data. The data value .75 is a possible outlier.

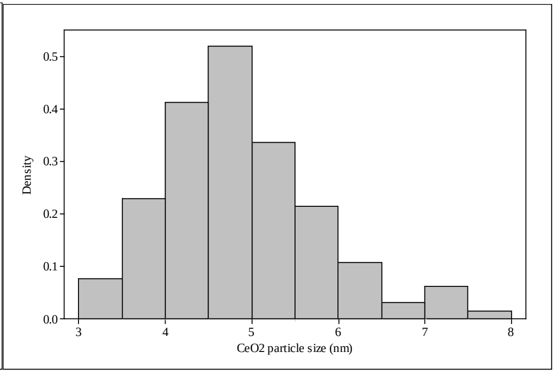
12. The sample size for this data set is $n = 5 + 15 + 27 + 34 + 22 + 14 + 7 + 2 + 4 + 1 = 131$.

a. The first four intervals correspond to observations less than 5, so the proportion of values less than 5 is (5 + 15 + 27 + 34)/131 = 81/131 = .618.

b. The last four intervals correspond to observations at least 6, so the proportion of values at least 6 is (7 + 2 + 4 + 1)/131 = 14/131 = .107.

c. **& d.** The relative (percent) frequency and density histograms appear below. The distribution of CeO_2 sizes is not symmetric, but rather positively skewed. Notice that the relative frequency and density histograms are essentially identical, other than the vertical axis labeling, because the bin widths are all the same.





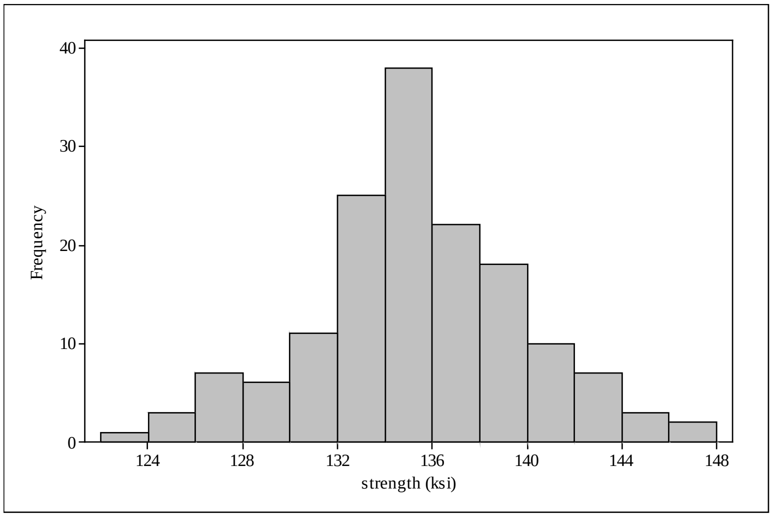
13.

a.

12	2	stem: tens
12	445	leaf: ones
12	6667777	
12	889999	
13	00011111111	
13	2222222222333333333333333	
13	44444444444444444455555555555555555555	
13	6666666666667777777777	
13	888888888888999999	
14	0000001111	
14	2333333	
14	444	
14	77	

The observations are highly concentrated at around 134 or 135, where the display suggests the typical value falls.

b.



The histogram of ultimate strengths is symmetric and unimodal, with the point of symmetry at approximately 135 ksi. There is a moderate amount of variation, and there are no gaps or outliers in the distribution.

14.

a.

2	23	stem: 1.0
3	2344567789	leaf: .10
4	01356889	
5	00001114455666789	
6	0000122223344456667789999	
7	00012233455555668	
8	02233448	
9	012233335666788	
10	2344455688	
11	2335999	
12	37	
13	8	
14	36	
15	0035	
16		
17		
18	9	

b. A representative is around 7.0.

c. The data exhibit a moderate amount of variation (this is subjective).

d. No, the data is skewed to the right, or positively skewed.

e. The value 18.9 appears to be an outlier, being more than two stem units from the previous value.

15.

American		French
	8	1
755543211000	9	00234566
9432	10	2356
6630	11	1369
850	12	223558
8	13	7
	14	
	15	8
2	16	

American movie times are unimodal strongly positively skewed, while French movie times appear to be bimodal. A typical American movie runs about 95 minutes, while French movies are typically either around 95 minutes or around 125 minutes. American movies are generally shorter than French movies and are less variable in length. Finally, both American and French movies occasionally run very long (outliers at 162 minutes and 158 minutes, respectively, in the samples).

16.

a.

Beams		Cylinders	
9	5	8	
88533	6	16	
98877643200	7	012488	
721	8	13359	stem: ones
770	9	278	leaf: tenths
7	10		
863	11	2	
	12	6	
	13		
	14	1	

The data appears to be slightly skewed to the right, or positively skewed. The value of 14.1 MPa appears to be an outlier. Three out of the twenty, or 15%, of the observations exceed 10 MPa.

b. The majority of observations are between 5 and 9 MPa for both beams and cylinders, with the modal class being 7.0-7.9 MPa. The observations for cylinders are more variable, or spread out, and the maximum value of the cylinder observations is higher.

c.

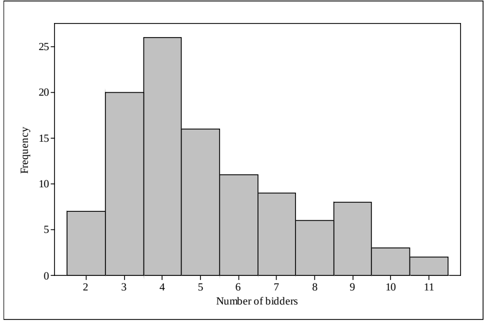
```
.  . .  :..  : .: . . .   :         .        .         .
-+---------+---------+---------+---------+---------+-----
6.0       7.5       9.0      10.5      12.0      13.5
                Cylinder strength (MPa)
```

17. The sample size for this data set is $n = 7 + 20 + 26 + \ldots + 3 + 2 = 108$.

a. "At most five bidders" means 2, 3, 4, or 5 bidders. The proportion of contracts that involved at most 5 bidders is (7 + 20 + 26 + 16)/108 = 69/108 = .639.
Similarly, the proportion of contracts that involved at least 5 bidders (5 through 11) is equal to (16 + 11 + 9 + 6 + 8 + 3 + 2)/108 = 55/108 = .509.

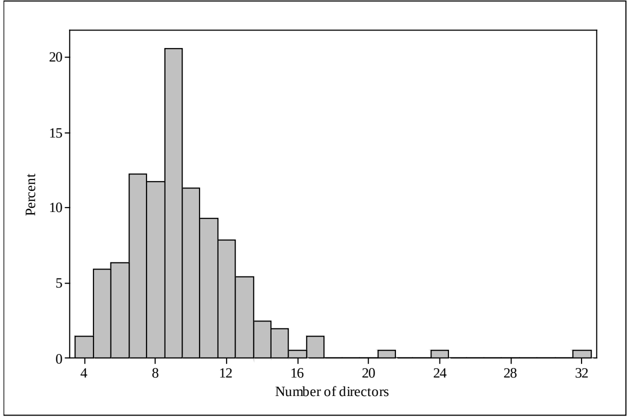
b. The number of contracts with between 5 and 10 bidders, inclusive, is 16 + 11 + 9 + 6 + 8 + 3 = 53, so the proportion is 53/108 = .491. "Strictly" between 5 and 10 means 6, 7, 8, or 9 bidders, for a proportion equal to (11 + 9 + 6 + 8)/108 = 34/108 = .315.

c. The distribution of number of bidders is positively skewed, ranging from 2 to 11 bidders, with a typical value of around 4-5 bidders.



18.

a. The most interesting feature of the histogram is the heavy presence of three very large outliers (21, 24, and 32 directors). Absent these three corporations, the distribution of number of directors would be roughly symmetric with a typical value of around 9.



Note: One way to have Minitab automatically construct a histogram from grouped data such as this is to use Minitab's ability to enter multiple copies of the same number by typing, for example, 42(9) to enter 42 copies of the number 9. The frequency data in this exercise was entered using the following Minitab commands:

```
MTB > set c1
DATA> 3(4) 12(5) 13(6) 25(7) 24(8) 42(9) 23(10) 19(11) 16(12)
11(13) 5(14) 4(15) 1(16) 3(17) 1(21) 1(24) 1(32)
DATA> end
```

b. The accompanying frequency distribution is nearly identical to the one in the textbook, except that the three largest values are compacted into the "≥ 18" category. If this were the originally-presented information, we could not create a histogram, because we would not know the upper boundary for the rectangle corresponding to the "≥ 18" category.

No. dir.	4	5	6	7	8	9	10	11
Freq.	3	12	13	25	24	42	23	19

No dir.	12	13	14	15	16	17	≥ 18
Freq.	16	11	5	4	1	3	3

c. The sample size is 3 + 12 + … + 3 + 1 + 1 + 1 = 204. So, the proportion of these corporations that have at most 10 directors is (3 + 12 + 13 + 25 + 24 + 42 + 23)/204 = 142/204 = .696.

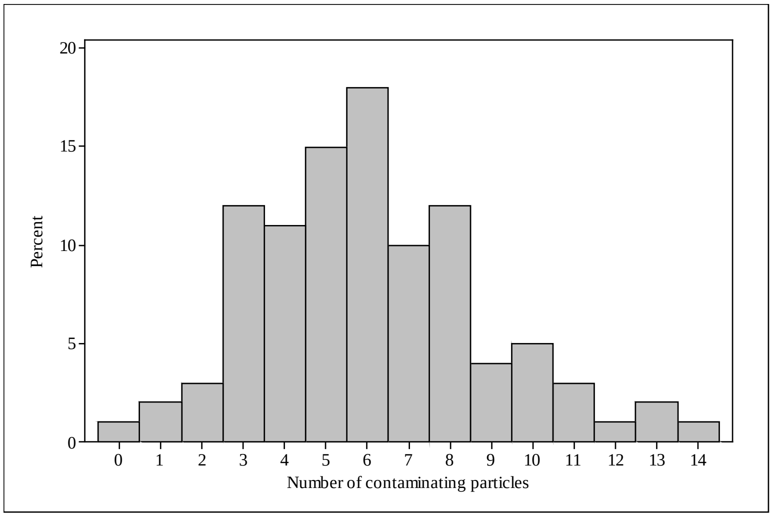
d. Similarly, the proportion of these corporations with more than 15 directors is (1 + 3 + 1 + 1 + 1)/204 = 7/204 = .034.

19.

a. From this frequency distribution, the proportion of wafers that contained at least one particle is (100-1)/100 = .99, or 99%. Note that it is much easier to subtract 1 (which is the number of wafers that contain 0 particles) from 100 than it would be to add all the frequencies for 1, 2, 3,… particles. In a similar fashion, the proportion containing at least 5 particles is (100 - 1-2-3-12-11)/100 = 71/100 = .71, or, 71%.

b. The proportion containing between 5 and 10 particles is (15+18+10+12+4+5)/100 = 64/100 = .64, or 64%. The proportion that contain strictly between 5 and 10 (meaning strictly *more* than 5 and strictly *less* than 10) is (18+10+12+4)/100 = 44/100 = .44, or 44%.

c. The following histogram was constructed using Minitab. The histogram is *almost* symmetric and unimodal; however, the distribution has a few smaller modes and has a very slight positive skew.



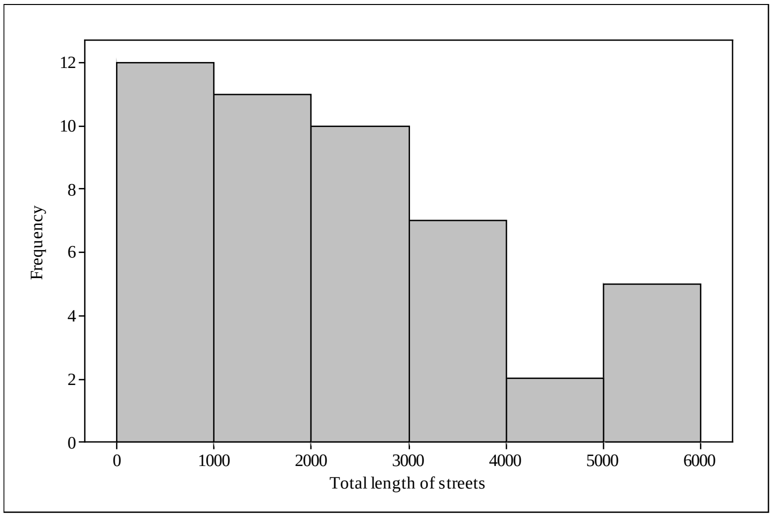
20.

a. The following stem-and-leaf display was constructed:

0	123334555599	
1	00122234688	stem: thousands
2	1112344477	leaf: hundreds
3	0113338	
4	37	
5	23778	

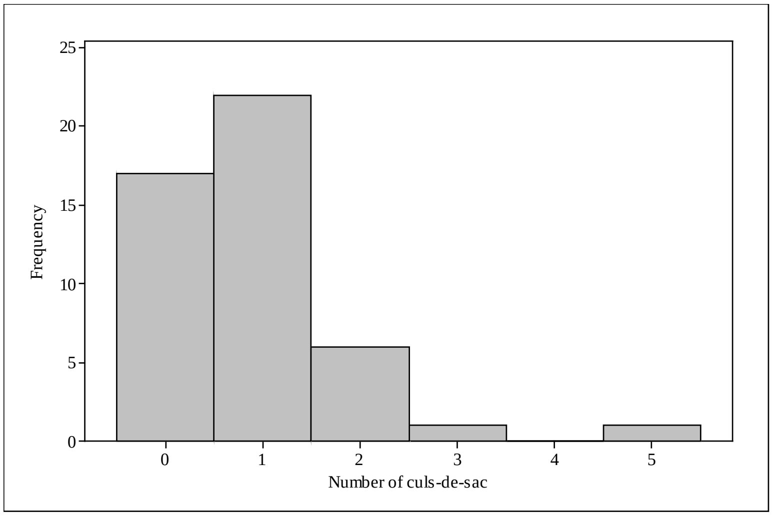
A typical data value is somewhere in the low 2000's. The display is bimodal (the stem at 5 would be considered a mode, the stem at 0 another) and has a positive skew.

b. A histogram of this data, using classes boundaries of 0, 1000, 2000, …, 6000 is shown below. The proportion of subdivisions with total length less than 2000 is (12+11)/47 = .489, or 48.9%. Between 2000 and 4000, the proportion is (10+7)/47 = .362, or 36.2%. The histogram shows the same general shape as depicted by the stem-and-leaf in part (a).

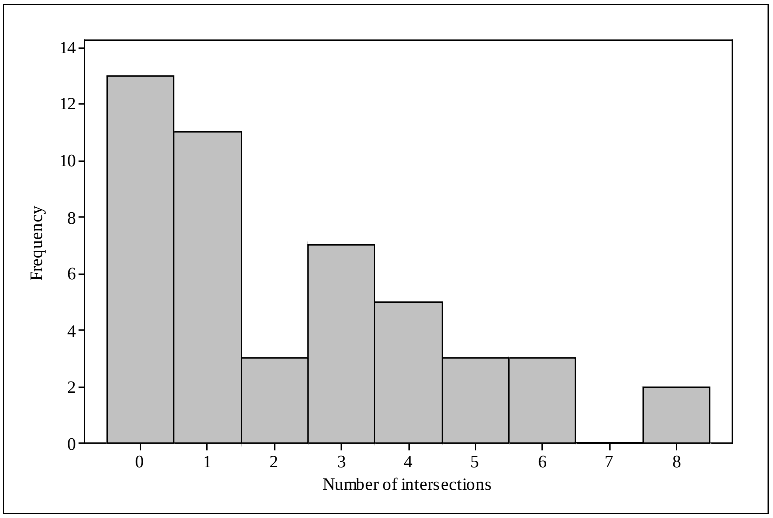


21.

a. A histogram of the *y* data appears below. From this histogram, the number of subdivisions having no cul-de-sacs (i.e., $y = 0$) is 17/47 = .362, or 36.2%. The proportion having at least one cul-de-sac ($y \geq 1$) is (47 – 17)/47 = 30/47 = .638, or 63.8%. Note that subtracting the number of cul-de-sacs with $y = 0$ from the total, 47, is an easy way to find the number of subdivisions with $y \geq 1$.

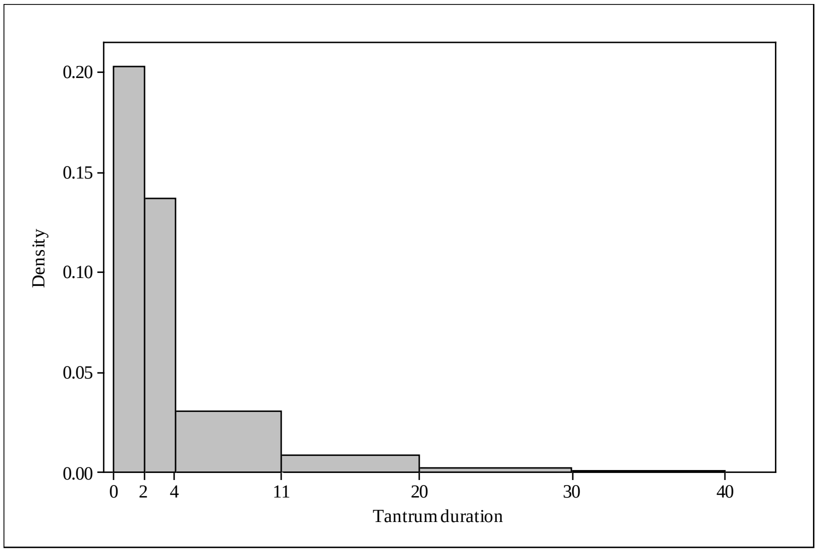


b. A histogram of the z data appears below. From this histogram, the number of subdivisions with at most 5 intersections (i.e., $z \le 5$) is 42/47 = .894, or 89.4%. The proportion having fewer than 5 intersections (i.e., $z < 5$) is 39/47 = .830, or 83.0%.



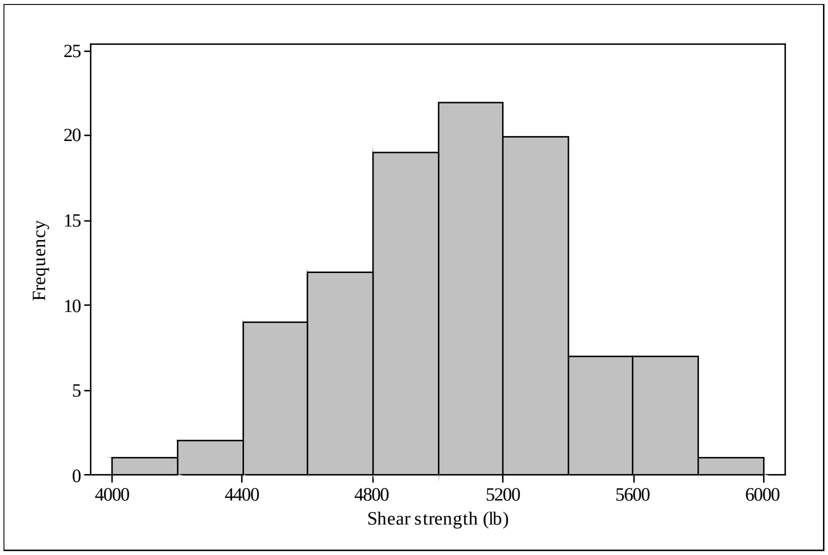
22. A very large percentage of the data values are greater than 0, which indicates that most, but not all, runners do slow down at the end of the race. The histogram is also positively skewed, which means that some runners slow down a *lot* compared to the others. A typical value for this data would be in the neighborhood of 200 seconds. The proportion of the runners who ran the last 5 km faster than they did the first 5 km is very small, about 1% or so.

23. Note: since the class intervals have unequal length, we must use a *density scale*.



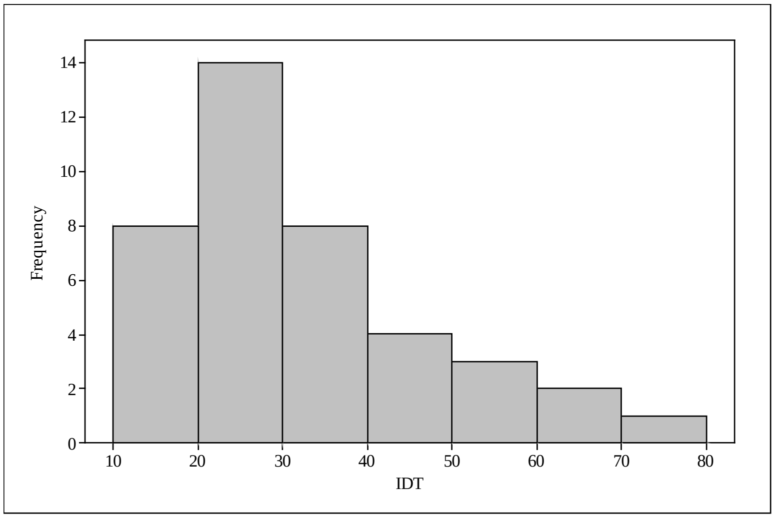
The distribution of tantrum durations is unimodal and heavily positively skewed. Most tantrums last between 0 and 11 minutes, but a few last more than half an hour! With such heavy skewness, it’s difficult to give a representative value.

24. The distribution of shear strengths is roughly symmetric and bell-shaped, centered at about 5000 lbs and ranging from about 4000 to 6000 lbs.

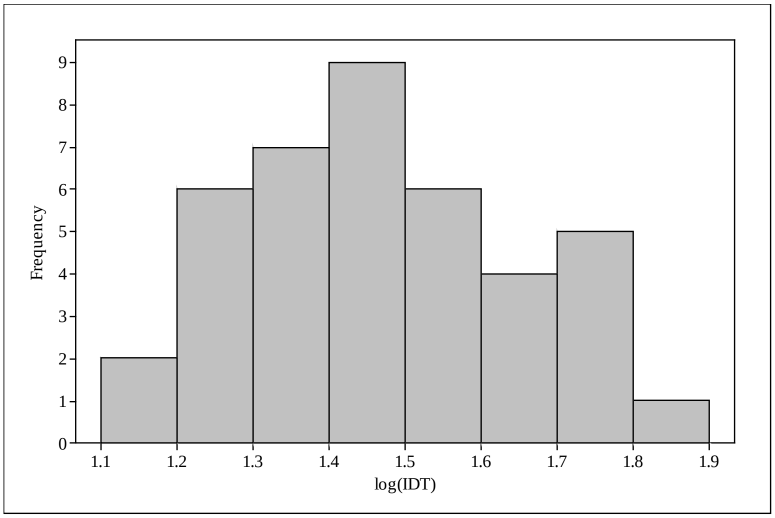


25. The transformation creates a much more symmetric, mound-shaped histogram.

Histogram of original data:



Histogram of transformed data:



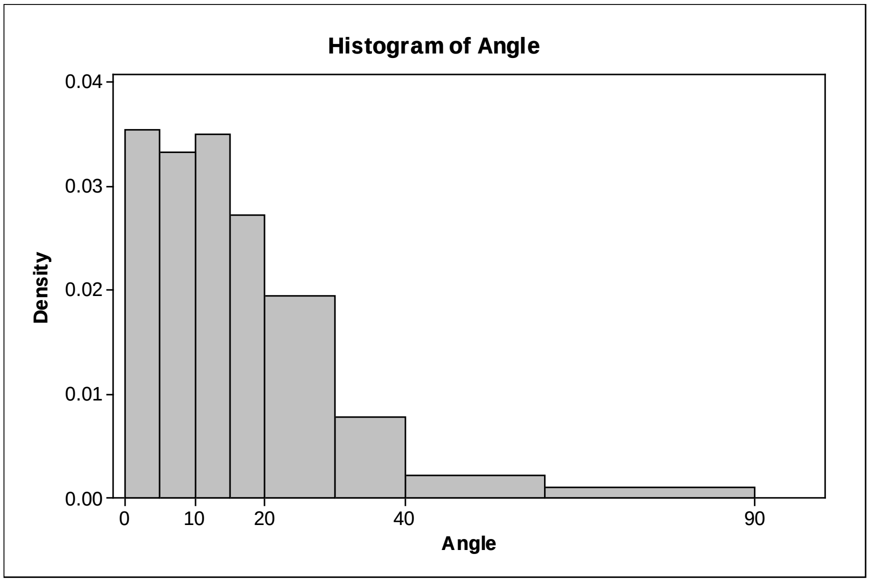
26.

a. Yes: the proportion of sampled angles smaller than 15° is .177 + .166 + .175 = .518.

b. The proportion of sampled angles at least 30° is .078 + .044 + .030 = .152.

c. The proportion of angles between 10° and 25° is roughly .175 + .136 + (.194)/2 = .408.

d. The distribution of misorientation angles is heavily positively skewed. Though angles can range from 0° to 90°, nearly 85% of all angles are less than 30°. Without more precise information, we cannot tell if the data contain outliers.

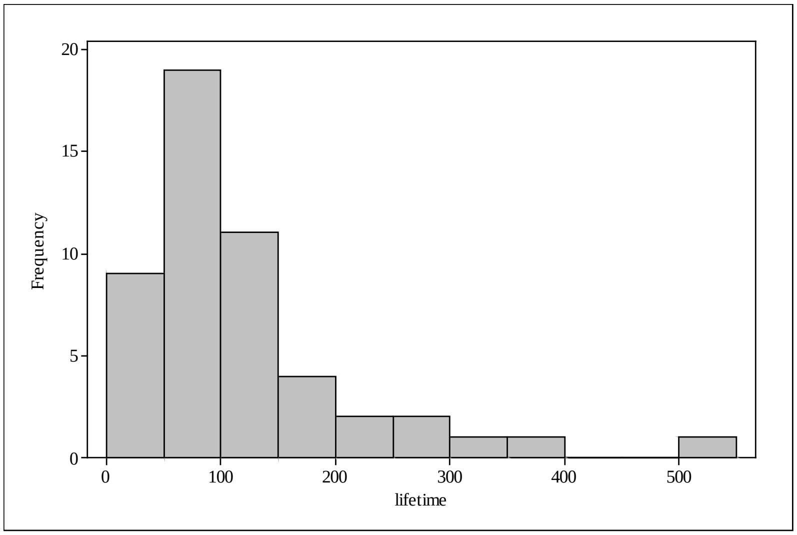


27.

a. The endpoints of the class intervals overlap. For example, the value 50 falls in both of the intervals 0–50 and 50–100.

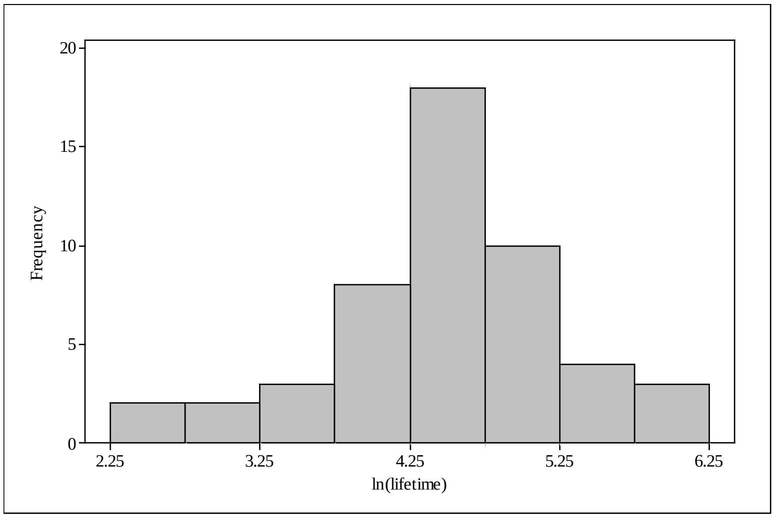
b. The lifetime distribution is positively skewed. A representative value is around 100. There is a great deal of variability in lifetimes and several possible candidates for outliers.

Class Interval	Frequency	Relative Frequency
0–< 50	9	0.18
50–<100	19	0.38
100–<150	11	0.22
150–<200	4	0.08
200–<250	2	0.04
250–<300	2	0.04
300–<350	1	0.02
350–<400	1	0.02
400–<450	0	0.00
450–<500	0	0.00
500–<550	1	0.02
	50	1.00



c. There is much more symmetry in the distribution of the transformed values than in the values themselves, and less variability. There are no longer gaps or obvious outliers.

Class Interval	Frequency	Relative Frequency
2.25–<2.75	2	0.04
2.75–<3.25	2	0.04
3.25–<3.75	3	0.06
3.75–<4.25	8	0.16
4.25–<4.75	18	0.36
4.75–<5.25	10	0.20
5.25–<5.75	4	0.08
5.75–<6.25	3	0.06



d. The proportion of lifetime observations in this sample that are less than 100 is .18 + .38 = .56, and the proportion that is at least 200 is .04 + .04 + .02 + .02 + .02 = .14.

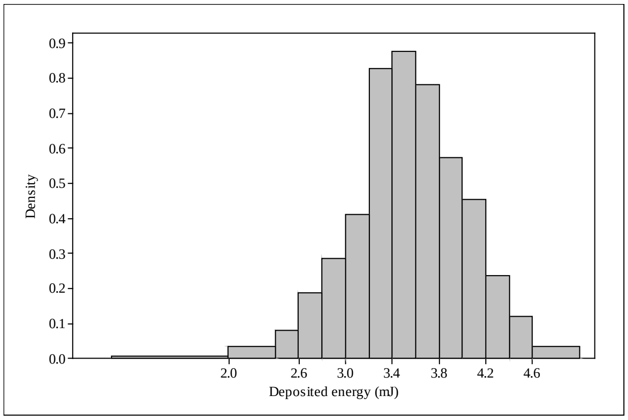
28. The sample size for this data set is *n* = 804.

a. (5 + 11 + 13 + 30 + 46)/804 = 105/804 = .131.

b. (73 + 38 + 19 + 11)/804 = 141/804 = .175.

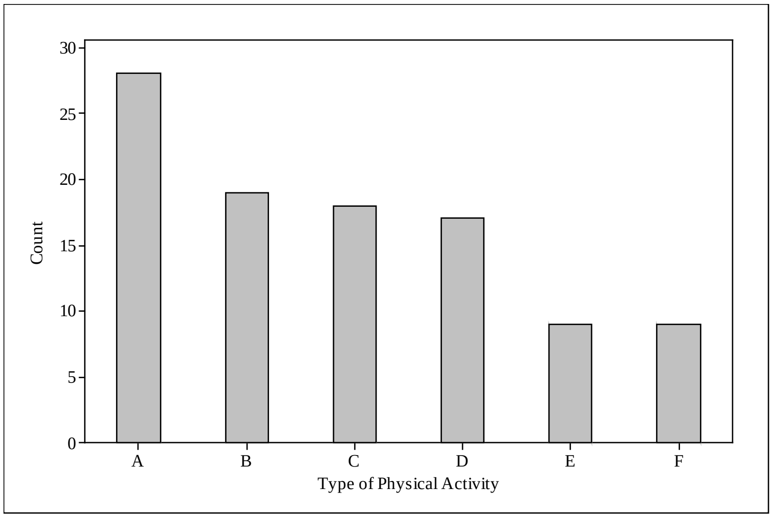
c. The number of trials resulting in deposited energy of 3.6 mJ or more is 126 + 92 + 73 + 38 + 19 + 11 = 359. Additionally, 141 trials resulted in deposited energy within the interval 3.4-<3.6. If we assume that roughly half of these were in the interval 3.5-<3.6 (since 3.5 is the midpoint), then our estimated frequency is 359 + (141)/2 = 429.5, for a rough proportion equal to 429.5/804 = .534.

d. The deposited energy distribution is roughly symmetric or perhaps slightly negatively skewed (there is a somewhat long left tail). Notice that the histogram must be made on a density scale, since the interval widths are not all the same.

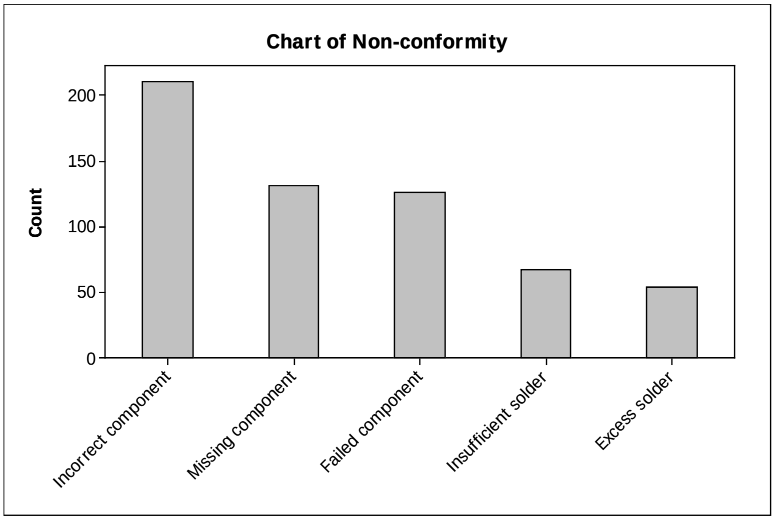


29.

Physical Activity	Frequency	Relative Frequency
A	28	.28
B	19	.19
C	18	.18
D	17	.17
E	9	.09
F	9	.09
	100	1.00



30.



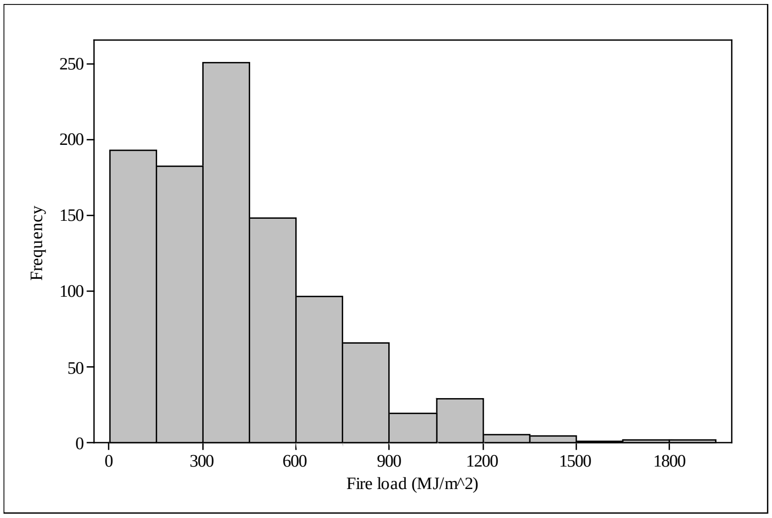
31.

Class	Frequency	Cum. Freq.	Cum. Rel. Freq.
0.0–<4.0	2	2	0.050
4.0–<8.0	14	16	0.400
8.0–<12.0	11	27	0.675
12.0–<16.0	8	35	0.875
16.0–<20.0	4	39	0.975
20.0–<24.0	0	39	0.975
24.0–<28.0	1	40	1.000

32.

a. Cumulative percents must be restored to relative frequencies. Then the histogram may be constructed (see below). The relative frequency distribution is almost unimodal and exhibits a large positive skew. The typical middle value is somewhere between 400 and 450, although the skewness makes it difficult to pinpoint more exactly than this.

Class	Rel. Freq.	Class	Rel. Freq.
0–< 150	.193	900–<1050	.019
150–< 300	.183	1050–<1200	.029
300–< 450	.251	1200–<1350	.005
450–< 600	.148	1350–<1500	.004
600–< 750	.097	1500–<1650	.001
750–< 900	.066	1650–<1800	.002
		1800–<1950	.002



b. The proportion of the fire loads less than 600 is .193 + .183 + .251 + .148 = .775. The proportion of loads that are at least 1200 is .005 + .004 + .001 + .002 + .002 = .014.

c. The proportion of loads between 600 and 1200 is 1 – .775 – .014 = .211.

Section 1.3

33.

a. Using software, $\bar{x}$ = 640.5 ($640,500) and $\tilde{x}$ = 582.5 ($582,500). The average sale price for a home in this sample was $640,500. Half the sales were for less than $582,500, while half were for more than $582,500.

b. Changing that one value lowers the sample mean to 610.5 ($610,500) but has no effect on the sample median.

c. After removing the two largest and two smallest values, $\bar{x}_{tr(20)}$ = 591.2 ($591,200).

d. A 10% trimmed mean from removing just the highest and lowest values is $\bar{x}_{tr(10)}$ = 596.3. To form a 15% trimmed mean, take the average of the 10% and 20% trimmed means to get $\bar{x}_{tr(15)}$ = (591.2 + 596.3)/2 = 593.75 ($593,750).

34.

a. For urban homes, $\bar{x}$ = 21.55 EU/mg; for farm homes, $\bar{x}$ = 8.56 EU/mg. The average endotoxin concentration in urban homes is more than double the average endotoxin concentration in farm homes.

b. For urban homes, $\tilde{x}$ = 17.00 EU/mg; for farm homes, $\tilde{x}$ = 8.90 EU/mg. The median endotoxin concentration in urban homes is nearly double the median endotoxin concentration in farm homes. The mean and median endotoxin concentration for urban homes are so different because the few large values, especially the extreme value of 80.0, raise the mean but not the median.

c. For urban homes, deleting the smallest (x = 4.0) and largest (x = 80.0) values gives a trimmed mean of $\bar{x}_{tr}$ = 153/9 = 17 EU/mg. The corresponding trimming percentage is 100(1/11) ≈ 9.1%. The trimmed mean is less than the mean of the entire sample, since the sample was positively skewed. Coincidentally, the median and trimmed mean are equal.

For farm homes, deleting the smallest (x = 0.3) and largest (x = 21.0) values gives a trimmed mean of $\bar{x}_{tr}$ = 107.1/13 = 8.24 EU/mg. The corresponding trimming percentage is 100(1/15) ≈ 6.7%. The trimmed mean is below, though not far from, the mean and median of the entire sample.

35. The sample size is n = 15.

a. The sample mean is $\bar{x}$ = 18.55/15 = 1.237 µg/g and the sample median is $\tilde{x}$ = the 8th ordered value = .56 µg/g. These values are very different due to the heavy positive skewness in the data.

b. A 1/15 trimmed mean is obtained by removing the largest and smallest values and averaging the remaining 13 numbers: (.22 + … + 3.07)/13 = 1.162. Similarly, a 2/15 trimmed mean is the average of the middle 11 values: (.25 + … + 2.25)/11 = 1.074. Since the average of 1/15 and 2/15 is .1 (10%), a 10% trimmed mean is given by the midpoint of these two trimmed means: (1.162 + 1.074)/2 = 1.118 µg/g.

c. The median of the data set will remain .56 so long as that's the 8th ordered observation. Hence, the value .20 could be increased to as high as .56 without changing the fact that the 8th ordered observation is .56. Equivalently, .20 could be increased by as much as .36 without affecting the value of the sample median.

36.

a. A stem-and leaf display of this data appears below:

32	55
33	49
34	
35	6699
36	34469
37	03345
38	9
39	2347
40	23
41	
42	4

stem: ones
leaf: tenths

The display is reasonably symmetric, so the mean and median will be close.

b. The sample mean is $\bar{x}$ = 9638/26 = 370.7 sec, while the sample median is $\tilde{x}$ = (369+370)/2 = 369.50 sec.

c. The largest value (currently 424) could be increased by any amount. Doing so will not change the fact that the middle two observations are 369 and 370, and hence, the median will not change. However, the value x = 424 cannot be changed to a number less than 370 (a change of 424 – 370 = 54) since that will change the middle two values.

d. Expressed in minutes, the mean is (370.7 sec)/(60 sec) = 6.18 min, while the median is 6.16 min.

37. $\bar{x} = 12.01$, $\tilde{x} = 11.35$, $\bar{x}_{tr(10)} = 11.46$. The median or the trimmed mean would be better choices than the mean because of the outlier 21.9.

38.

a. The reported values are (in increasing order) 110, 115, 120, 120, 125, 130, 130, 135, and 140. Thus the median of the reported values is 125.

b. 127.6 is reported as 130, so the median is now 130, a very substantial change. When there is rounding or grouping, the median can be highly sensitive to small change.

39.

a. $\Sigma x_i = 16.475$ so $\bar{x} = \dfrac{16.475}{16} = 1.0297$; $\tilde{x} = \dfrac{(1.007 + 1.011)}{2} = 1.009$

b. 1.394 can be decreased until it reaches 1.011 (i.e. by 1.394 – 1.011 = 0.383), the largest of the 2 middle values. If it is decreased by more than 0.383, the median will change.

40. $\tilde{x} = 60.8$, $\bar{x}_{tr(25)} = 59.3083$, $\bar{x}_{tr(10)} = 58.3475$, $\bar{x} = 58.54$. All four measures of center have about the same value.

41.

a. $x/n = 7/10 = .7$

b. $\bar{x} = .70$ = the sample proportion of successes

c. To have x/n equal .80 requires $x/25 = .80$ or $x = (.80)(25) = 20$. There are 7 successes (S) already, so another 20 – 7 = 13 would be required.

42.

a. $\bar{y} = \dfrac{\Sigma y_i}{n} = \dfrac{\Sigma(x_i + c)}{n} = \dfrac{\Sigma x_i}{n} + \dfrac{nc}{n} = \bar{x} + c$

$\tilde{y} =$ the median of $(x_1 + c, x_2 + c, ..., x_n + c) =$ median of $(x_1, x_2, ..., x_n) + c = \tilde{x} + c$

b. $\bar{y} = \dfrac{\Sigma y_i}{n} = \dfrac{\Sigma(x_i \cdot c)}{n} = \dfrac{c\Sigma x_i}{n} = c\bar{x}$

$\tilde{y} =$ the median of $(cx_1, cx_2, ..., cx_n) = c \cdot$ the median of $(x_1, x_2, ..., x_n) = c\tilde{x}$

43. The median and certain trimmed means can be calculated, while the mean cannot — the exact values of the "100+" observations are required to calculate the mean. $\tilde{x} = \dfrac{(57 + 79)}{2} = 68.0$, $\bar{x}_{tr(20)} = 66.2$, $\bar{x}_{tr(30)} = 67.5$.

Section 1.4

44.

a. The maximum and minimum values are 182.6 and 180.3, respectively, so the range is 182.6 – 180.3 = 2.3°C.

b. Note: If we apply the hint and subtract 180 from each observation, the mean will be 1.41, and the middle two columns will not change. The sum and sum of squares will change, but those effects will cancel and the answer below will stay the same.

	x_i	$(x_i - \overline{x})$	$(x_i - \overline{x})^2$	x_i^2
	180.5	–0.90833	0.82507	32580.3
	181.7	0.29167	0.08507	33014.9
	180.9	–0.50833	0.25840	32724.8
	181.6	0.19167	0.03674	32978.6
	182.6	1.19167	1.42007	33342.8
	181.6	0.19167	0.03674	32978.6
	181.3	–0.10833	0.01174	32869.7
	182.1	0.69167	0.47840	33160.4
	182.1	0.69167	0.47840	33160.4
	180.3	–1.10833	1.22840	32508.1
	181.7	0.29167	0.08507	33014.9
	180.5	–0.90833	0.82507	32580.3
sums:	2176.9	0	5.769167	394913.6
	$\overline{x}$ = 181.41			

$s^2 = \sum_{i=1}^{n} (x_i - \overline{x})^2 / (n-1)$ = 5.769167/(12 – 1) = 0.52447.

c. $s = \sqrt{0.52447}$ = 0.724.

d. $s^2 = \dfrac{\Sigma x^2 - (\Sigma x)^2 / n}{n-1} = \dfrac{394913.6 - (2176.9)^2 / 12}{11} = 0.52447$.

45.

a. $\overline{x}$ = 115.58. The deviations from the mean are 116.4 – 115.58 = .82, 115.9 – 115.58 = .32, 114.6 –115.58 = –.98, 115.2 – 115.58 = –.38, and 115.8 – 115.58 = .22. Notice that the deviations from the mean sum to zero, as they should.

b. $s^2 = [(.82)^2 + (.32)^2 + (-.98)^2 + (-.38)^2 + (.22)^2]/(5 - 1)$ = 1.928/4 = .482, so s = .694.

c. Σx_i^2 = 66795.61, so $s^2 = S_{xx}/(n - 1) = \left(\Sigma x_i^2 - (\Sigma x_i)^2 / n\right) / (n-1)$ = (66795.61 –(577.9)^2/5)/4 = 1.928/4 = .482.

d. The new sample values are: 16.4 15.9 14.6 15.2 15.8. While the new mean is 15.58, all the deviations are the same as in part (a), and the variance of the transformed data is identical to that of part (b).

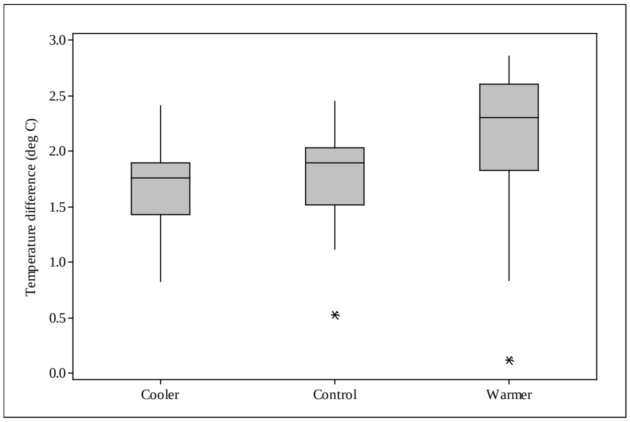
46.

a. Since all three distributions are somewhat skewed and two contain outliers (see **d**), medians are the more appropriate central measures. The medians are
Cooler: 1.760°C Control: 1.900°C Warmer: 2.305°C
The median difference between air and soil temperature increases as the conditions of the minichambers transition from cooler to warmer (1.76 < 1.9 < 2.305).

b. With the aid of software, the standard deviations are
Cooler: 0.401°C Control: 0.531°C Warmer: 0.778°C
For the 15 observations under the "cooler" conditions, the typical deviation between an observed temperature difference and the <u>mean</u> temperature difference (1.760°C) is roughly 0.4°C. A similar interpretation applies to the other two standard deviations.
We see that, according to the standard deviations, variability increases as the conditions of the minichambers transition from cooler to warmer (0.401 < 0.531 < 0.778).

c. Apply the definitions of lower fourth, upper fourth, and fourth spread to the sorted data within each condition.
Cooler: lower fourth = (1.43 + 1.57)/2 = 1.50, upper fourth = (1.88 + 1.90)/2 = 1.89,
$f_s = 1.89 - 1.50 = 0.39$°C
Control: lower fourth = (1.52 + 1.78)/2 = 1.65, upper fourth = (2.00 + 2.03)/2 = 2.015,
$f_s = 2.015 - 1.65 = 0.365$°C
Warmer: lower fourth = 1.91, upper fourth = 2.60,
$f_s = 2.60 - 1.91 = 0.69$°C
The fourth spreads do <u>not</u> communicate the same message as the standard deviations did. The fourth spreads indicate that variability is quite similar under the cooler and control settings, while variability is much larger under the warmer setting. The disparity between the results of **b** and **c** can be partly attributed to the skewness and outliers in the data, which unduly affect the standard deviations.

d. As noted earlier, the temperature difference distributions are negatively skewed under all three conditions. The control and warmer data sets each have a single outlier. The boxplots confirm that median temperature difference increases as we transition from cooler to warmer, that cooler and control variability are similar, and that variability under the warmer condition is quite a bit larger.



47.

a. From software, $\tilde{x}$ = 14.7% and $\overline{x}$ = 14.88%. The sample average alcohol content of these 10 wines was 14.88%. Half the wines have alcohol content below 14.7% and half are above 14.7% alcohol.

b. Working long-hand, $\Sigma(x_i - \overline{x})^2$ = (14.8 – 14.88)^2 + … + (15.0 – 14.88)^2 = 7.536. The sample variance equals s^2 = $\Sigma(x_i - \overline{x})^2$ = 7.536/(10 – 1) = 0.837.

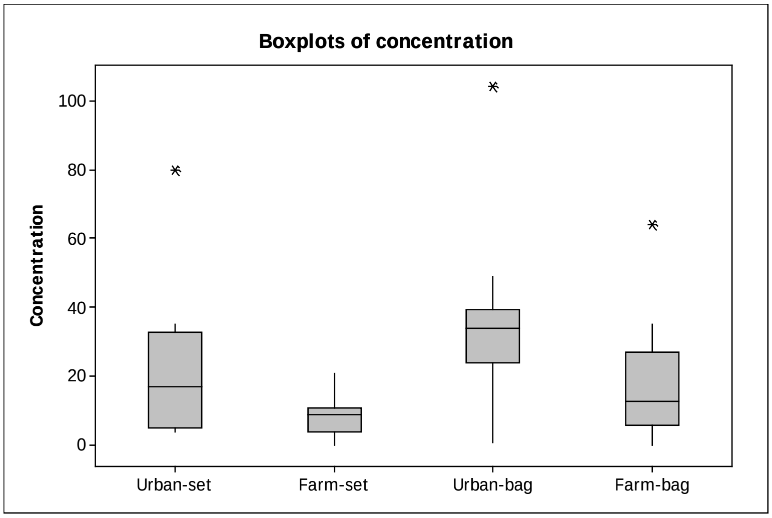
c. Subtracting 13 from each value will not affect the variance. The 10 new observations are 1.8, 1.5, 3.1, 1.2, 2.9, 0.7, 3.2, 1.6, 0.8, and 2.0. The sum and sum of squares of these 10 new numbers are Σy_i = 18.8 and Σy_i^2 = 42.88. Using the sample variance shortcut, we obtain s^2 = [42.88 – (18.8)^2/10]/(10 – 1) = 7.536/9 = 0.837 again.

48.

a. Using the sums provided for urban homes, S_{xx} = 10,079 – (237.0)^2/11 = 4972.73, so s = $\sqrt{\frac{4972.73}{11-1}}$ = 22.3 EU/mg. Similarly for farm homes, S_{xx} = 518.836 and s = 6.09 EU/mg. The endotoxin concentration in an urban home “typically” deviates from the average of 21.55 by about 22.3 EU/mg. The endotoxin concentration in a farm home “typically” deviates from the average of 8.56 by about 6.09 EU/mg. (These interpretations are very loose, especially since the distributions are not symmetric.) In any case, the variability in endotoxin concentration is far greater in urban homes than in farm homes.

b. The upper and lower fourths of the urban data are 28.0 and 5.5, respectively, for a fourth spread of 22.5 EU/mg. The upper and lower fourths of the farm data are 10.1 and 4, respectively, for a fourth spread of 6.1 EU/mg. Again, we see that the variability in endotoxin concentration is much greater for urban homes than for farm homes.

c. Consider the box plots below. The endotoxin concentration in urban homes generally exceeds that in farm homes, whether measured by settled dust or bag dust. The endotoxin concentration in bag dust generally exceeds that of settled dust, both in urban homes and in farm homes. Settled dust in farm homes shows far less variability than any other scenario.



49.

a. $\Sigma x_i = 2.75 + \cdots + 3.01 = 56.80$, $\Sigma x_i^2 = 2.75^2 + \cdots + 3.01^2 = 197.8040$

b. $s^2 = \dfrac{197.8040 - (56.80)^2/17}{16} = \dfrac{8.0252}{16} = .5016, \quad s = .708$

50. From software or from the sums provided, $\bar{x}$ = 20179/27 = 747.37 and $s = \sqrt{\dfrac{24657511 - (20179)^2/27}{26}} = 606.89$. The maximum award should be $\bar{x} + 2s$ = 747.37 + 2(606.89) = 1961.16, or $1,961,160. This is quite a bit less than the $3.5 million that was awarded originally.

51.

a. From software, s^2 = 1264.77 min^2 and s = 35.56 min. Working by hand, $\Sigma x = 2563$ and $\Sigma x^2 = 368501$, so

$$s^2 = \frac{368501 - (2563)^2/19}{19-1} = 1264.766 \text{ and } s = \sqrt{1264.766} = 35.564$$

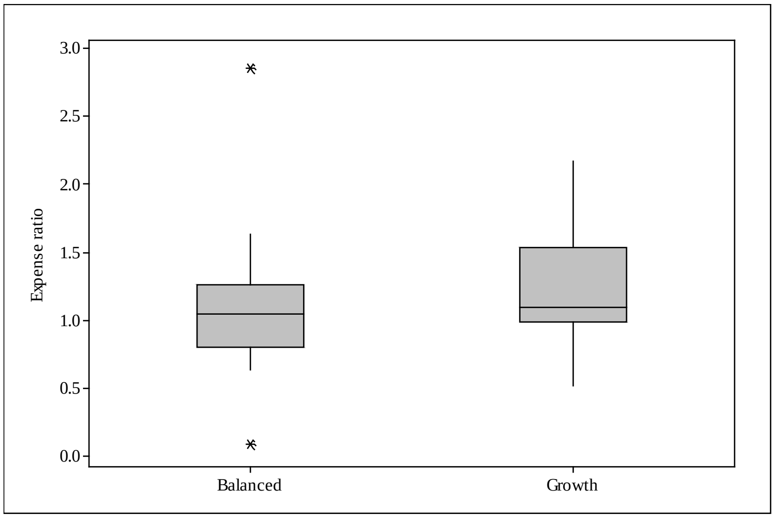
b. If y = time in hours, then $y = cx$ where $c = \frac{1}{60}$. So, $s_y^2 = c^2 s_x^2 = \left(\frac{1}{60}\right)^2 1264.766 = .351\,\text{hr}^2$ and $s_y = cs_x = \left(\frac{1}{60}\right)35.564 = .593\,\text{hr}$.

52. Let d denote the fifth deviation. Then $.3 + .9 + 1.0 + 1.3 + d = 0$ or $3.5 + d = 0$, so $d = -3.5$. One sample for which these are the deviations is $x_1 = 3.8$, $x_2 = 4.4$, $x_3 = 4.5$, $x_4 = 4.8$, $x_5 = 0$. (These were obtained by adding 3.5 to each deviation; adding any other number will produce a different sample with the desired property.)

53.

a. Using software, for the sample of balanced funds we have $\bar{x} = 1.121, \tilde{x} = 1.050, s = 0.536$; for the sample of growth funds we have $\bar{x} = 1.244, \tilde{x} = 1.100, s = 0.448$.

b. The distribution of expense ratios for this sample of balanced funds is fairly symmetric, while the distribution for growth funds is positively skewed. These balanced and growth mutual funds have similar median expense ratios (1.05% and 1.10%, respectively), but expense ratios are generally higher for growth funds. The lone exception is a balanced fund with a 2.86% expense ratio. (There is also one unusually low expense ratio in the sample of balanced funds, at 0.09%.)



54.

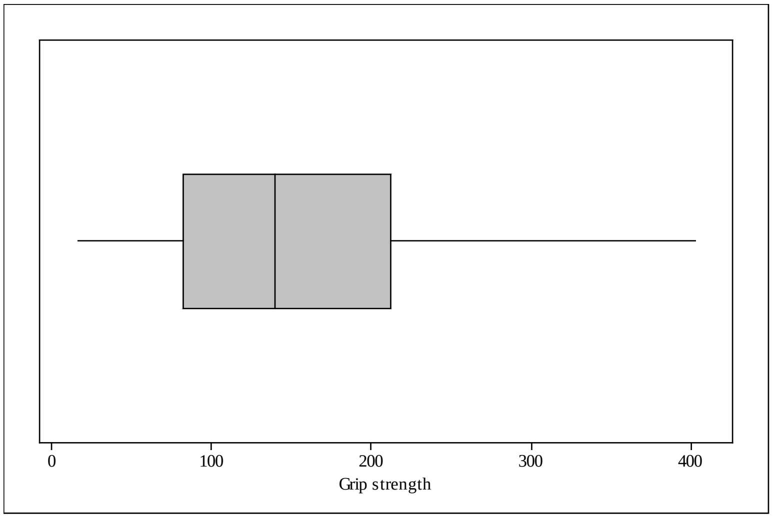
a. Minitab provides the stem-and-leaf display below. Grip strengths for this sample of 42 individuals are positively skewed, and there is one high outlier at 403 N.

```
 6    0  111234
 14   0  55668999
(10)  1  0011223444           Stem = 100s
 18   1  567889               Leaf = 10s
 12   2  01223334
 4    2  59
 2    3  2
 1    3
 1    4  0
```

b. Each half has 21 observations. The lower fourth is the 11th observation, 87 N. The upper fourth is the 32nd observation (11th from the top), 210 N. The fourth spread is the difference: f_s = 210 – 87 = 123 N.

c. min = 16; lower fourth = 87; median = 140; upper fourth = 210; max = 403

The boxplot tells a similar story: grip strengths are slightly positively skewed, with a median of 140N and a fourth spread of 123 N.



d. inner fences: 87 – 1.5(123) = –97.5, 210 + 1.5(123) = 394.5
outer fences: 87 – 3(123) = –282, 210 + 3(123) = 579
Grip strength can't be negative, so low outliers are impossible here. A mild high outlier is above 394.5 N and an extreme high outlier is above 579 N. The value 403 N is a mild outlier by this criterion. (Note: some software uses slightly different rules to define outliers — using quartiles and interquartile range — which result in 403 N not being classified as an outlier.)

e. The fourth spread is unaffected unless that observation drops below the current upper fourth, 210. That's a decrease of 403 – 210 = 193 N.

55.

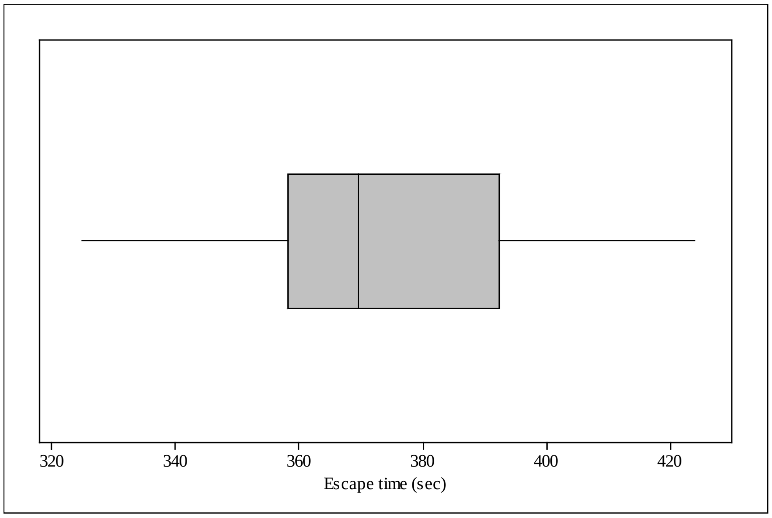
a. Lower half of the data set: 325 325 334 339 356 356 359 359 363 364 364 366 369, whose median, and therefore the lower fourth, is 359 (the 7th observation in the sorted list).

Upper half of the data set: 370 373 373 374 375 389 392 393 394 397 402 403 424, whose median, and therefore the upper fourth is 392.

So, $f_s = 392 - 359 = 33$.

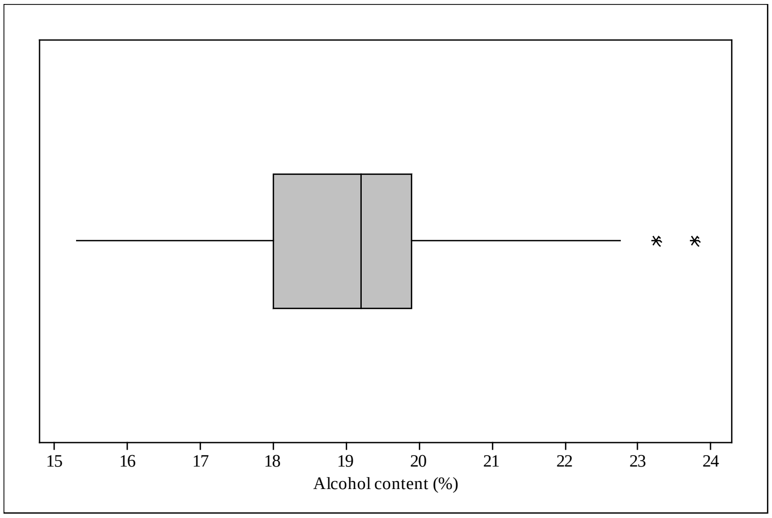
b. inner fences: $359 - 1.5(33) = 309.5$, $392 + 1.5(33) = 441.5$
To be a mild outlier, an observation must be below 309.5 or above 441.5. There are none in this data set. Clearly, then, there are also no extreme outliers.

c. A boxplot of this data appears below. The distribution of escape times is roughly symmetric with no outliers. Notice the box plot "hides" the fact that the distribution contains two gaps, which can be seen in the stem-and-leaf display.



d. Not until the value $x = 424$ is lowered below the upper fourth value of 392 would there be any change in the value of the upper fourth (and, thus, of the fourth spread). That is, the value $x = 424$ could not be decreased by more than $424 - 392 = 32$ seconds.

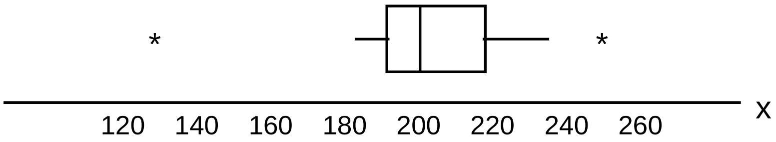
56. The alcohol content distribution of this sample of 35 port wines is roughly symmetric except for two high outliers. The median alcohol content is 19.2% and the fourth spread is 1.42%. [upper fourth = (19.90 + 19.62)/2 = 19.76; lower fourth = (18.00 + 18.68)/2 = 18.34] The two outliers were 23.25% and 23.78%, indicating two port wines with unusually high alcohol content.



57.

a. $f_s = 216.8 - 196.0 = 20.8$
inner fences: 196 – 1.5(20.8) = 164.6, 216.8 + 1.5(20.8) = 248
outer fences: 196 – 3(20.8) = 133.6, 216.8 + 3(20.8) = 279.2
Of the observations listed, 125.8 is an extreme low outlier and 250.2 is a mild high outlier.

b. A boxplot of this data appears below. There is a bit of positive skew to the data but, except for the two outliers identified in part (a), the variation in the data is relatively small.



58. The most noticeable feature of the comparative boxplots is that machine 2's sample values have considerably more variation than does machine 1's sample values. However, a typical value, as measured by the median, seems to be about the same for the two machines. The only outlier that exists is from machine 1.

59.

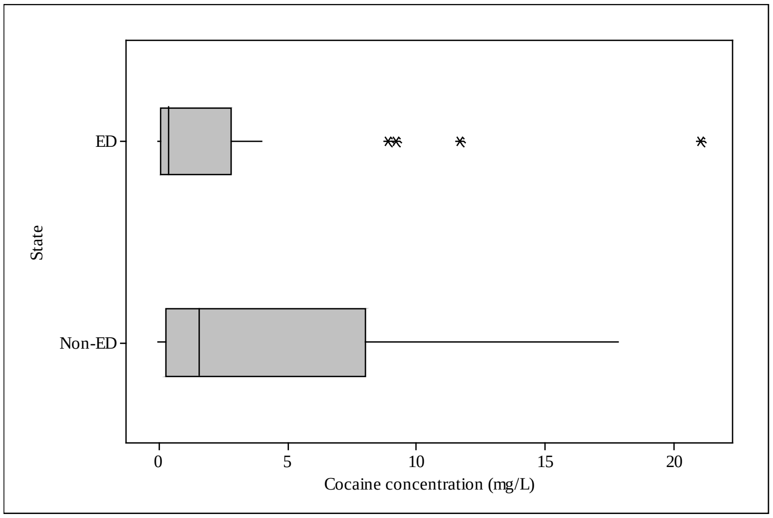
a. If you aren't using software, don't forget to *sort* the data first!
ED: median = .4, lower fourth = (.1 + .1)/2 = .1, upper fourth = (2.7 + 2.8)/2 = 2.75, fourth spread = 2.75 – .1 = 2.65

Non-ED: median = (1.5 + 1.7)/2 = 1.6, lower fourth = .3, upper fourth = 7.9, fourth spread = 7.9 – .3 = 7.6.

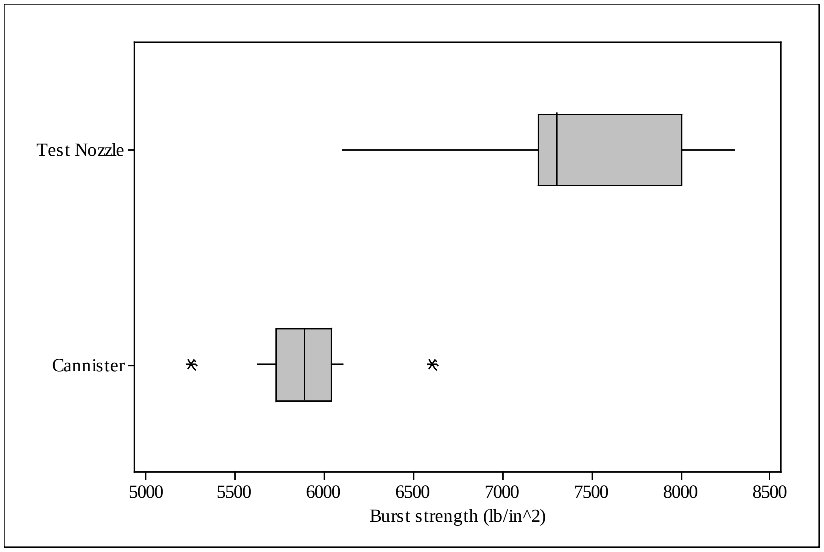
b. *ED*: mild outliers are less than .1 – 1.5(2.65) = –3.875 or greater than 2.75 + 1.5(2.65) = 6.725. Extreme outliers are less than .1 – 3(2.65) = –7.85 or greater than 2.75 + 3(2.65) = 10.7. So, the two largest observations (11.7, 21.0) are extreme outliers and the next two largest values (8.9, 9.2) are mild outliers. There are no outliers at the lower end of the data.

Non-ED: mild outliers are less than .3 – 1.5(7.6) = –11.1 or greater than 7.9 + 1.5(7.6) = 19.3. Note that there are no mild outliers in the data, hence there cannot be any extreme outliers, either.

c. A comparative boxplot appears below. The outliers in the ED data are clearly visible. There is noticeable positive skewness in both samples; the Non-ED sample has more variability then the Ed sample; the typical values of the ED sample tend to be smaller than those for the Non-ED sample.



60. A comparative boxplot (created in Minitab) of this data appears below. The burst strengths for the test nozzle closure welds are quite different from the burst strengths of the production canister nozzle welds. The test welds have much higher burst strengths and the burst strengths are much more variable. The production welds have more consistent burst strength and are consistently lower than the test welds. The production welds data does contain 2 outliers.



61. Outliers occur in the 6a.m. data. The distributions at the other times are fairly symmetric. Variability and the "typical" gasoline-vapor coefficient values increase somewhat until 2p.m., then decrease slightly.

Supplementary Exercises

62. To simplify the math, subtract the mean from each observation; i.e., let $y_i = x_i - \overline{x} =$ $x_i - 76831$. Then $y_1 = 76683 - 76831 = –148$ and $y_4 = 77048 - 76831 = 217$; by rescaling, $\overline{y} = \overline{x} - 76831 = 0$, so $y_2 + y_3 = –(y_1 + y_4) = –69$. Also,

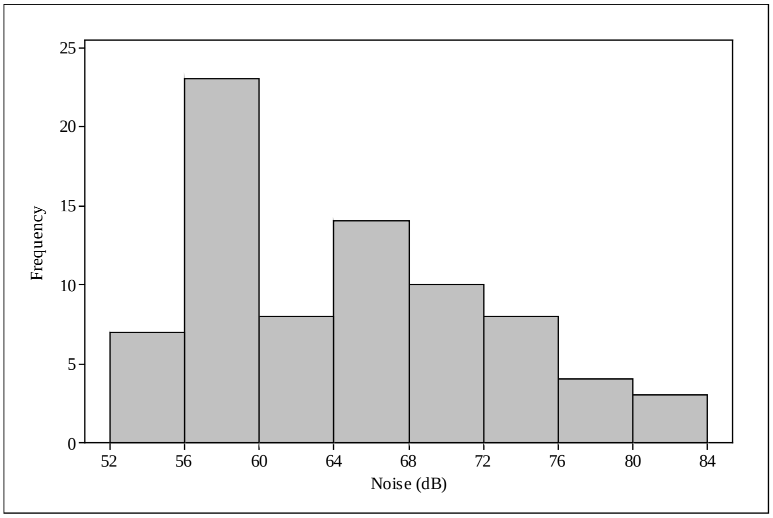
$$180 = s = \sqrt{\frac{\Sigma(x_i - \overline{x})^2}{n-1}} = \sqrt{\frac{\Sigma y_i^2}{3}} \quad \Rightarrow \quad \Sigma y_i^2 = 3(180)^2 = 97200$$

so $y_2^2 + y_3^2 = 97200 - (y_1^2 + y_4^2) = 97200 - ((-148)^2 + (217)^2) = 28207$.

To solve the equations $y_2 + y_3 = –69$ and $y_2^2 + y_3^2 = 28207$, substitute $y_3 = –69 – y_2$ into the second equation and use the quadratic formula to solve. This gives $y_2 = 79.14$ or –148.14 (one is y_2 and one is y_3).

Finally, x_2 and x_3 are given by $y_2 + 76831$ and $y_3 + 76831$, or 79,610 and 76,683.

63. As seen in the histogram below, this noise distribution is bimodal (but close to unimodal) with a positive skew and no outliers. The mean noise level is 64.89 dB and the median noise level is 64.7 dB. The fourth spread of the noise measurements is about 70.4 – 57.8 = 12.6 dB.

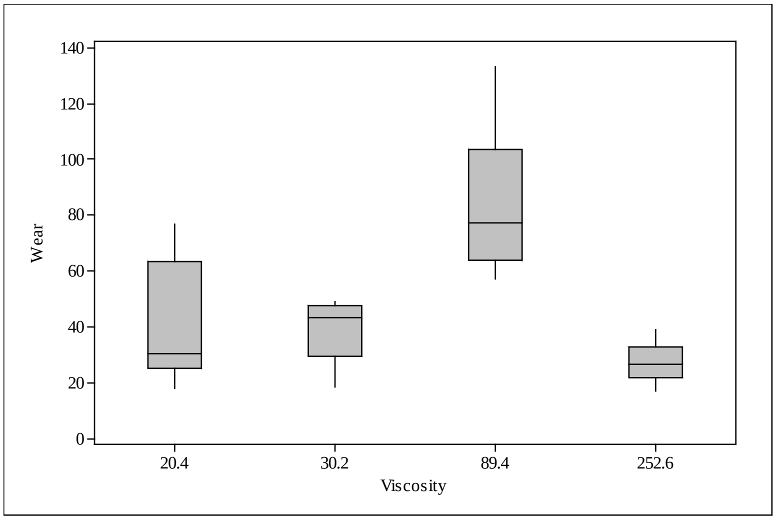


64.

a. The sample coefficient of variation is similar for the three highest oil viscosity levels (29.66, 32.30, 27.86) but is much higher for the lowest viscosity (56.01). At low viscosity, it appears that there is much more variation in volume wear relative to the average or "typical" amount of wear.

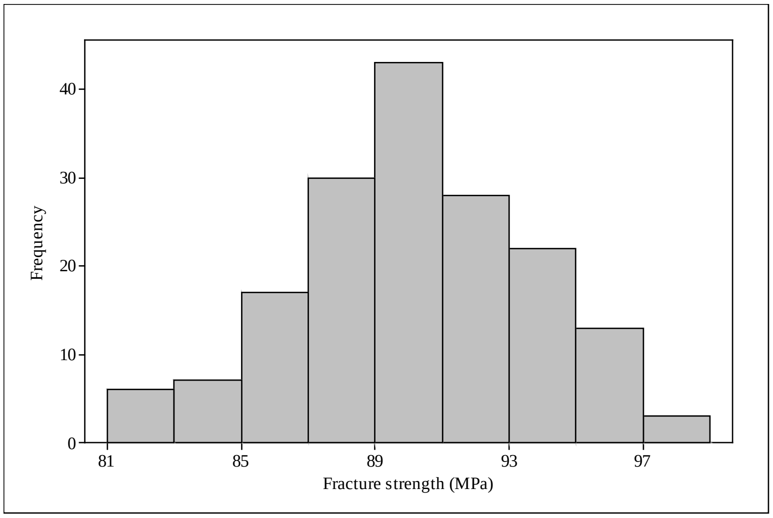
	Wear		
Viscosity	$\overline{x}$	s	cv
20.4	40.17	22.50	56.01
30.2	38.83	11.52	29.66
89.4	84.10	27.20	32.30
252.6	27.10	7.55	27.86

b. Volume wear varies dramatically by viscosity level. At very high viscosity, wear is typically the least and the least variable. Volume wear is actually by far the highest at a "medium" viscosity level and also has the greatest variability at this viscosity level. "Lower" viscosity levels correspond to less wear than a medium level, though there is much greater (relative) variation at a very low viscosity level.



65.

a. The histogram appears below. A representative value for this data would be around 90 MPa. The histogram is reasonably symmetric, unimodal, and somewhat bell-shaped with a fair amount of variability (*s* ≈ 3 or 4 MPa).

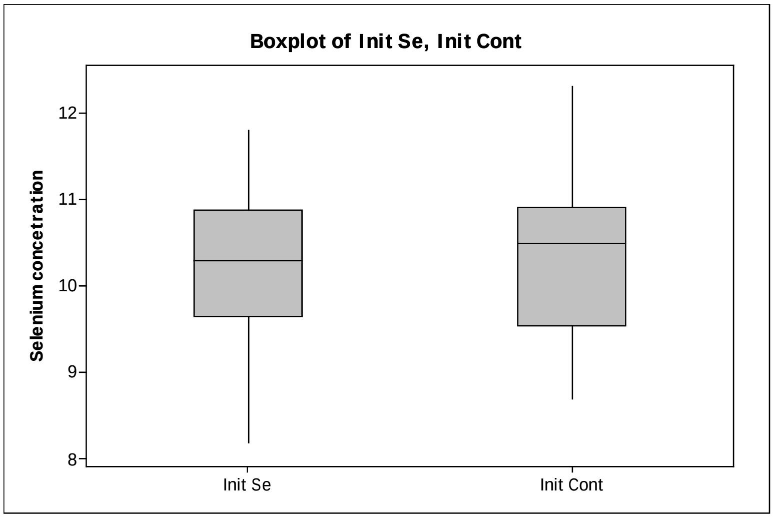


b. The proportion of the observations that are at least 85 is 1 – (6+7)/169 = .9231. The proportion less than 95 is 1 – (13+3)/169 = .9053.

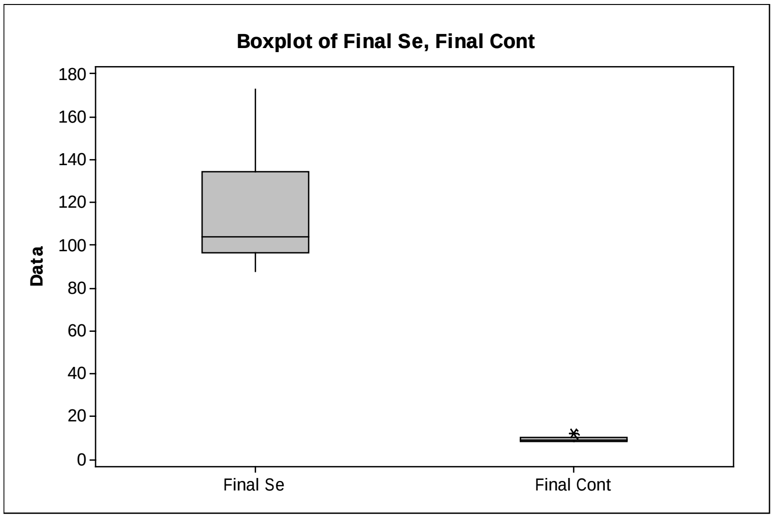
c. 90 is the midpoint of the class 89–<91, which contains 43 observations (a relative frequency of 43/169 = .2544). Therefore about half of this frequency, .1272, should be added to the relative frequencies for the classes to the left of $x = 90$. That is, the approximate proportion of observations that are less than 90 is .0355 + .0414 + .1006 + .1775 + .1272 = .4822.

66.

a. The initial Se concentrations in the treatment and control groups are not that different. The differences in the box plots below are minor. The median initial Se concentrations for the treatment and control groups are 10.3 mg/L and 10.5 mg/L, respectively, each with fourth spread of about 1.25 mg/L. So, the two groups of cows are comparable at the beginning of the study.

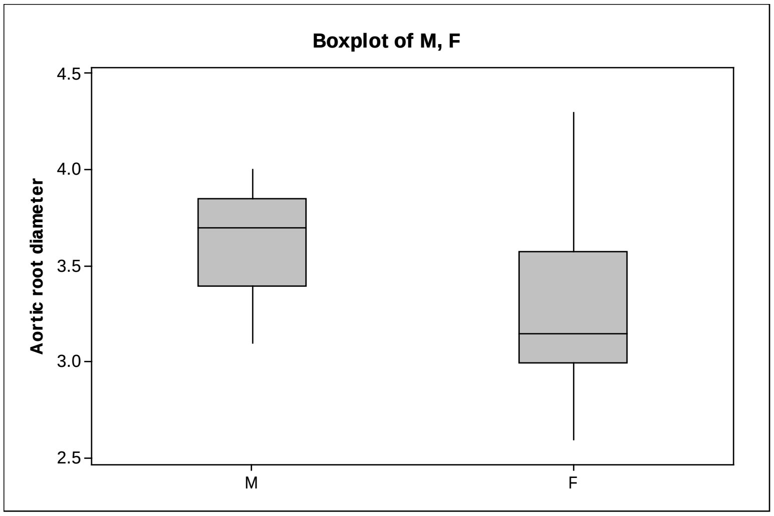


b. The final Se concentrations of the two groups are extremely different, as evidenced by the box plots below. Whereas the median final Se concentration for the control group is 9.3 mg/L (actually slightly lower than the initial concentration), the median Se concentration in the treatment group is now 103.9 mg/L, nearly a 10-fold increase.



67.

a. Aortic root diameters for males have mean 3.64 cm, median 3.70 cm, standard deviation 0.269 cm, and fourth spread 0.40. The corresponding values for females are $\bar{x}$ = 3.28 cm, $\tilde{x}$ = 3.15 cm, s = 0.478 cm, and f_s = 0.50 cm. Aortic root diameters are typically (though not universally) somewhat smaller for females than for males, and females show more variability. The distribution for males is negatively skewed, while the distribution for females is positively skewed (see graphs below).



b. For females (n = 10), the 10% trimmed mean is the average of the middle 8 observations: $\bar{x}_{tr(10)}$ = 3.24 cm. For males (n = 13), the 1/13 trimmed mean is 40.2/11 = 3.6545, and the 2/13 trimmed mean is 32.8/9 = 3.6444. Interpolating, the 10% trimmed mean is $\bar{x}_{tr(10)}$ = 0.7(3.6545) + 0.3(3.6444) = 3.65 cm. (10% is three-tenths of the way from 1/13 to 2/13).

68.

a.
$$\frac{d}{dc}\left\{\sum(x_i-c)^2\right\}=\sum\frac{d}{dc}(x_i-c)^2=-2\sum(x_i-c)=0\Rightarrow\sum(x_i-c)=0\Rightarrow$$
$$\sum x_i-\sum c=0\Rightarrow\sum x_i-nc=0\Rightarrow nc=\sum x_i\Rightarrow c=\frac{\sum x_i}{n}=\bar{x}$$

b. Since $c=\bar{x}$ minimizes $\Sigma(x_i-c)^2$, $\Sigma(x_i-\bar{x})^2<\Sigma(x_i-\mu)^2$.

69.

a.

$$\overline{y}=\frac{\sum y_i}{n}=\frac{\sum\left(ax_i+b\right)}{n}=\frac{a\sum x_i+\sum b}{n}=\frac{a\sum x_i+nb}{n}=a\overline{x}+b$$

$$s_y^2=\frac{\sum\left(y_i-\overline{y}\right)^2}{n-1}=\frac{\sum\left(ax_i+b-(a\overline{x}+b)\right)^2}{n-1}=\frac{\sum\left(ax_i-a\overline{x}\right)^2}{n-1}$$

$$=\frac{a^2\sum\left(x_i-\overline{x}\right)^2}{n-1}=a^2s_x^2$$

b.

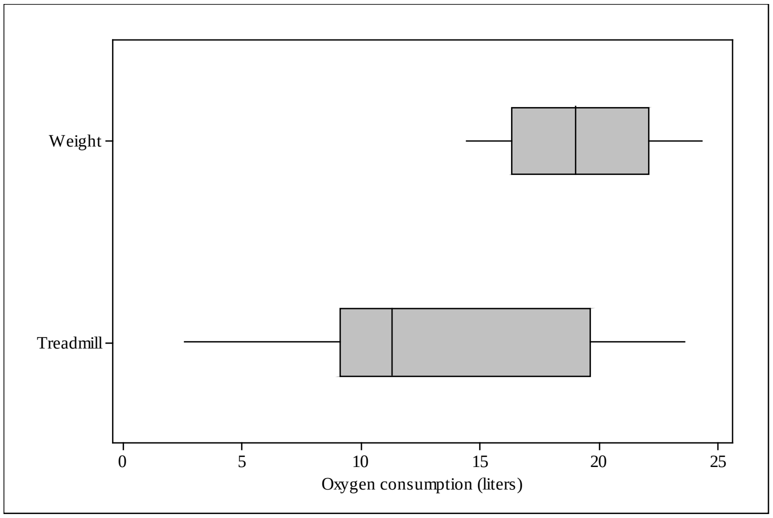
$$x={}^\circ\mathrm{C},\ y={}^\circ\mathrm{F}$$

$$\overline{y}=\frac{9}{5}\left(87.3\right)+32=189.14^\circ\mathrm{F}$$

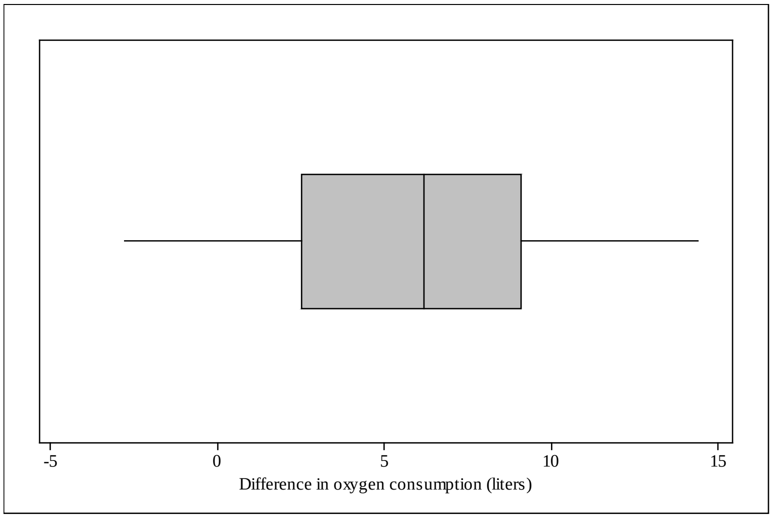
$$s_y=\sqrt{s_y^2}=\sqrt{\left(\frac{9}{5}\right)^2\left(1.04\right)^2}=\sqrt{3.5044}=1.872^\circ\mathrm{F}$$

70.

a. There is a significant difference in the variability of the two samples. The weight training produced much higher oxygen consumption, on average, than the treadmill exercise, with the median consumptions being approximately 20 and 11 liters, respectively.



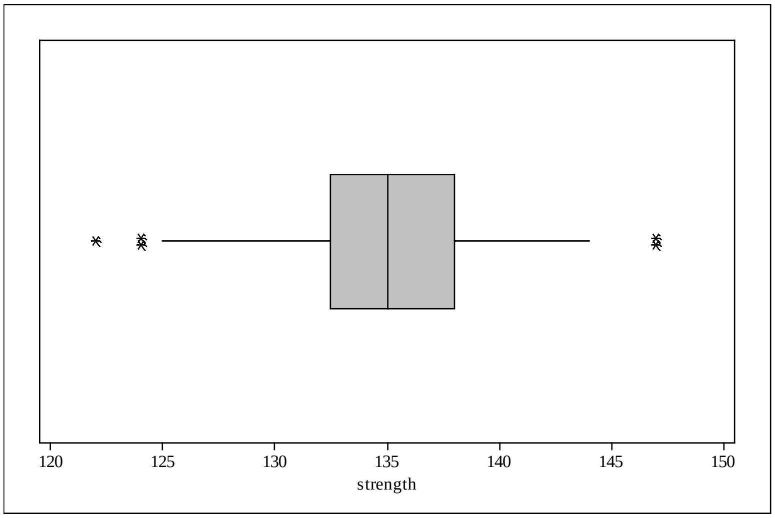
b. The differences in oxygen consumption (weight minus treadmill) for the 15 subjects are 3.3, 9.1, 10.4, 9.1, 6.2, 2.5, 2.2, 8.4, 8.7, 14.4, 2.5, –2.8, –0.4, 5.0, and 11.5. The majority of the differences are positive, which suggests that the weight training produced higher oxygen consumption for most subjects than the treadmill did. The median difference is about 6 liters.



71.

a. The mean, median, and trimmed mean are virtually identical, which suggests symmetry. If there are outliers, they are balanced. The range of values is only 25.5, but half of the values are between 132.95 and 138.25.

b. See the comments for (a). In addition, using 1.5(Q3 – Q1) as a yardstick, the two largest and three smallest observations are mild outliers.

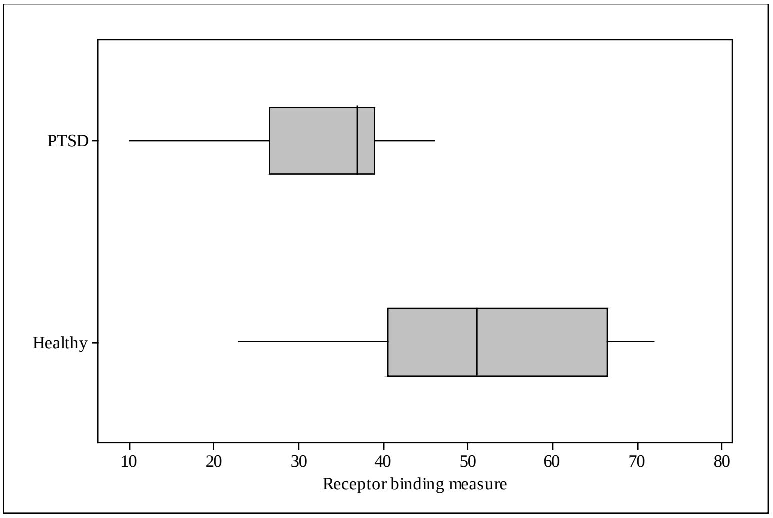


72. A table of summary statistics, a stem and leaf display, and a comparative boxplot are below. The healthy individuals have higher receptor binding measure, on average, than the individuals with PTSD. There is also more variation in the healthy individuals' values. The distribution of values for the healthy is reasonably symmetric, while the distribution for the PTSD individuals is negatively skewed. The box plot indicates that there are no outliers, and confirms the above comments regarding symmetry and skewness.

	PTSD	Healthy
Mean	32.92	52.23
Median	37	51
Std Dev	9.93	14.86
Min	10	23
Max	46	72

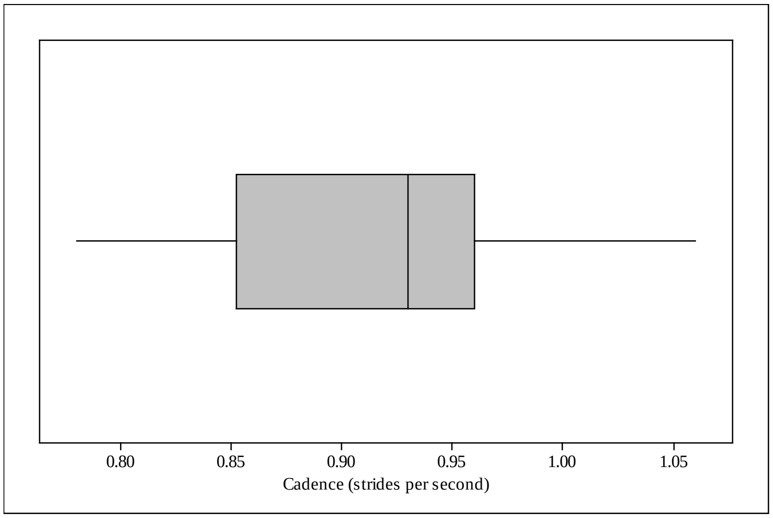
Healthy		PTSD
	1	0
3	2	058
9	3	1578899
7310	4	26
81	5	
9763	6	
2	7	

stem = tens
leaf = ones



73. From software, $\overline{x}$ = .9255, s = .0809; $\tilde{x}$ = .93, f_s = .1. The cadence observations are slightly skewed (mean = .9255 strides/sec, median = .93 strides/sec) and show a small amount of variability (standard deviation = .0809, fourth spread = .1). There are no apparent outliers in the data.

7|8 stem = tenths
8|11556 leaf = hundredths
9|2233335566
0|0566



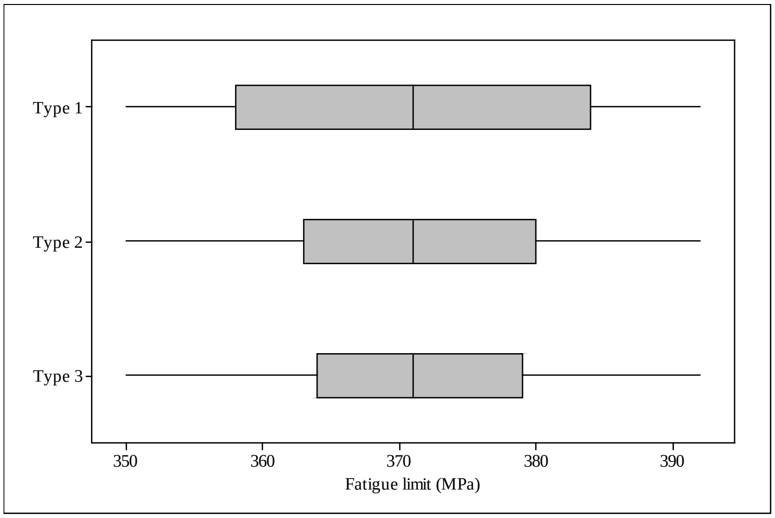
74.

a. The mode is .93. It occurs four times in the data set.

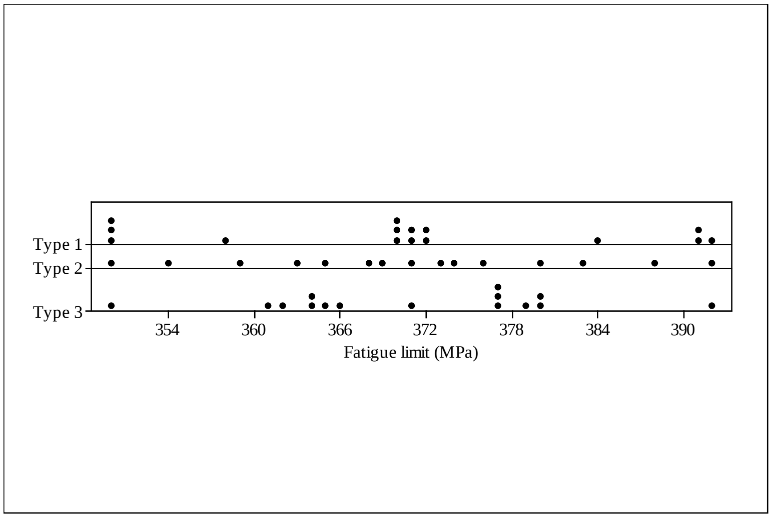
b. The *modal category* is the one in which the most observations occur; i.e., the modal category has the highest frequency. In a survey, the modal category is the most popular answer.

75.

a. The median is the same (371) in each plot and all three data sets are very symmetric. In addition, all three have the same minimum value (350) and same maximum value (392). Moreover, all three data sets have the same lower (364) and upper quartiles (378). So, all three boxplots will be *identical*. (Slight differences in the boxplots below are due to the way Minitab software interpolates to calculate the quartiles.)



b. A comparative dotplot is shown below. These graphs show that there are differences in the variability of the three data sets. They also show differences in the way the values are distributed in the three data sets, especially big differences in the presence of gaps and clusters.



c. The boxplot in (a) is not capable of detecting the differences among the data sets. The primary reason is that boxplots give up some detail in describing data because they use only five summary numbers for comparing data sets.

76. The measures that are sensitive to outliers are: the mean and the midrange. The mean is sensitive because all values are used in computing it. The midrange is sensitive because it uses only the most extreme values in its computation.

The median, the trimmed mean, and the midfourth are not sensitive to outliers.

The median is the most resistant to outliers because it uses only the middle value (or values) in its computation.

The trimmed mean is somewhat resistant to outliers. The larger the trimming percentage, the more resistant the trimmed mean becomes.

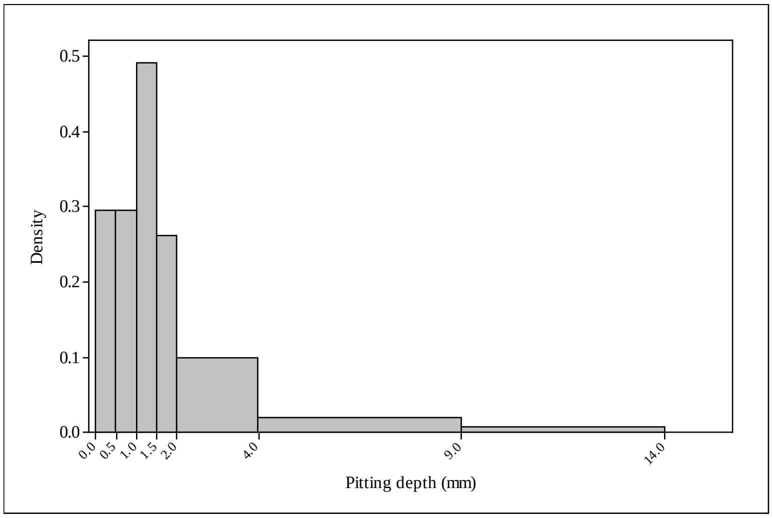
The midfourth, which uses the quartiles, is reasonably resistant to outliers because both quartiles are resistant to outliers.

77.

a.

```
0   444444444577888999           leaf = 1.0
1   00011111111124455669999      stem = 0.1
2   1234457
3   11355
4   17
5   3
6
7   67
8   1

HI  10.44, 13.41
```

b. Since the intervals have unequal width, you must use a *density scale*.



c. Representative depths are quite similar for the three types of soils — between 1.5 and 2. Data from the C and CL soils shows much more variability than for the other two types. The boxplots for the first three types show substantial positive skewness both in the middle 50% and overall. The boxplot for the SYCL soil shows negative skewness in the middle 50% and mild positive skewness overall. Finally, there are multiple outliers for the first three types of soils, including extreme outliers.

78.

a. Since the constant $\bar{x}$ is subtracted from each *x* value to obtain each *y* value, and addition or subtraction of a constant doesn't affect variability, $s_y^2 = s_x^2$ and $s_y = s_x$.

b. Let *c* = 1/*s*, where *s* is the sample standard deviation of the *x*'s (and also, by part (a), of the *y*'s). Then $z_i = cy_i \Rightarrow s_z^2 = c^2 s_y^2 = (1/s)^2 s^2$ = 1 and s_z = 1. That is, the "standardized" quantities $z_1, \ldots, z_n$ have a sample variance and standard deviation of 1.

79.

a. $\sum_{i=1}^{n+1} x_i = \sum_{i=1}^{n} x_i + x_{n+1} = n\bar{x}_n + x_{n+1}$, so $\bar{x}_{n+1} = \frac{1}{n+1}\sum_{i=1}^{n+1} x_i = \frac{n\bar{x}_n + x_{n+1}}{n+1}$.

b. In the second line below, we artificially add and subtract $n\bar{x}_n^2$ to create the term needed for the sample variance:

$$ns_{n+1}^2 = \sum_{i=1}^{n+1}(x_i - \bar{x}_{n+1})^2 = \sum_{i=1}^{n+1} x_i^2 - (n+1)\bar{x}_{n+1}^2$$

$$= \sum_{i=1}^{n} x_i^2 + x_{n+1}^2 - (n+1)\bar{x}_{n+1}^2 = \left[\sum_{i=1}^{n} x_i^2 - n\bar{x}_n^2\right] + n\bar{x}_n^2 + x_{n+1}^2 - (n+1)\bar{x}_{n+1}^2$$

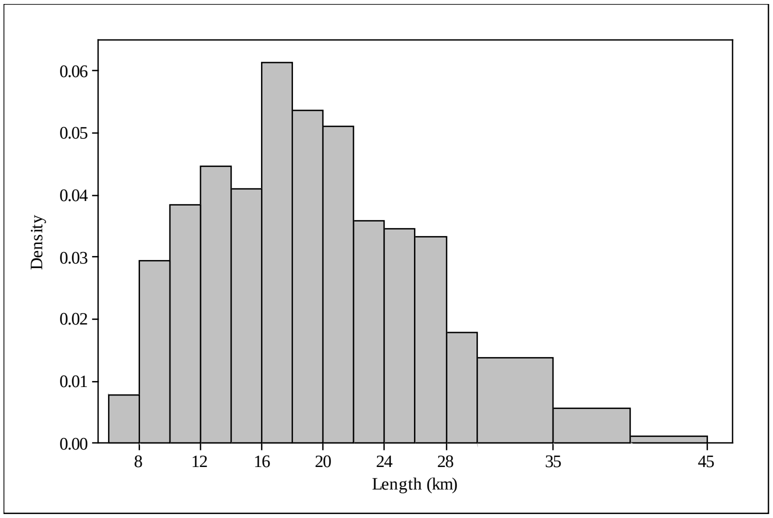
$$= (n-1)s_n^2 + \left\{x_{n+1}^2 + n\bar{x}_n^2 - (n+1)\bar{x}_{n+1}^2\right\}$$

Substitute the expression for $\bar{x}_{n+1}$ from part (a) into the expression in braces, and it simplifies to $\frac{n}{n+1}(x_{n+1} - \bar{x}_n)^2$, as desired.

c. First, $\bar{x}_{16} = \frac{15(12.58)+11.8}{16} = \frac{200.5}{16} = 12.53$. Then, solving (b) for s_{n+1}^2 gives $s_{n+1}^2 = \frac{n-1}{n}s_n^2 + \frac{1}{n+1}(x_{n+1} - \bar{x}_n)^2 = \frac{14}{15}(.512)^2 + \frac{1}{16}(11.8 - 12.58)^2$ = .238. Finally, the standard deviation is $s_{16} = \sqrt{.238} = .532$.

80.

a.



b. There are 391 observations. The proportion of route lengths less than 20 km is (6 + 23 + … + 42)/391 = 216/391 = .552. The proportion of route lengths at least 30 km is (27 + 11 + 2)/391 = 40/391 = .102.

c. First compute (.90)(391 + 1) = 352.8. Thus, the 90th percentile should be about the 352nd ordered value. The 352nd ordered value is the first value in the interval 30–<35. We do not know how the values in that interval are distributed, however, the smallest value (i.e., the 352nd value in the data set) cannot be smaller than 30. So, the 90th percentile is roughly 30.

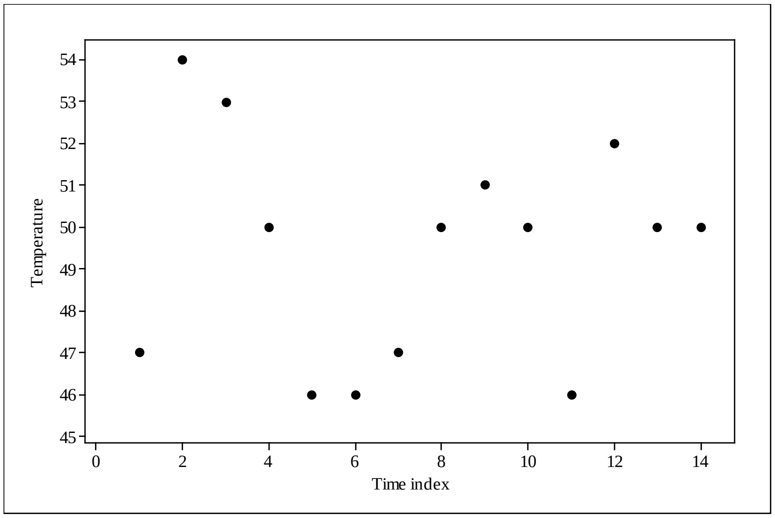
d. First compute (.50)(391 + 1) = 196. Thus the median (50th percentile) should be the 196th ordered value. The 196th observation lies in the interval 18–<20, which includes observations #175 to #216. The 196th observation is about in the middle of these. Thus, we would say the median is roughly 19.

81. Assuming that the histogram is unimodal, then there is evidence of positive skewness in the data since the median lies to the left of the mean (for a symmetric distribution, the mean and median would coincide).

For more evidence of skewness, compare the distances of the 5th and 95th percentiles from the median: median – 5th %ile = 500 – 400 = 100, while 95th %ile – median = 720 – 500 = 220. Thus, the largest 5% of the values (above the 95th percentile) are further from the median than are the lowest 5%. The same skewness is evident when comparing the 10th and 90th percentiles to the median, or comparing the maximum and minimum to the median.

82.

a. There is some evidence of a cyclical pattern.



b. A complete listing of the smoothed values appears below. To illustrate, with *α* = .1 we have $\overline{x}_2 = .1x_2 + .9\overline{x}_1 = (.1)(54) + (.9)(47) = 47.7$, $\overline{x}_3 = .1x_3 + .9\overline{x}_2$ = (.1)(.53) + (.9)(47.7) = 48.23 ≈ 48.2, etc. It's clear from the values below that *α* = .1 gives a smoother series.

t	$\overline{x}_t$ for α = .1	$\overline{x}_t$ for α = .5
1	47.0	47.0
2	47.7	50.5
3	48.2	51.8
4	48.4	50.9
5	48.2	48.4
6	48.0	47.2
7	47.9	47.1
8	48.1	48.6
9	48.4	49.8
10	48.5	49.9
11	48.3	47.9
12	48.6	50.0
13	48.8	50.0
14	48.9	50.0

c. As seen below, $\overline{x}_t$ depends on x_t and all previous values. As *k* increases, the coefficient on x_{t-k} decreases (further back in time implies less weight).

$$\overline{x}_t = \alpha x_t + (1-\alpha)\overline{x}_{t-1} = \alpha x_t + (1-\alpha)[\alpha x_{t-1} + (1-\alpha)\overline{x}_{t-2}]$$
$$= \alpha x_t + \alpha(1-\alpha)x_{t-1} + (1-\alpha)^2[\alpha x_{t-2} + (1-\alpha)\overline{x}_{t-3}] = \cdots$$
$$= \alpha x_t + \alpha(1-\alpha)x_{t-1} + \alpha(1-\alpha)^2 x_{t-2} + \cdots + \alpha(1-\alpha)^{t-2} x_2 + (1-\alpha)^{t-1} x_1$$

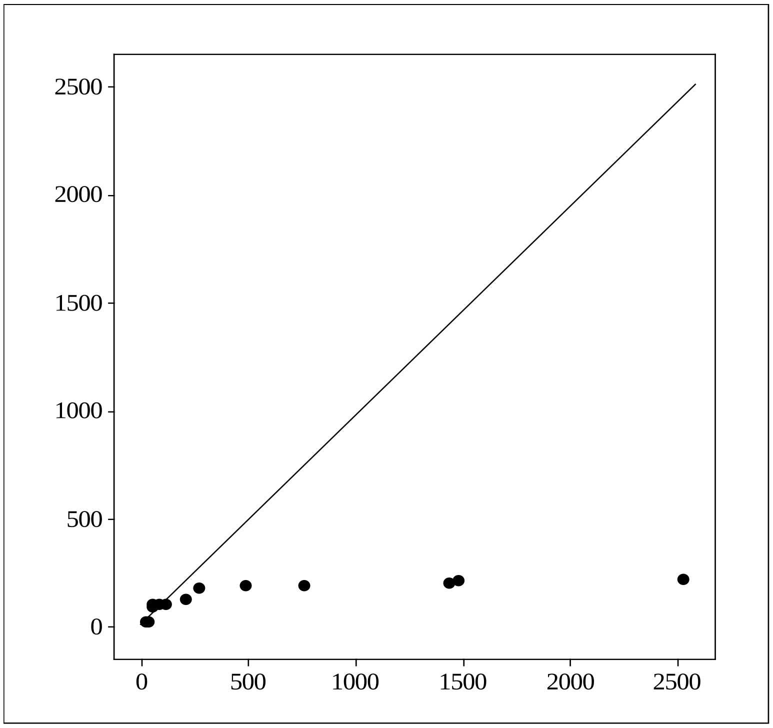
d. For large *t*, the smoothed series is not very sensitive to the initival value x_1, since the coefficient $(1 - \alpha)^{t-1}$ will be very small.

83.

a. When there is perfect symmetry, the smallest observation y_1 and the largest observation y_n will be equidistant from the median, so $y_n - \tilde{x} = \tilde{x} - y_1$. Similarly, the second-smallest and second-largest will be equidistant from the median, so $y_{n-1} - \tilde{x} = \tilde{x} - y_2$, and so on. Thus, the first and second numbers in each pair will be equal, so that each point in the plot will fall exactly on the 45° line.

When the data is positively skewed, y_n will be much further from the median than is y_1, so $y_n - \tilde{x}$ will considerably exceed $\tilde{x} - y_1$ and the point $(y_n - \tilde{x}, \tilde{x} - y_1)$ will fall considerably below the 45° line, as will the other points in the plot.

b. The median of these $n = 26$ observations is 221.6 (the midpoint of the 13th and 14th ordered values). The first point in the plot is (2745.6 – 221.6, 221.6 – 4.1) = (2524.0, 217.5). The others are: (1476.2, 213.9), (1434.4, 204.1), (756.4, 190.2), (481.8, 188.9), (267.5, 181.0), (208.4, 129.2), (112.5, 106.3), (81.2, 103.3), (53.1, 102.6), (53.1, 92.0), (33.4, 23.0), and (20.9, 20.9). The first number in each of the first seven pairs greatly exceeds the second number, so each of those points falls well below the 45° line. A substantial positive skew (stretched upper tail) is indicated.



84. As suggested in the hint, split the sum into two "halves" corresponding to the lowest *n*/2 observations and the highest *n*/2 observations (we'll use *L* and *U* to denote these).

$$\begin{aligned}\sum |x_i - \tilde{x}| &= \sum_L |x_i - \tilde{x}| + \sum_U |x_i - \tilde{x}| \\ &= \sum_L (\tilde{x} - x_i) + \sum_U (x_i - \tilde{x}) \\ &= \sum_L \tilde{x} - \sum_L x_i + \sum_U x_i - \sum_U \tilde{x}\end{aligned}$$

Each of these four sums covers exactly *n*/2 terms. The first and fourth sums are, therefore, both equal to (*n*/2) $\tilde{x}$; these cancel. The inner two sums may be re-written in terms of averages:

$$\begin{aligned}\sum |x_i - \tilde{x}| &= \sum_L \tilde{x} - \sum_L x_i + \sum_U x_i - \sum_U \tilde{x} = -\sum_L x_i + \sum_U x_i \\ &= -(n/2)\overline{x}_L + (n/2)\overline{x}_U \qquad \Rightarrow\end{aligned}$$

$$\sum |x_i - \tilde{x}| / n = (\overline{x}_U - \overline{x}_L)/2$$

When *n* is odd, the middle (ordered) value is exactly $\tilde{x}$. Using *L* and *U* to denote the lowest (*n* – 1)/2 observations and largest (*n* – 1)/2 observations, respectively, we may write $\sum |x_i - \tilde{x}| = \sum_L |x_i - \tilde{x}| + 0 + \sum_U |x_i - \tilde{x}|$, where the 0 comes from the middle (ordered) value, viz. $|\tilde{x} - \tilde{x}| = 0$. The rest of the derivation proceeds exactly as before, except that the surviving sums each have (*n* – 1)/2 terms in them, not *n*/2. As a result, for *n* odd we have $\sum |x_i - \tilde{x}| / (n-1) = (\overline{x}_U - \overline{x}_L)/2$.

CHAPTER 2

Section 2.1

1.

a. S = {1324, 1342, 1423, 1432, 2314, 2341, 2413, 2431, 3124, 3142, 4123, 4132, 3214, 3241, 4213, 4231}.

b. Event A contains the outcomes where 1 is first in the list:
A = {1324, 1342, 1423, 1432}.

c. Event B contains the outcomes where 2 is first or second:
B = {2314, 2341, 2413, 2431, 3214, 3241, 4213, 4231}.

d. The event $A \cup B$ contains the outcomes in A or B or both:
$A \cup B$ = {1324, 1342, 1423, 1432, 2314, 2341, 2413, 2431, 3214, 3241, 4213, 4231}.
$A \cap B = \varnothing$, since 1 and 2 can't both get into the championship game.
$A' = S - A$ = {2314, 2341, 2413, 2431, 3124, 3142, 4123, 4132, 3214, 3241, 4213, 4231}.

2.

a. $A = \{RRR, LLL, SSS\}$.

b. $B = \{RLS, RSL, LRS, LSR, SRL, SLR\}$.

c. $C = \{RRL, RRS, RLR, RSR, LRR, SRR\}$.

d. $D = \{RRL, RRS, RLR, RSR, LRR, SRR, LLR, LLS, LRL, LSL, RLL, SLL, SSR, SSL, SRS, SLS, RSS, LSS\}$

e. Event D' contains outcomes where either all cars go the same direction or they all go different directions:
$D' = \{RRR, LLL, SSS, RLS, RSL, LRS, LSR, SRL, SLR\}$.
Because event D totally encloses event C (see the lists above), the compound event $C \cup D$ is just event D:
$C \cup D = D = \{RRL, RRS, RLR, RSR, LRR, SRR, LLR, LLS, LRL, LSL, RLL, SLL, SSR, SSL, SRS, SLS, RSS, LSS\}$.
Using similar reasoning, we see that the compound event $C \cap D$ is just event C:
$C \cap D = C = \{RRL, RRS, RLR, RSR, LRR, SRR\}$.

3.

a. $A = \{SSF, SFS, FSS\}$.

b. $B = \{SSS, SSF, SFS, FSS\}$.

c. For event C to occur, the system must have component 1 working (S in the first position), then at least one of the other two components must work (at least one S in the second and third positions): $C = \{SSS, SSF, SFS\}$.

d. $C' = \{SFF, FSS, FSF, FFS, FFF\}$.
$A \cup C = \{SSS, SSF, SFS, FSS\}$.
$A \cap C = \{SSF, SFS\}$.
$B \cup C = \{SSS, SSF, SFS, FSS\}$. Notice that B contains C, so $B \cup C = B$.
$B \cap C = \{SSS\ SSF, SFS\}$. Since B contains C, $B \cap C = C$.

4.

a. The $2^4 = 16$ possible outcomes have been numbered here for later reference.

	Home Mortgage Number			
Outcome	1	2	3	4
1	*F*	*F*	*F*	*F*
2	*F*	*F*	*F*	*V*
3	*F*	*F*	*V*	*F*
4	*F*	*F*	*V*	*V*
5	*F*	*V*	*F*	*F*
6	*F*	*V*	*F*	*V*
7	*F*	*V*	*V*	*F*
8	*F*	*V*	*V*	*V*
9	*V*	*F*	*F*	*F*
10	*V*	*F*	*F*	*V*
11	*V*	*F*	*V*	*F*
12	*V*	*F*	*V*	*V*
13	*V*	*V*	*F*	*F*
14	*V*	*V*	*F*	*V*
15	*V*	*V*	*V*	*F*
16	*V*	*V*	*V*	*V*

b. Outcome numbers 2, 3, 5, 9 above.

c. Outcome numbers 1, 16 above.

d. Outcome numbers 1, 2, 3, 5, 9 above.

e. In words, the union of (c) and (d) is the event that either all of the mortgages are variable, or that at most one of them is variable-rate: outcomes 1, 2, 3, 5, 9, 16. The intersection of (c) and (d) is the event that all of the mortgages are fixed-rate: outcome 1.

f. The union of (b) and (c) is the event that either exactly three are fixed, or that all four are the same: outcomes 1, 2, 3, 5, 9, 16. The intersection of (b) and (c) is the event that exactly three are fixed and all four are the same type. This cannot happen (the events have no outcomes in common), so the intersection of (b) and (c) is $\varnothing$.

5.

a. The $3^3 = 27$ possible outcomes are numbered below for later reference.

Outcome Number	Outcome	Outcome Number	Outcome
1	111	15	223
2	112	16	231
3	113	17	232
4	121	18	233
5	122	19	311
6	123	20	312
7	131	21	313
8	132	22	321
9	133	23	322
10	211	24	323
11	212	25	331
12	213	26	332
13	221	27	333
14	222		

b. Outcome numbers 1, 14, 27 above.

c. Outcome numbers 6, 8, 12, 16, 20, 22 above.

d. Outcome numbers 1, 3, 7, 9, 19, 21, 25, 27 above.

6.

a. S = {123, 124, 125, 213, 214, 215, 13, 14, 15, 23, 24, 25, 3, 4, 5}.

b. A = {3, 4, 5}.

c. B = {125, 215, 15, 25, 5}.

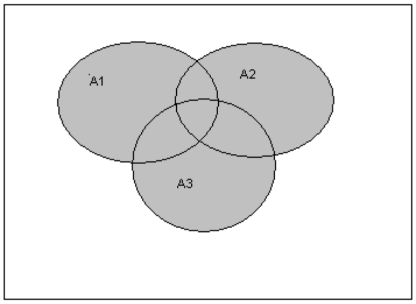
d. C = {23, 24, 25, 3, 4, 5}.

7.

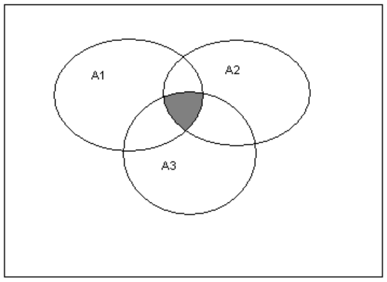
a. *S* = {*BBBAAAA*, *BBABAAA*, *BBAABAA*, *BBAAABA*, *BBAAAAB*, *BABBAAA*, *BABABAA*, *BABAABA*, *BABAAAB*, *BAABBAA*, *BAABABA*, *BAABAAB*, *BAAABBA*, *BAAABAB*, *BAAAABB*, *ABBBAAA*, *ABBABAA*, *ABBAABA*, *ABBAAAB*, *ABABBAA*, *ABABABA*, *ABABAAB*, *ABAABBA*, *ABAABAB*, *ABAAABB*, *AABBBAA*, *AABBABA*, *AABBAAB*, *AABABBA*, *AABABAB*, *AABAABB*, *AAABBBA*, *AAABBAB*, *AAABABB*, *AAAABBB*}.

b. *AAAABBB*, *AAABABB*, *AAABBAB*, *AABAABB*, *AABABAB*.

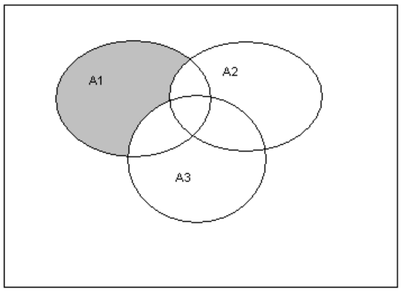
8.



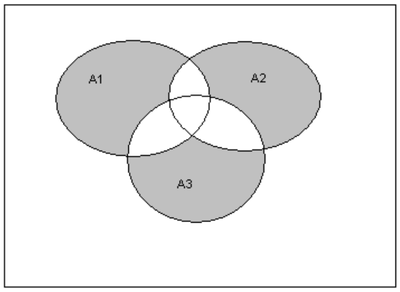
a. $A_1 \cup A_2 \cup A_3$



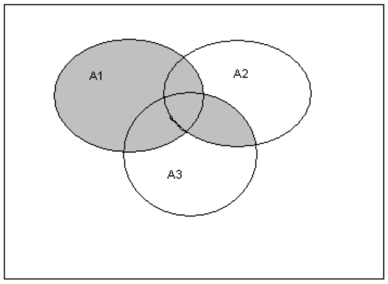
b. $A_1 \cap A_2 \cap A_3$



c. $A_1 \cap A_2' \cap A_3'$



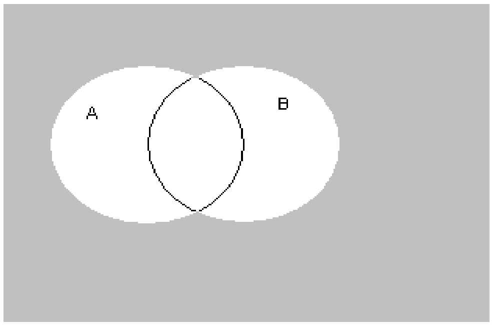
d. $(A_1 \cap A_2' \cap A_3') \cup (A_1' \cap A_2 \cap A_3') \cup (A_1' \cap A_2' \cap A_3)$

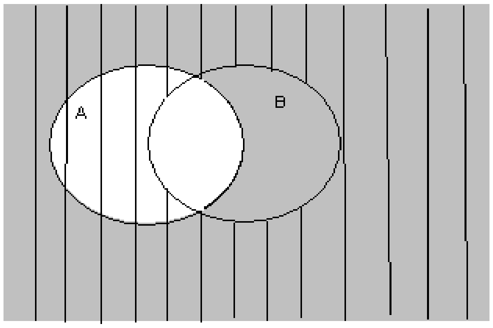


e. $A_1 \cup (A_2 \cap A_3)$

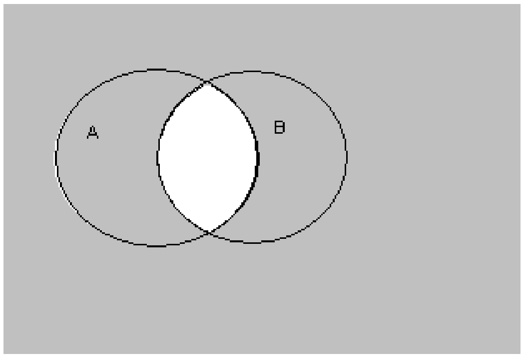
9.

a. In the diagram on the left, the shaded area is $(A \cup B)'$. On the right, the shaded area is A', the striped area is B', and the intersection $A' \cap B'$ occurs where there is both shading <u>and</u> stripes. These two diagrams display the same area.





b. In the diagram below, the shaded area represents $(A \cap B)'$. Using the right-hand diagram from (a), the <u>union</u> of A' and B' is represented by the areas that have either shading <u>or</u> stripes (or both). Both of the diagrams display the same area.



10.

a. Many examples exist; e.g., A = {Chevy, Buick}, B = {Ford, Lincoln}, C = {Toyota} are three mutually exclusive events.

b. No. Let E = {Chevy, Buick}, F = {Buick, Ford}, G = {Toyota}. These events are <u>not</u> mutually exclusive (E and F have an outcome in common), yet there is no outcome common to all three events.

Section 2.2

11.

a. .07.

b. .15 + .10 + .05 = .30.

c. Let A = the selected individual owns shares in a stock fund. Then $P(A) = .18 + .25 = .43$. The desired probability, that a selected customer does not shares in a stock fund, equals $P(A') = 1 – P(A) = 1 – .43 = .57$. This could also be calculated by adding the probabilities for all the funds that are not stocks.

12.

a. No, this is not possible. Since event $A \cap B$ is contained within event B, it must be the case that $P(A \cap B) \le P(B)$. However, .5 > .4.

b. By the addition rule, $P(A \cup B) = .5 + .4 – .3 = .6$.

c. $P(\text{neither } A \text{ nor } B) = P(A' \cap B') = P((A \cup B)') = 1 – P(A\cup B) = 1 – .6 = .4$.

d. The event of interest is $A\cap B'$; from a Venn diagram, we see $P(A \cap B') = P(A) – P(A \cap B) = .5 – .3 = .2$.

e. From a Venn diagram, we see that the probability of interest is $P(\text{exactly one}) = P(\text{at least one}) – P(\text{both}) = P(A \cup B) – P(A \cap B) = .6 – .3 = .3$.

13.

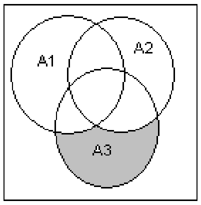
a. $A_1 \cup A_2$ = "awarded either #1 or #2 (or both)": from the addition rule, $P(A_1 \cup A_2) = P(A_1) + P(A_2) – P(A_1 \cap A_2) = .22 + .25 – .11 = .36$.

b. $A_1' \cap A_2'$ = "awarded neither #1 or #2": using the hint and part (a), $P(A_1' \cap A_2') = P((A_1 \cup A_2)') = 1 - P(A_1 \cup A_2) = 1 – .36 = .64$.

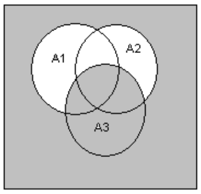
c. $A_1 \cup A_2 \cup A_3$ = "awarded at least one of these three projects": using the addition rule for 3 events, $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) =$ $.22 +.25 + .28 – .11 – .05 – .07 + .01 = .53$.

d. $A_1' \cap A_2' \cap A_3'$ = "awarded none of the three projects": $P(A_1' \cap A_2' \cap A_3') = 1 – P(\text{awarded at least one}) = 1 – .53 = .47$.

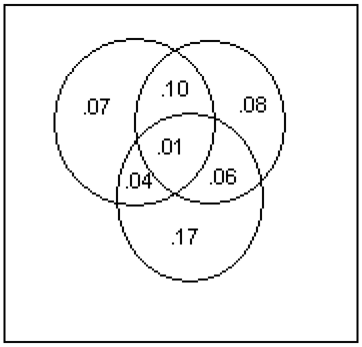
e. $A_1' \cap A_2' \cap A_3$ = "awarded #3 but neither #1 nor #2": from a Venn diagram, $P(A_1' \cap A_2' \cap A_3) = P(A_3) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) =$ .28 – .05 – .07 + .01 = .17. The last term addresses the "double counting" of the two subtractions.



f. $(A_1' \cap A_2') \cup A_3$ = "awarded neither of #1 and #2, or awarded #3": from a Venn diagram, $P((A_1' \cap A_2') \cup A_3) = P(\text{none awarded}) + P(A_3)$ = .47 (from **d**) + .28 = 75.



Alternatively, answers to **a-f** can be obtained from probabilities on the accompanying Venn diagram:



14. Let A = an adult consumes coffee and B = an adult consumes carbonated soda. We're told that $P(A) = .55$, $P(B) = .45$, and $P(A \cup B) = .70$.

a. The addition rule says $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, so $.70 = .55 + .45 - P(A \cap B)$ or $P(A \cap B) = .55 + .45 - .70 = .30$.

b. There are two ways to read this question. We can read "does not (consume at least one)," which means the adult consumes neither beverage. The probability is then $P(\text{neither } A \text{ nor } B) = P(A' \cap B') = 1 - P(A \cup B) = 1 - .70 = .30$.

The other reading, and this is presumably the intent, is "there is at least one beverage the adult does not consume, i.e. $A' \cup B'$. The probability is $P(A' \cup B') = 1 - P(A \cap B) = 1 - .30$ from **a** = .70. (It's just a coincidence this equals $P(A \cup B)$.)

Both of these approaches use *deMorgan's laws*, which say that $P(A' \cap B') = 1 - P(A \cup B)$ and $P(A' \cup B') = 1 - P(A \cap B)$.

15.

a. Let E be the event that at most one purchases an electric dryer. Then E' is the event that at least two purchase electric dryers, and $P(E') = 1 – P(E) = 1 – .428 = .572$.

b. Let A be the event that all five purchase gas, and let B be the event that all five purchase electric. All other possible outcomes are those in which at least one of each type of clothes dryer is purchased. Thus, the desired probability is $1 – [P(A) – P(B)] =$
$1 – [.116 + .005] = .879$.

16.

a. There are six simple events, corresponding to the outcomes *CDP*, *CPD*, *DCP*, *DPC*, *PCD*, and *PDC*. Since the same cola is in every glass, these six outcomes are equally likely to occur, and the probability assigned to each is $\frac{1}{6}$.

b. $P(C \text{ ranked first}) = P(\{CPD, CDP\}) = \frac{1}{6}+\frac{1}{6}=\frac{2}{6} = .333$.

c. $P(C \text{ ranked first and } D \text{ last}) = P(\{CPD\}) = \frac{1}{6}$.

17.

a. The probabilities do not add to 1 because there are other software packages besides SPSS and SAS for which requests could be made.

b. $P(A') = 1 – P(A) = 1 – .30 = .70$.

c. Since A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B) = .30 + .50 = .80$.

d. By deMorgan's law, $P(A' \cap B') = P((A \cup B)') = 1 – P(A \cup B) = 1 – .80 = .20$.
In this example, deMorgan's law says the event "neither A nor B" is the complement of the event "either A or B." (That's true regardless of whether they're mutually exclusive.)

18. The only reason we'd need at least two selections to find a $10 bill is if the <u>first</u> selection was <u>not</u> a $10 bill bulb. There are 4 + 6 = 10 non-$10 bills out of 5 + 4 + 6 = 15 bills in the wallet, so the probability of this event is simply 10/15, or 2/3.

19. Let A be that the selected joint was found defective by inspector A, so $P(A) = \frac{724}{10{,}000}$. Let B be analogous for inspector B, so $P(B) = \frac{751}{10{,}000}$. The event "at least one of the inspectors judged a joint to be defective is $A \cup B$, so $P(A \cup B) = \frac{1159}{10{,}000}$.

a. By deMorgan's law, $P(\text{neither } A \text{ nor } B) = P(A' \cap B') = 1 – \text{P}(\text{A} \cup \text{B}) = 1 – \frac{1159}{10{,}000} = \frac{8841}{10{,}000} = .8841$.

b. The desired event is $B \cap A'$. From a Venn diagram, we see that $P(B \cap A') = P(B) – P(A \cap B)$. From the addition rule, $P(A \cup B) = P(A) + P(B) – P(A \cap B)$ gives $P(A \cap B) = .0724 + .0751 – .1159 = .0316$. Finally, $P(B \cap A') = P(B) – P(A \cap B) = .0751 – .0316 = .0435$.

20.

a. Let S_1, S_2 and S_3 represent day, swing, and night shifts, respectively. Let C_1 and C_2 represent unsafe conditions and unrelated to conditions, respectively. Then the simple events are S_1C_1, S_1C_2, S_2C_1, S_2C_2, S_3C_1, S_3C_2.

b. $P(C_1)$= P({S_1C_1, S_2C_1, S_3C_1})= .10 + .08 + .05 = .23.

c. $P(S_1')$ = 1 – P({S_1C_1, S_1C_2}) = 1 – (.10 + .35) = .55.

21. In what follows, the first letter refers to the auto deductible and the second letter refers to the homeowner's deductible.

a. $P(MH)$ = .10.

b. P(low auto deductible) = P({LN, LL, LM, LH}) = .04 + .06 + .05 + .03 = .18. Following a similar pattern, P(low homeowner's deductible) = .06 + .10 + .03 = .19.

c. P(same deductible for both) = P({LL, MM, HH}) = .06 + .20 + .15 = .41.

d. P(deductibles are different) = 1 – P(same deductible for both) = 1 – .41 = .59.

e. P(at least one low deductible) = P({LN, LL, LM, LH, ML, HL}) = .04 + .06 + .05 + .03 + .10 + .03 = .31.

f. P(neither deductible is low) = 1 – P(at least one low deductible) = 1 – .31 = .69.

22. Let A = motorist must stop at first signal and B = motorist must stop at second signal. We're told that $P(A)$ = .4, $P(B)$ = .5, and $P(A \cup B)$ = .6.

a. From the addition rule, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, so .6 = .4 + .5 – $P(A \cap B)$, from which $P(A \cap B)$ = .4 + .5 – .6 = .3.

b. From a Venn diagram, $P(A \cap B') = P(A) - P(A \cap B)$ = .4 – .3 = .1.

c. From a Venn diagram, P(stop at exactly one signal) = $P(A \cup B) - P(A \cap B)$ = .6 – .3 = .3. Or, P(stop at exactly one signal) = $P([A \cap B'] \cup [A' \cap B]) = P(A \cap B') + P(A' \cap B) = [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)]$ = [.4 – .3] + [.5 – .3] = .1 + .2 = .3.

23. Assume that the computers are numbered 1-6 as described and that computers 1 and 2 are the two laptops. There are 15 possible outcomes: (1,2) (1,3) (1,4) (1,5) (1,6) (2,3) (2,4) (2,5) (2,6) (3,4) (3,5) (3,6) (4,5) (4,6) and (5,6).

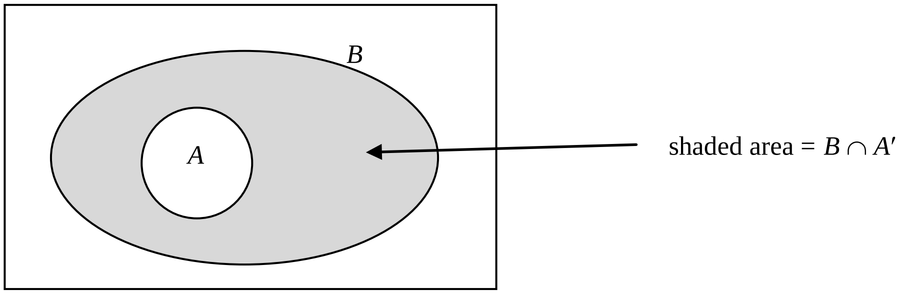
a. P(both are laptops) = P({(1,2)}) = $\frac{1}{15}$ =.067.

b. P(both are desktops) = P({(3,4) (3,5) (3,6) (4,5) (4,6) (5,6)}) = $\frac{6}{15}$ = .40.

c. P(at least one desktop) = 1 – P(no desktops) = 1 – P(both are laptops) = 1 – .067 = .933.

d. P(at least one of each type) = 1 – P(both are the same) = 1 – [P(both are laptops) + P(both are desktops)] = 1 – [.067 + .40] = .533.

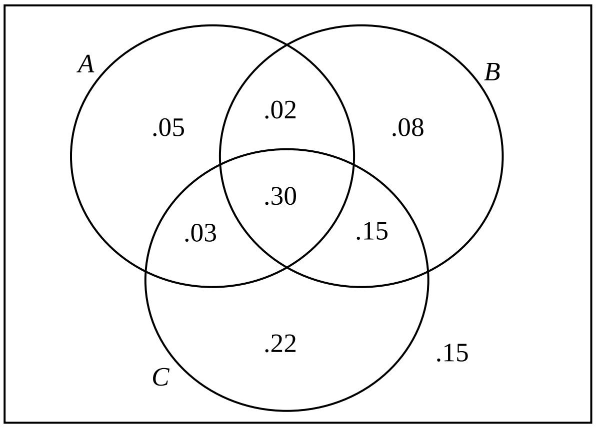
24. Since A is contained in B, we may write $B = A \cup (B \cap A')$, the union of two mutually exclusive events. (See diagram for these two events.) Apply the axioms:
$P(B) = P(A \cup (B \cap A')) = P(A) + P(B \cap A')$ by Axiom 3. Then, since $P(B \cap A') \geq 0$ by Axiom 1, $P(B) = P(A) + P(B \cap A') \geq P(A) + 0 = P(A)$. This proves the statement.



For general events A and B (i.e., not necessarily those in the diagram), it's always the case that $A \cap B$ is contained in A as well as in B, while A and B are both contained in $A \cup B$. Therefore, $P(A \cap B) \leq P(A) \leq P(A \cup B)$ and $P(A \cap B) \leq P(B) \leq P(A \cup B)$.

25. By rearranging the addition rule, $P(A \cap B) = P(A) + P(B) - P(A \cup B) = .40 + .55 - .63 = .32$. By the same method, $P(A \cap C) = .40 + .70 - .77 = .33$ and $P(B \cap C) = .55 + .70 - .80 = .45$. Finally, rearranging the addition rule for 3 events gives
$P(A \cap B \cap C) = P(A \cup B \cup C) - P(A) - P(B) - P(C) + P(A \cap B) + P(A \cap C) + P(B \cap C) = .85 - .40 - .55 - .70 + .32 + .33 + .45 = .30$.

These probabilities are reflected in the Venn diagram below.



a. $P(A \cup B \cup C) = .85$, as given.

b. P(none selected) = 1 – P(at least one selected) = $1 - P(A \cup B \cup C) = 1 - .85 = .15$.

c. From the Venn diagram, P(only automatic transmission selected) = .22.

d. From the Venn diagram, P(exactly one of the three) = .05 + .08 + .22 = .35.

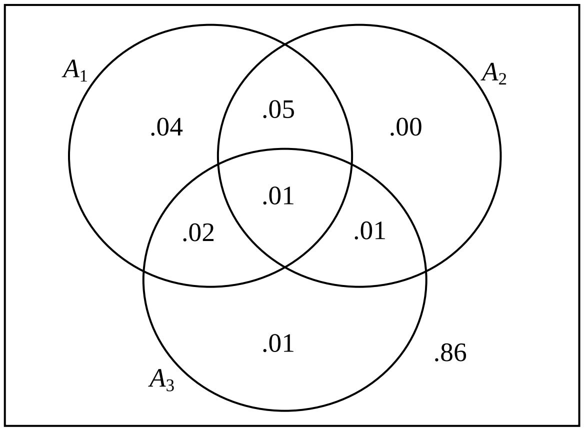
26. These questions can be solved algebraically, or with the Venn diagram below.

a. $P(A_1') = 1 - P(A_1)$ = 1 – .12 = .88.

b. The addition rule says $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Solving for the intersection ("and") probability, you get $P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$ = .12 + .07 – .13 = .06.

c. A Venn diagram shows that $P(A \cap B') = P(A) - P(A \cap B)$. Applying that here with $A = A_1 \cap A_2$ and B = A_3, you get $P([A_1 \cap A_2] \cap A_3') = P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3) =$ .06 – .01 = .05.

d. The event "at most two defects" is the complement of "all three defects," so the answer is just 1 – $P(A_1 \cap A_2 \cap A_3)$ = 1 – .01 = .99.



27. There are 10 equally likely outcomes: {A, B} {A, Co} {A, Cr} {A,F} {B, Co} {B, Cr} {B, F} {Co, Cr} {Co, F} and {Cr, F}.

a. P({A, B}) = $\frac{1}{10}$ = .1.

b. P(at least one C) = P({A, Co} or {A, Cr} or {B, Co} or {B, Cr} or {Co, Cr} or {Co, F} or {Cr, F}) = $\frac{7}{10}$ = .7.

c. Replacing each person with his/her years of experience, P(at least 15 years) = P({3, 14} or {6, 10} or {6, 14} or {7, 10} or {7, 14} or {10, 14}) = $\frac{6}{10}$ = .6.

28. Recall there are 27 equally likely outcomes.

a. P(all the same station) = P((1,1,1) or (2,2,2) or (3,3,3)) = $\frac{3}{27} = \frac{1}{9}$.

b. P(at most 2 are assigned to the same station) = 1 – P(all 3 are the same) = 1 – $\frac{1}{9} = \frac{8}{9}$.

c. P(all different stations) = P((1,2,3) or (1,3,2) or (2,1,3) or (2,3,1) or (3,1,2) or (3,2,1)) = $\frac{6}{27} = \frac{2}{9}$.

Section 2.3

29.

a. There are 26 letters, so allowing repeats there are (26)(26) = $(26)^2$ = 676 possible 2-letter domain names. Add in the 10 digits, and there are 36 characters available, so allowing repeats there are (36)(36) = $(36)^2$ = 1296 possible 2-character domain names.

b. By the same logic as part **a**, the answers are $(26)^3$ = 17,576 and $(36)^3$ = 46,656.

c. Continuing, $(26)^4$ = 456,976; $(36)^4$ = 1,679,616.

d. P(4-character sequence is already owned) = 1 – P(4-character sequence still available) = 1 – 97,786/$(36)^4$ = .942.

30.

a. Because order is important, we'll use $P_{3,8} = (8)(7)(6) = 336$.

b. Order doesn't matter here, so we use $\binom{30}{6}$ = 593,775.

c. The number of ways to choose 2 zinfandels from the 8 available is $\binom{8}{2}$. Similarly, the number of ways to choose the merlots and cabernets are $\binom{10}{2}$ and $\binom{12}{2}$, respectively. Hence, the total number of options (using the Fundamental Counting Principle) equals $\binom{8}{2}\binom{10}{2}\binom{12}{2}$ = (28)(45)(66) = 83,160.

d. The numerator comes from part **c** and the denominator from part **b**: $\frac{83{,}160}{593{,}775}$ = .140.

e. We use the same denominator as in part **d**. The number of ways to choose all zinfandel is $\binom{8}{6}$, with similar answers for all merlot and all cabernet. Since these are disjoint events, P(all same) = P(all zin) + P(all merlot) + P(all cab) = $\dfrac{\binom{8}{6}+\binom{10}{6}+\binom{12}{6}}{\binom{30}{6}} = \dfrac{1162}{593{,}775} = .002$.

31.

a. Use the Fundamental Counting Principle: (9)(5) = 45.

b. By the same reasoning, there are (9)(5)(32) = 1440 such sequences, so such a policy could be carried out for 1440 successive nights, or almost 4 years, without repeating exactly the same program.

32.

a. Since there are 5 receivers, 4 CD players, 3 speakers, and 4 turntables, the total number of possible selections is (5)(4)(3)(4) = 240.

b. We now only have 1 choice for the receiver and CD player: (1)(1)(3)(4) = 12.

c. Eliminating Sony leaves 4, 3, 3, and 3 choices for the four pieces of equipment, respectively: (4)(3)(3)(3) = 108.

d. From **a**, there are 240 possible configurations. From **c**, 108 of them involve zero Sony products. So, the number of configurations with at least one Sony product is 240 – 108 = 132.

e. Assuming all 240 arrangements are equally likely, $P(\text{at least one Sony}) = \frac{132}{240} = .55$.

Next, P(exactly one component Sony) = P(only the receiver is Sony) + P(only the CD player is Sony) + P(only the turntable is Sony). Counting from the available options gives

P(exactly one component Sony) = $\frac{(1)(3)(3)(3)+(4)(1)(3)(3)+(4)(3)(3)(1)}{240} = \frac{99}{240} = .413$.

33.

a. Since there are 15 players and 9 positions, and order matters in a line-up (catcher, pitcher, shortstop, etc. are different positions), the number of possibilities is $P_{9,15}$ = (15)(14)…(7) or 15!/(15–9)! = 1,816,214,440.

b. For each of the starting line-ups in part (a), there are 9! possible batting orders. So, multiply the answer from (a) by 9! to get (1,816,214,440)(362,880) = 659,067,881,472,000.

c. Order still matters: There are $P_{3,5}$ = 60 ways to choose three left-handers for the outfield and $P_{6,10}$ = 151,200 ways to choose six right-handers for the other positions. The total number of possibilities is = (60)(151,200) = 9,072,000.

34.

a. Since order doesn't matter, the number of ways to randomly select 5 keyboards from the 25 available is $\binom{25}{5}$ = 53,130.

b. Sample in two stages. First, there are 6 keyboards with an electrical defect, so the number of ways to select exactly 2 of them is $\binom{6}{2}$. Next, the remaining 5 – 2 = 3 keyboards in the sample must have mechanical defects; as there are 19 such keyboards, the number of ways to randomly select 3 is $\binom{19}{3}$. So, the number of ways to achieve both of these in the sample of 5 is the product of these two counting numbers: $\binom{6}{2}\binom{19}{3}$ = (15)(969) = 14,535.

c. Following the analogy from **b**, the number of samples with exactly 4 mechanical defects is $\binom{19}{4}\binom{6}{1}$, and the number with exactly 5 mechanical defects is $\binom{19}{5}\binom{6}{0}$. So, the number of samples with <u>at least</u> 4 mechanical defects is $\binom{19}{4}\binom{6}{1} + \binom{19}{5}\binom{6}{0}$, and the probability of this event is

$$\frac{\binom{19}{4}\binom{6}{1}+\binom{19}{5}\binom{6}{0}}{\binom{25}{5}} = \frac{34{,}884}{53{,}130} = .657.$$ (The denominator comes from **a**.)

35.

a. There are $\binom{10}{5}$ = 252 ways to select 5 workers from the day shift. In other words, of all the ways to select 5 workers from among the 24 available, 252 such selections result in 5 day-shift workers. Since the grand total number of possible selections is $\binom{24}{5}$ = 42504, the probability of randomly selecting 5 day-shift workers (and, hence, no swing or graveyard workers) is 252/42504 = .00593.

b. Similar to **a**, there are $\binom{8}{5}$ = 56 ways to select 5 swing-shift workers and $\binom{6}{5}$ = 6 ways to select 5 graveyard-shift workers. So, there are 252 + 56 + 6 = 314 ways to pick 5 workers from the same shift. The probability of this randomly occurring is 314/42504 = .00739.

c. P(at least two shifts represented) = 1 – P(all from same shift) = 1 – .00739 = .99261.

d. There are several ways to approach this question. For example, let A_1 = "day shift is unrepresented," A_2 = "swing shift is unrepresented," and A_3 = "graveyard shift is unrepresented." Then we want $P(A_1 \cup A_2 \cup A_3)$.

$N(A_1)$ = N(day shift unrepresented) = N(all from swing/graveyard) = $\binom{8+6}{5}$ = 2002, since there are 8 + 6 = 14 total employees in the swing and graveyard shifts. Similarly, $N(A_2)$ = $\binom{10+6}{5}$ = 4368 and $N(A_3)$ = $\binom{10+8}{5}$ = 8568. Next, $N(A_1 \cap A_2)$ = N(all from graveyard) = 6 from **b**. Similarly, $N(A_1 \cap A_3)$ = 56 and $N(A_2 \cap A_3)$ = 252. Finally, $N(A_1 \cap A_2 \cap A_3)$ = 0, since at least one shift must be represented. Now, apply the addition rule for 3 events:

$$P(A_1 \cup A_2 \cup A_3) = \frac{2002+4368+8568-6-56-252+0}{42504} = \frac{14624}{42504} = .3441.$$

36. There are $\binom{5}{2}$ = 10 possible ways to select the positions for B's votes: *BBAAA, BABAA, BAABA, BAAAB, ABBAA, ABABA, ABAAB, AABBA, AABAB*, and *AAABB*. Only the last two have A ahead of B throughout the vote count. Since the outcomes are equally likely, the desired probability is 2/10 = .20.

37.

a. By the Fundamental Counting Principle, with $n_1 = 3$, $n_2 = 4$, and $n_3 = 5$, there are (3)(4)(5) = 60 runs.

b. With $n_1 = 1$ (just one temperature), $n_2 = 2$, and $n_3 = 5$, there are (1)(2)(5) = 10 such runs.

c. For each of the 5 specific catalysts, there are (3)(4) = 12 pairings of temperature and pressure. Imagine we separate the 60 possible runs into those 5 sets of 12. The number of ways to select exactly one run from each of these 5 sets of 12 is $\binom{12}{1}^5 = 12^5$. Since there are $\binom{60}{5}$ ways to select the 5 runs overall, the desired probability is $\binom{12}{1}^5 / \binom{60}{5} = 12^5 / \binom{60}{5}$ = .0456.

38.

a. A sonnet has 14 lines, each of which may come from any of the 10 pages. Order matters, and we're sampling with replacement, so the number of possibilities is 10 × 10 × … × 10 = 10^{14}.

b. Similarly, the number of sonnets you could create avoiding the first and last pages (so, only using lines from the middle 8 sonnets) is 8^{14}. Thus, the probability that a randomly-created sonnet would not use any lines from the first or last page is $8^{14}/10^{14} = .8^{14}$ = .044.

39. In **a-c**, the size of the sample space is $N = \binom{5+6+4}{3} = \binom{15}{3}$ = 455.

a. There are four 23W bulbs available and 5+6 = 11 non-23W bulbs available. The number of ways to select exactly two of the former (and, thus, exactly one of the latter) is $\binom{4}{2}\binom{11}{1}$ = 6(11) = 66. Hence, the probability is 66/455 = .145.

b. The number of ways to select three 13W bulbs is $\binom{5}{3}$ = 10. Similarly, there are $\binom{6}{3}$ = 20 ways to select three 18W bulbs and $\binom{4}{3}$ = 4 ways to select three 23W bulbs. Put together, there are 10 + 20 + 4 = 34 ways to select three bulbs of the same wattage, and so the probability is 34/455 = .075.

c. The number of ways to obtain one of each type is $\binom{5}{1}\binom{6}{1}\binom{4}{1}$ = (5)(6)(4) = 120, and so the probability is 120/455 = .264.

d. Rather than consider many different options (choose 1, choose 2, etc.), re-frame the problem this way: at least 6 draws are required to get a 23W bulb iff a random sample of <u>five</u> bulbs fails to produce a 23W bulb. Since there are 11 non-23W bulbs, the chance of getting no 23W bulbs in a sample of size 5 is $\binom{11}{5}/\binom{15}{5}$ = 462/3003 = .154.

40.

a. If the *A*'s were distinguishable from one another, and similarly for the *B*'s, *C*'s and *D*'s, then there would be 12! possible chain molecules. Six of these are:

$A_1A_2A_3B_2C_3C_1D_3C_2D_1D_2B_3B_1$ $A_1A_3A_2B_2C_3C_1D_3C_2D_1D_2B_3B_1$
$A_2A_1A_3B_2C_3C_1D_3C_2D_1D_2B_3B_1$ $A_2A_3A_1B_2C_3C_1D_3C_2D_1D_2B_3B_1$
$A_3A_1A_2B_2C_3C_1D_3C_2D_1D_2B_3B_1$ $A_3A_2A_1B_2C_3C_1D_3C_2D_1D_2B_3B_1$

These 6 (=3!) differ only with respect to ordering of the 3 *A*'s. In general, groups of 6 chain molecules can be created such that within each group only the ordering of the *A*'s is different. When the A subscripts are suppressed, each group of 6 "collapses" into a single molecule (*B*'s, *C*'s and *D*'s are still distinguishable).

At this point there are (12!/3!) different molecules. Now suppressing subscripts on the *B*'s, *C*'s, and *D*'s in turn gives $\frac{12!}{(3!)^4} = 369,600$ chain molecules.

b. Think of the group of 3 *A*'s as a single entity, and similarly for the *B*'s, *C*'s, and *D*'s. Then there are 4! = 24 ways to order these triplets, and thus 24 molecules in which the *A*'s are contiguous, the *B*'s, *C*'s, and *D*'s also. The desired probability is $\frac{24}{369,600} = .00006494$.

41.

a. (10)(10)(10)(10) = 10^4 = 10,000. These are the strings 0000 through 9999.

b. Count the number of prohibited sequences. There are (i) 10 with all digits identical (0000, 1111, …, 9999); (ii) 14 with sequential digits (0123, 1234, 2345, 3456, 4567, 5678, 6789, and 7890, plus these same seven descending); (iii) 100 beginning with 19 (1900 through 1999). That's a total of 10 + 14 + 100 = 124 impermissible sequences, so there are a total of 10,000 – 124 = 9876 permissible sequences. The chance of randomly selecting one is just $\frac{9876}{10,000}$ = .9876.

c. All PINs of the form 8xx1 are legitimate, so there are (10)(10) = 100 such PINs. With someone randomly selecting 3 such PINs, the chance of guessing the correct sequence is 3/100 = .03.

d. Of all the PINs of the form 1xx1, eleven is prohibited: 1111, and the ten of the form 19x1. That leaves 89 possibilities, so the chances of correctly guessing the PIN in 3 tries is 3/89 = .0337.

42.

a. If Player X sits out, the number of possible teams is $\binom{3}{1}\binom{4}{2}\binom{4}{2}$ = 108. If Player X plays guard, we need one <u>more</u> guard, and the number of possible teams is $\binom{3}{1}\binom{4}{1}\binom{4}{2}$ = 72. Finally, if Player X plays forward, we need one <u>more</u> forward, and the number of possible teams is $\binom{3}{1}\binom{4}{2}\binom{4}{1}$ = 72. So, the total possible number of teams from this group of 12 players is 108 + 72 + 72 = 252.

b. Using the idea in **a**, consider all possible scenarios. If Players X and Y both sit out, the number of possible teams is $\binom{3}{1}\binom{5}{2}\binom{5}{2}$ = 300. If Player X plays while Player Y sits out, the number of possible teams is $\binom{3}{1}\binom{5}{1}\binom{5}{2}+\binom{3}{1}\binom{5}{2}\binom{5}{1}$ = 150 + 150 = 300. Similarly, there are 300 teams with Player X benched and Player Y in. Finally, there are three cases when X and Y both play: they're both guards, they're both forwards, or they split duties. The number of ways to select the rest of the team under these scenarios is $\binom{3}{1}\binom{5}{0}\binom{5}{2}+\binom{3}{1}\binom{5}{2}\binom{5}{0}+\binom{3}{1}\binom{5}{1}\binom{5}{1}$ = 30 + 30 + 75 = 135.

Since there are $\binom{15}{5}$ = 3003 ways to randomly select 5 players from a 15-person roster, the probability of randomly selecting a legitimate team is $\frac{300+300+135}{3003}=\frac{735}{3003}$ = .245.

43. There are $\binom{52}{5}$ = 2,598,960 five-card hands. The number of 10-high straights is (4)(4)(4)(4)(4) = 4^5 = 1024 (any of four 6s, any of four 7s, etc.). So, P(10 high straight) = $\frac{1024}{2{,}598{,}960}=.000394$. Next, there ten "types of straight: A2345, 23456, …, 910JQK, 10JQKA. So, P(straight) = $10\times\frac{1024}{2{,}598{,}960}=.00394$. Finally, there are only 40 straight flushes: each of the ten sequences above in each of the 4 suits makes (10)(4) = 40. So, P(straight flush) = $\frac{40}{2{,}598{,}960}=.00001539$.

44. $\binom{n}{k}=\frac{n!}{k!(n-k)!}=\frac{n!}{(n-k)!k!}=\binom{n}{n-k}$

The number of subsets of size k equals the number of subsets of size $n – k$, because to each subset of size k there corresponds exactly one subset of size $n – k$: the $n – k$ objects not in the subset of size k. The combinations formula counts the number of ways to split n objects into two subsets: one of size k, and one of size $n – k$.

Section 2.4

45.

a. $P(A)$ = .106 + .141 + .200 = .447, $P(C)$ =.215 + .200 + .065 + .020 = .500, and $P(A \cap C)$ = .200.

b. $P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{.200}{.500} = .400$. If we know that the individual came from ethnic group 3, the probability that he has Type A blood is .40. $P(C|A) = \frac{P(A \cap C)}{P(A)} = \frac{.200}{.447}$ = .447. If a person has Type A blood, the probability that he is from ethnic group 3 is .447.

c. Define D = "ethnic group 1 selected." We are asked for $P(D|B')$. From the table, $P(D \cap B')$ = .082 + .106 + .004 = .192 and $P(B')$ = 1 – $P(B)$ = 1 – [.008 + .018 + .065] = .909. So, the desired probability is $P(D|B') = \frac{P(D \cap B')}{P(B')} = \frac{.192}{.909} = .211$.

46. Let A be that the individual is more than 6 feet tall. Let B be that the individual is a professional basketball player. Then $P(A|B)$ = the probability of the individual being more than 6 feet tall, knowing that the individual is a professional basketball player, while $P(B|A)$ = the probability of the individual being a professional basketball player, knowing that the individual is more than 6 feet tall. $P(A|B)$ will be larger. Most professional basketball players are tall, so the probability of an individual in that reduced sample space being more than 6 feet tall is very large. On the other hand, the number of individuals that are pro basketball players is small in relation to the number of males more than 6 feet tall.

47.

a. Apply the addition rule for three events: $P(A \cup B \cup C)$ = .6 + .4 + .2 – .3 – .15 – .1 + .08 = .73.

b. $P(A \cap B \cap C') = P(A \cap B) - P(A \cap B \cap C)$ = .3 – .08 = .22.

c. $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{.3}{.6} = .50$ and $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.3}{.4} = .75$. Half of students with Visa cards also have a MasterCard, while three-quarters of students with a MasterCard also have a Visa card.

d. $P(A \cap B \mid C) = \frac{P([A \cap B] \cap C)}{P(C)} = \frac{P(A \cap B \cap C)}{P(C)} = \frac{.08}{.2}$ = .40.

e. $P(A \cup B \mid C) = \frac{P([A \cup B] \cap C)}{P(C)} = \frac{P([A \cap C] \cup [B \cap C])}{P(C)}$. Use a distributive law:

$$= \frac{P(A \cap C) + P(B \cap C) - P([A \cap C] \cap [B \cap C])}{P(C)} = \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)} =$$

$$\frac{.15 + .1 - .08}{.2} = .85.$$

48.

a. $P(A_2 \mid A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)} = \frac{.06}{.12}$ = .50. The numerator comes from Exercise 26.

b. $P(A_1 \cap A_2 \cap A_3 \mid A_1) = \frac{P([A_1 \cap A_2 \cap A_3] \cap A_1)}{P(A_1)} = \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1)} = \frac{.01}{.12}$ = .0833. The numerator simplifies because $A_1 \cap A_2 \cap A_3$ is a subset of A_1, so their intersection is just the smaller event.

c. For this example, you definitely need a Venn diagram. The seven pieces of the partition inside the three circles have probabilities .04, .05, .00, .02, .01, .01, and .01. Those add to .14 (so the chance of no defects is .86).
Let E = "exactly one defect." From the Venn diagram, $P(E)$ = .04 + .00 + .01 = .05. From the addition above, P(at least one defect) = $P(A_1 \cup A_2 \cup A_3)$ = .14. Finally, the answer to the question is $P(E \mid A_1 \cup A_2 \cup A_3) = \frac{P(E \cap [A_1 \cup A_2 \cup A_3])}{P(A_1 \cup A_2 \cup A_3)} = \frac{P(E)}{P(A_1 \cup A_2 \cup A_3)} = \frac{.05}{.14}$ = .3571. The numerator simplifies because E is a subset of $A_1 \cup A_2 \cup A_3$.

d. $P(A_3' \mid A_1 \cap A_2) = \frac{P(A_3' \cap [A_1 \cap A_2])}{P(A_1 \cap A_2)} = \frac{.05}{.06}$ = .8333. The numerator is Exercise 26(c), while the denominator is Exercise 26(b).

49.

a. P(small cup) = .14 + .20 = .34. P(decaf) = .20 + .10 + .10 = .40.

b. P(decaf | small) = $\frac{P(\text{small} \cap \text{decaf})}{P(\text{small})} = \frac{.20}{.34}$ = .588. 58.8% of all people who purchase a small cup of coffee choose decaf.

c. P(small | decaf) = $\frac{P(\text{small} \cap \text{decaf})}{P(\text{decaf})} = \frac{.20}{.40}$ = .50. 50% of all people who purchase decaf coffee choose the small size.

50.

a. P(**M** ∩ **LS** ∩ **PR**) = .05, directly from the table of probabilities.

b. P(**M** ∩ **Pr**) = P(**M** ∩ **LS** ∩ **PR**) + P(**M** ∩ **SS** ∩ **PR**) = .05 + .07 = .12.

c. P(**SS**) = sum of 9 probabilities in the **SS** table = .56. P(**LS**) = 1 – .56 = .44.

d. From the two tables, P(**M**) = .08 + .07 + .12 + .10 + .05 + .07 = .49. P(**Pr**) = .02 + .07 + .07 + .02 + .05 + .02 = .25.

e. P(**M**|**SS** ∩ **Pl**) = $\frac{P(\mathbf{M} \cap \mathbf{SS} \cap \mathbf{Pl})}{P(\mathbf{SS} \cap \mathbf{Pl})} = \frac{.08}{.04+.08+.03} = .533$.

f. P(**SS**|**M** ∩ **Pl**) = $\frac{P(\mathbf{SS} \cap \mathbf{M} \cap \mathbf{Pl})}{P(\mathbf{M} \cap \mathbf{Pl})} = \frac{.08}{.08+.10} = .444$. P(**LS**|**M** ∩ **Pl**) = 1 – P(**SS**|**M** ∩ **Pl**) = 1 – .444 = .556.

51.

a. Let A = child has a food allergy, and R = child has a history of severe reaction. We are told that $P(A)$ = .08 and $P(R \mid A)$ = .39. By the multiplication rule, $P(A \cap R) = P(A) \times P(R \mid A) = (.08)(.39) = .0312$.

b. Let M = the child is allergic to multiple foods. We are told that $P(M \mid A)$ = .30, and the goal is to find $P(M)$. But notice that M is actually a subset of A: you can't have multiple food allergies without having at least one such allergy! So, apply the multiplication rule again:

$P(M) = P(M \cap A) = P(A) \times P(M \mid A) = (.08)(.30) = .024$.

52. We know that $P(A_1 \cup A_2) = .07$ and $P(A_1 \cap A_2) = .01$, and that $P(A_1) = P(A_2)$ because the pumps are identical. There are two solution methods. The first doesn't require explicit reference to q or r: Let A_1 be the event that #1 fails and A_2 be the event that #2 fails.

Apply the addition rule: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \Rightarrow .07 = 2P(A_1) - .01 \Rightarrow P(A_1) = .04$.

Otherwise, we assume that $P(A_1) = P(A_2) = q$ and that $P(A_1 \mid A_2) = P(A_2 \mid A_1) = r$ (the goal is to find q). Proceed as follows: $.01 = P(A_1 \cap A_2) = P(A_1)\, P(A_2 \mid A_1) = qr$ and $.07 = P(A_1 \cup A_2) =$ $P(A_1 \cap A_2) + P(A_1' \cap A_2) + P(A_1 \cap A_2')$ $= .01 + q(1 - r) + q(1 - r) \Rightarrow q(1 - r) = .03$.

These two equations give $2q - .01 = .07$, from which $q = .04$ (and $r = .25$).

53. $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(B)}{P(A)}$ (since B is contained in A, $A \cap B = B$)

$= \dfrac{.05}{.60} = .0833$

54.

a. $P(A_2 \mid A_1) = \dfrac{P(A_1 \cap A_2)}{P(A_1)} = \dfrac{.11}{.22} = .50$. If the firm is awarded project 1, there is a 50% chance they will also be awarded project 2.

b. $P(A_2 \cap A_3 \mid A_1) = \dfrac{P(A_1 \cap A_2 \cap A_3)}{P(A_1)} = \dfrac{.01}{.22} = .0455$. If the firm is awarded project 1, there is a 4.55% chance they will also be awarded projects 2 and 3.

c. $P(A_2 \cup A_3 \mid A_1) = \dfrac{P[A_1 \cap (A_2 \cup A_3)]}{P(A_1)} = \dfrac{P[(A_1 \cap A_2) \cup (A_1 \cap A_3)]}{P(A_1)}$

$= \dfrac{P(A_1 \cap A_2) + P(A_1 \cap A_3) - P(A_1 \cap A_2 \cap A_3)}{P(A_1)} = \dfrac{.15}{.22} = .682$. If the firm is awarded project 1, there is a 68.2% chance they will also be awarded at least one of the other two projects.

d. $P(A_1 \cap A_2 \cap A_3 \mid A_1 \cup A_2 \cup A_3) = \dfrac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cup A_2 \cup A_3)} = \dfrac{.01}{.53} = .0189$. If the firm is awarded at least one of the projects, there is a 1.89% chance they will be awarded all three projects.

55. Let A = {carries Lyme disease} and B = {carries HGE}. We are told $P(A) = .16$, $P(B) = .10$, and $P(A \cap B \mid A \cup B) = .10$. From this last statement and the fact that $A \cap B$ is contained in $A \cup B$,

$.10 = \frac{P(A \cap B)}{P(A \cup B)} \Rightarrow P(A \cap B) = .10P(A \cup B) = .10[P(A) + P(B) - P(A \cap B)] = .10[.10 + .16 - P(A \cap B)] \Rightarrow$ $1.1P(A \cap B) = .026 \Rightarrow P(A \cap B) = .02364$.

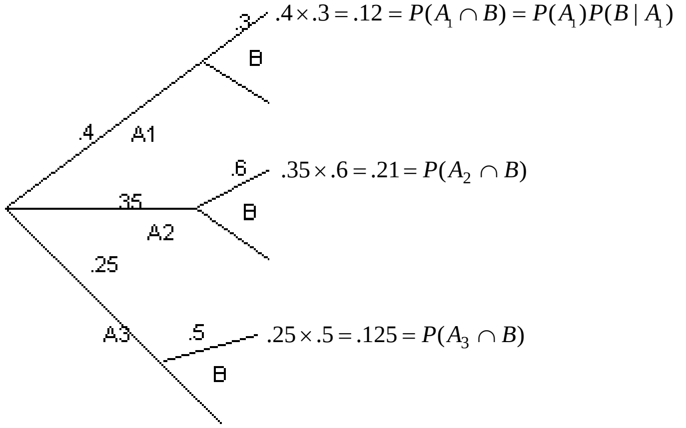
Finally, the desired probability is $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{.02364}{.10} = .2364$.

56. $$P(A \mid B) + P(A' \mid B) = \frac{P(A \cap B)}{P(B)} + \frac{P(A' \cap B)}{P(B)} = \frac{P(A \cap B) + P(A' \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

57. $P(B \mid A) > P(B)$ iff $P(B \mid A) + P(B' \mid A) > P(B) + P(B' \mid A)$ iff $1 > P(B) + P(B' \mid A)$ by Exercise 56 (with the letters switched). This holds iff $1 - P(B) > P(B' \mid A)$ iff $P(B') > P(B' \mid A)$, QED.

58. $P(A \cup B \mid C) = \frac{P[(A \cup B) \cap C)}{P(C)} = \frac{P[(A \cap C) \cup (B \cap C)]}{P(C)} = \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)} = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$

59. The required probabilities appear in the tree diagram below.

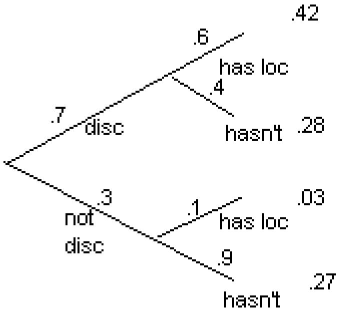


a. $P(A_2 \cap B) = .21$.

b. By the law of total probability, $P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) = .455$.

c. Using Bayes' theorem, $P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{.12}{.455} = .264$; $P(A_2 \mid B) = \frac{.21}{.455} = .462$; $P(A_3 \mid B) = 1 - .264 - .462 = .274$. Notice the three probabilities sum to 1.

60. The tree diagram below shows the probability for the four disjoint options; e.g., *P*(the flight is discovered and has a locator) = *P*(discovered)*P*(locator | discovered) = (.7)(.6) = .42.



a. *P*(not discovered | has locator) = $\frac{P(\text{not discovered} \cap \text{has locator})}{P(\text{has locator})} = \frac{.03}{.03+.42} = .067$.

b. *P*(discovered | no locator) = $\frac{P(\text{discovered} \cap \text{no locator})}{P(\text{no locator})} = \frac{.28}{.55} = .509$.

61. The initial ("prior") probabilities of 0, 1, 2 defectives in the batch are .5, .3, .2. Now, let's determine the probabilities of 0, 1, 2 defectives in the sample based on these three cases.

- If there are 0 defectives in the batch, clearly there are 0 defectives in the sample.

P(0 def in sample | 0 def in batch) = 1.

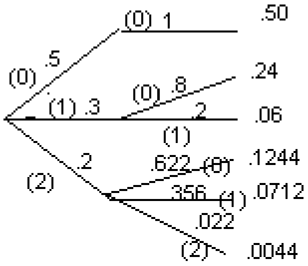
- If there is 1 defective in the batch, the chance it's discovered in a sample of 2 equals 2/10 = .2, and the probability it isn't discovered is 8/10 = .8.

P(0 def in sample | 1 def in batch) = .8, *P*(1 def in sample | 1 def in batch) = .2.

- If there are 2 defectives in the batch, the chance both are discovered in a sample of 2 equals $\frac{2}{10}\times\frac{1}{9} = .022$; the chance neither is discovered equals $\frac{8}{10}\times\frac{7}{9} = .622$; and the chance exactly 1 is discovered equals 1 – (.022 + .622) = .356.

P(0 def in sample | 2 def in batch) = .622, *P*(1 def in sample | 2 def in batch) = .356,
P(2 def in sample | 2 def in batch) = .022.

These calculations are summarized in the tree diagram below. Probabilities at the endpoints are intersectional probabilities, e.g. *P*(2 def in batch ∩ 2 def in sample) = (.2)(.022) = .0044.



a. Using the tree diagram and Bayes' rule,

P(0 def in batch | 0 def in sample) = $\dfrac{.5}{.5+.24+.1244} = .578$

P(1 def in batch | 0 def in sample) = $\dfrac{.24}{.5+.24+.1244} = .278$

P(2 def in batch | 0 def in sample) = $\dfrac{.1244}{.5+.24+.1244} = .144$

b. P(0 def in batch | 1 def in sample) = 0

P(1 def in batch | 1 def in sample) = $\dfrac{.06}{.06+.0712} = .457$

P(2 def in batch | 1 def in sample) = $\dfrac{.0712}{.06+.0712} = .543$

62. Let B = blue cab was involved, $G = B'$ = green cab was involved, and W = witness claims to have seen a blue cab. Before any witness statements, $P(B) = .15$ and $P(G)$. The witness' reliability can be coded as follows: $P(W \mid B) = .8$ (correctly identify blue), $P(W' \mid G) = .8$ (correctly identify green), and by taking complements $P(W' \mid B) = P(W \mid G) = .2$ (the two ways to mis-identify a color at night).

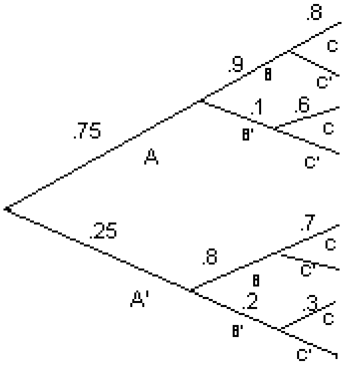
The goal is to determine $P(B \mid W)$, the chance a blue cab was involved given that's what the witness claims to have seen. Apply Bayes' Theorem:

$$P(B \mid W) = \frac{P(B)P(W \mid B)}{P(B)P(W \mid B) + P(B')P(W \mid B')} = \frac{(.15)(.8)}{(.15)(.8)+(.85)(.2)} = .4138.$$

The "posterior" probability that the cab was really blue is actually less than 50%. That's because there are so many more green cabs on the street, that it's more likely the witness mis-identified a green cab (.85 × .2) than that the witness correctly identified a blue cab (.15 × .8).

63.

a.



b. From the top path of the tree diagram, $P(A \cap B \cap C) = (.75)(.9)(.8) = .54$.

c. Event $B \cap C$ occurs twice on the diagram: $P(B \cap C) = P(A \cap B \cap C) + P(A' \cap B \cap C) = .54 + (.25)(.8)(.7) = .68$.

d. $P(C) = P(A \cap B \cap C) + P(A' \cap B \cap C) + P(A \cap B' \cap C) + P(A' \cap B' \cap C) = .54 + .045 + .14 + .015 = .74$.

e. Rewrite the conditional probability first: $P(A \mid B \cap C) = \dfrac{P(A \cap B \cap C)}{P(B \cap C)} = \dfrac{.54}{.68} = .7941$.

64. A tree diagram can help. We know that *P*(short) = .6, *P*(medium) = .3, *P*(long) = .1; also, *P*(Word | short) = .8, *P*(Word | medium) = .5, *P*(Word | long) = .3.

a. Use the law of total probability: *P*(Word) = (.6)(.8) + (.3)(.5) + (.1)(.3) = .66.

b. *P*(small | Word) = $\dfrac{P(\text{small } \cap \text{Word})}{P(\text{Word})} = \dfrac{(.6)(.8)}{.66}$ = .727. Similarly, *P*(medium | Word) = $\dfrac{(.3)(.5)}{.66}$ = .227, and *P*(long | Word) = .045. (These sum to .999 due to rounding error.)

65. A tree diagram can help. We know that *P*(day) = .2, *P*(1-night) = .5, *P*(2-night) = .3; also, *P*(purchase | day) = .1, *P*(purchase | 1-night) = .3, and *P*(purchase | 2-night) = .2.

Apply Bayes' rule: e.g., *P*(day | purchase) = $\dfrac{P(\text{day} \cap \text{purchase})}{P(\text{purchase})} = \dfrac{(.2)(.1)}{(.2)(.1)+(.5)(.3)+(.3)(.2)} = \dfrac{.02}{.23}$ = .087.

Similarly, *P*(1-night | purchase) = $\dfrac{(.5)(.3)}{.23}$ = .652 and *P*(2-night | purchase) = .261.

66. Let *E*, *C*, and *L* be the events associated with e-mail, cell phones, and laptops, respectively. We are told $P(E) = 40\%$, $P(C) = 30\%$, $P(L) = 25\%$, $P(E \cap C) = 23\%$, $P(E' \cap C' \cap L') = 51\%$, $P(E \mid L) = 88\%$, and $P(L \mid C) = 70\%$.

a. $P(C \mid E) = P(E \cap C)/P(E) = .23/.40 = .575$.

b. Use Bayes' rule: $P(C \mid L) = P(C \cap L)/P(L) = P(C)P(L \mid C)/P(L) = .30(.70)/.25 = .84$.

c. $P(C|E \cap L) = P(C \cap E \cap L)/P(E \cap L)$.
For the denominator, $P(E \cap L) = P(L)P(E \mid L) = (.25)(.88) = .22$.
For the numerator, use $P(\text{E} \cup C \cup L) = 1 - P(E' \cap C' \cap L') = 1 - .51 = .49$ and write
$P(E \cup C \cup L) = P(C) + P(E) + P(L) - P(E \cap C) - P(C \cap L) - P(E \cap L) + P(C \cap E \cap L)$
$\Rightarrow .49 = .30 + .40 + .25 - .23 - .30(.70) - .22 + P(C \cap E \cap L) \Rightarrow P(C \cap E \cap L) = .20$.
So, finally, $P(C|E \cap L) = .20/.22 = .9091$.

67. Let *T* denote the event that a randomly selected person is, in fact, a terrorist. Apply Bayes' theorem, using *P*(*T*) = 1,000/300,000,000 = .0000033:

$P(T \mid +) = \dfrac{P(T)P(+ \mid T)}{P(T)P(+ \mid T) + P(T')P(+ \mid T')} = \dfrac{(.0000033)(.99)}{(.0000033)(.99) + (1 - .0000033)(1 - .999)}$ = .003289. That is to say, roughly 0.3% of all people "flagged" as terrorists would be actual terrorists in this scenario.

68. Let's see how we can implement the hint. If she's flying airline #1, the chance of 2 late flights is (30%)(10%) = 3%; the two flights being "unaffected" by each other means we can multiply their probabilities. Similarly, the chance of 0 late flights on airline #1 is (70%)(90%) = 63%. Since percents add to 100%, the chance of exactly 1 late flight on airline #1 is 100% – (3% + 63%) = 34%. A similar approach works for the other two airlines: the probability of exactly 1 late flight on airline #2 is 35%, and the chance of exactly 1 late flight on airline #3 is 45%.

The initial ("prior") probabilities for the three airlines are $P(A_1)$ = 50%, $P(A_2)$ = 30%, and $P(A_3)$ = 20%. Given that she had exactly 1 late flight (call that event B), the conditional ("posterior") probabilities of the three airlines can be calculated using Bayes' Rule:

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)} = \frac{(.5)(.34)}{(.5)(.34) + (.3)(.35) + (.2)(.45)} = \frac{.170}{.365} =$$

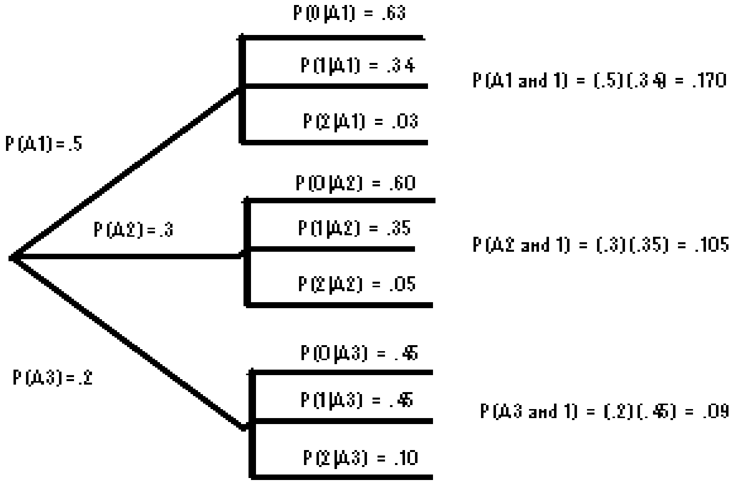
.4657;

$$P(A_2 \mid B) = \frac{P(A_2)P(B \mid A_2)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)} = \frac{(.3)(.35)}{.365} = .2877; \text{ and}$$

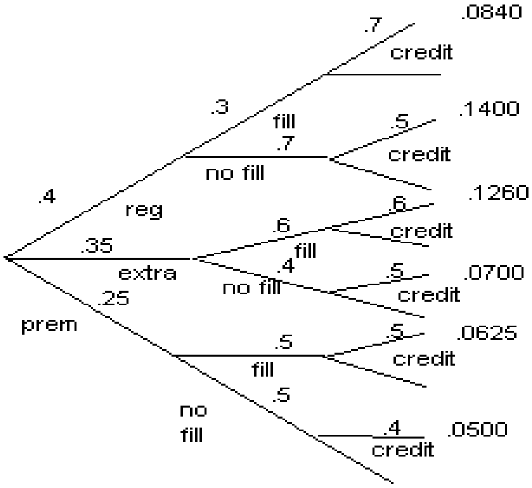
$$P(A_3 \mid B) = \frac{P(A_3)P(B \mid A_3)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)} = \frac{(.2)(.45)}{.365} = .2466.$$

Notice that, except for rounding error, these three posterior probabilities add to 1.

The tree diagram below shows these probabilities.



69. The tree diagram below summarizes the information in the exercise (plus the previous information in Exercise 59). Probabilities for the branches corresponding to paying with credit are indicated at the far right. (“extra” = “plus”)



a. P(plus $\cap$ fill $\cap$ credit) = (.35)(.6)(.6) = .1260.

b. P(premium $\cap$ no fill $\cap$ credit) = (.25)(.5)(.4) = .05.

c. From the tree diagram, P(premium $\cap$ credit) = .0625 + .0500 = .1125.

d. From the tree diagram, P(fill $\cap$ credit) = .0840 + .1260 + .0625 = .2725.

e. P(credit) = .0840 + .1400 + .1260 + .0700 + .0625 + .0500 = .5325.

f. P(premium | credit) = $\dfrac{P(\text{premium} \cap \text{credit})}{P(\text{credit})} = \dfrac{.1125}{.5325} = .2113$.

Section 2.5

70. Using the definition, two events A and B are independent if $P(A \mid B) = P(A)$;
$P(A \mid B) = .6125$; $P(A) = .50$; $.6125 \neq .50$, so A and B are not independent.
Using the multiplication rule, the events are independent if $P(A \cap B)=P(A)P(B)$;
$P(A \cap B) = .25$; $P(A)P(B) = (.5)(.4) = .2$. $.25 \neq .2$, so A and B are not independent.

71.

a. Since the events are independent, then A' and B' are independent, too. (See the paragraph below Equation 2.7.) Thus, $P(B'|A') = P(B') = 1 - .7 = .3$.

b. Using the addition rule, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = .4 + .7 - (.4)(.7) = .82$. Since A and B are independent, we are permitted to write $P(A \cap B) = P(A)P(B) = (.4)(.7)$.

c. $P(AB' \mid A \cup B) = \dfrac{P(AB' \cap (A \cup B))}{P(A \cup B)} = \dfrac{P(AB')}{P(A \cup B)} = \dfrac{P(A)P(B')}{P(A \cup B)} = \dfrac{(.4)(1-.7)}{.82} = \dfrac{.12}{.82} = .146$.

72. $P(A_1 \cap A_2) = .11$ while $P(A_1)P(A_2) = .055$, so A_1 and A_2 are not independent.
$P(A_1 \cap A_3) = .05$ while $P(A_1)P(A_3) = .0616$, so A_1 and A_3 are not independent.
$P(A_2 \cap A_3) = .07$ and $P(A_2)P(A_3) = .07$, so A_2 and A_3 are independent.

73. From a Venn diagram, $P(B) = P(A' \cap B) + P(A \cap B) = P(B) \Rightarrow P(A' \cap B) = P(B) - P(A \cap B)$. If A and B are independent, then $P(A' \cap B) = P(B) - P(A)P(B) = [1 - P(A)]P(B) = P(A')P(B)$. Thus, A' and B are independent.

Alternatively, $P(A' \mid B) = \dfrac{P(A' \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)} = \dfrac{P(B) - P(A)P(B)}{P(B)} = 1 - P(A) = P(A')$.

74. Using subscripts to differentiate between the selected individuals,
$P(O_1 \cap O_2) = P(O_1)P(O_2) = (.45)(.45) = .2025$.
$P(\text{two individuals match}) = P(A_1 \cap A_2) + P(B_1 \cap B_2) + P(AB_1 \cap AB_2) + P(O_1 \cap O_2) =$
$.40^2 + .11^2 + .04^2 + .45^2 = .3762$.

75. Let event E be the event that an error was signaled incorrectly.
We want $P(\text{at least one signaled incorrectly}) = P(E_1 \cup \ldots \cup E_{10})$. To use independence, we need intersections, so apply deMorgan's law: $= P(E_1 \cup \ldots \cup E_{10}) = 1 - P(E_1' \cap \cdots \cap E_{10}')$. $P(E') = 1 - .05 = .95$, so for 10 independent points, $P(E_1' \cap \cdots \cap E_{10}') = (.95)\ldots(.95) = (.95)^{10}$. Finally, $P(E_1 \cup E_2 \cup \ldots \cup E_{10}) = 1 - (.95)^{10} = .401$. Similarly, for 25 points, the desired probability is $1 - (P(E'))^{25} = 1 - (.95)^{25} = .723$.

76. Follow the same logic as in Exercise 75: If the probability of an event is p, and there are n independent "trials," the chance this event never occurs is $(1 – p)^n$, while the chance of at least one occurrence is $1 – (1 – p)^n$. With p = 1/9,000,000,000 and n = 1,000,000,000, this calculates to 1 – .9048 = .0952.

Note: For extremely small values of p, $(1 – p)^n \approx 1 – np$. So, the probability of at least one occurrence under these assumptions is roughly $1 – (1 – np) = np$. Here, that would equal 1/9.

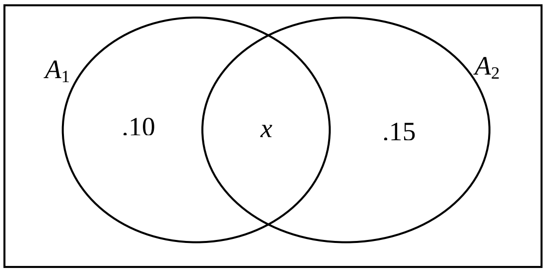
77. Let p denote the probability that a rivet is defective.

a. .15 = P(seam needs reworking) = 1 – P(seam doesn't need reworking) =
1 – P(no rivets are defective) = 1 – P(1st isn't def $\cap \ldots \cap$ 25th isn't def) =
$1 – (1 – p)\ldots(1 – p) = 1 – (1 – p)^{25}$.
Solve for p: $(1 – p)^{25} = .85 \Rightarrow 1 – p = (.85)^{1/25} \Rightarrow p = 1 – .99352 = .00648$.

b. The desired condition is $.10 = 1 – (1 – p)^{25}$. Again, solve for p: $(1 – p)^{25} = .90 \Rightarrow$
$p = 1 – (.90)^{1/25} = 1 – .99579 = .00421$.

78. P(at least one opens) = 1 – P(none open) = $1 – (.04)^5$ = .999999897.
P(at least one fails to open) = 1 – P(all open) = $1 – (.96)^5$ = .1846.

79. Let A_1 = older pump fails, A_2 = newer pump fails, and $x = P(A_1 \cap A_2)$. The goal is to find x. From the Venn diagram below, $P(A_1) = .10 + x$ and $P(A_2) = .05 + x$. Independence implies that $x = P(A_1 \cap A_2) = P(A_1)P(A_2) = (.10 + x)(.05 + x)$. The resulting quadratic equation, $x^2 – .85x + .005 = 0$, has roots $x = .0059$ and $x = .8441$. The latter is impossible, since the probabilities in the Venn diagram would then exceed 1. Therefore, $x = .0059$.



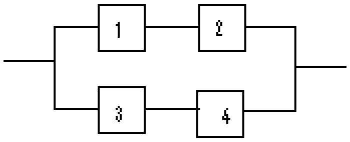
80. Let A_i denote the event that component #i works (i = 1, 2, 3, 4). Based on the design of the system, the event "the system works" is $(A_1 \cup A_2) \cup (A_3 \cap A_4)$. We'll eventually need $P(A_1 \cup A_2)$, so work that out first: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = (.9) + (.9) - (.9)(.9) = .99$. The third term uses independence of events. Also, $P(A_3 \cap A_4)$ = (.8)(.8) = .64, again using independence.

Now use the addition rule and independence for the system:

$$\begin{aligned} P((A_1 \cup A_2) \cup (A_3 \cap A_4)) &= P(A_1 \cup A_2) + P(A_3 \cap A_4) - P((A_1 \cup A_2) \cap (A_3 \cap A_4)) \\ &= P(A_1 \cup A_2) + P(A_3 \cap A_4) - P(A_1 \cup A_2) \times P(A_3 \cap A_4) \\ &= (.99) + (.64) - (.99)(.64) = .9964 \end{aligned}$$

(You could also use deMorgan's law in a couple of places.)

81. Using the hints, let $P(A_i) = p$, and $x = p^2$. Following the solution provided in the example, P(system lifetime exceeds t_0) = $p^2 + p^2 - p^4 = 2p^2 - p^4 = 2x - x^2$. Now, set this equal to .99: $2x - x^2 = .99 \Rightarrow x^2 - 2x + .99 = 0 \Rightarrow x = 0.9$ or $1.1 \Rightarrow p = 1.049$ or $.9487$. Since the value we want is a probability and cannot exceed 1, the correct answer is $p = .9487$.



82. A = {(3,1)(3,2)(3,3)(3,4)(3,5)(3,6)} $\Rightarrow P(A) = \frac{6}{36} = \frac{1}{6}$; B = {(1,4)(2,4)(3,4)(4,4)(5,4)(6,4)} $\Rightarrow P(B) = \frac{1}{6}$; and C = {(1,6)(2,5)(3,4)(4,3)(5,2)(6,1)} $\Rightarrow P(C) = \frac{1}{6}$.

$A \cap B$ = {(3,4)} $\Rightarrow P(A \cap B) = \frac{1}{36} = P(A)P(B)$; $A \cap C$ = {(3,4)} $\Rightarrow P(A \cap C) = \frac{1}{36} = P(A)P(C)$; and $B \cap C$ = {(3,4)} $\Rightarrow P(B \cap C) = \frac{1}{36} = P(B)P(C)$. Therefore, these three events are pairwise independent.

However, $A \cap B \cap C$ = {(3,4)} $\Rightarrow P(A \cap B \cap C) = \frac{1}{36}$, while $P(A)P(B)P(C) = \ = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$, so $P(A \cap B \cap C) \neq P(A)P(B)P(C)$ and these three events are not mutually independent.

83. We'll need to know P(both detect the defect) = 1 – P(at least one doesn't) = 1 – .2 = .8.

a. $P(1^{st}$ detects $\cap$ 2^{nd} doesn't) = $P(1^{st}$ detects) – $P(1^{st}$ does $\cap$ 2^{nd} does) = .9 – .8 = .1.
Similarly, $P(1^{st}$ doesn't $\cap$ 2^{nd} does) = .1, so P(exactly one does)= .1 + .1= .2.

b. P(neither detects a defect) = 1 – [P(both do) + P(exactly 1 does)] = 1 – [.8+.2] = 0. That is, under this model there is a 0% probability neither inspector detects a defect. As a result, P(all 3 escape) = (0)(0)(0) = 0.

84. We'll make repeated use of the independence of the A_is and their complements.

a. $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$ = (.95)(.98)(.80) = .7448.

b. This is the complement of part **a**, so the answer is 1 – .7448 = .2552.

c. $P(A_1' \cap A_2' \cap A_3') = P(A_1')P(A_2')P(A_3')$ = (.05)(.02)(.20) = .0002.

d. $P(A_1' \cap A_2 \cap A_3) = P(A_1')P(A_2)P(A_3)$ = (.05)(.98)(.80) = .0392.

e. $P([A_1' \cap A_2 \cap A_3] \cup [A_1 \cap A_2' \cap A_3] \cup [A_1 \cap A_2 \cap A_3'])$ = (.05)(.98)(.80) + (.95)(.02)(.80) + (.95)(.98)(.20) = .07302.

f. This is just a little joke — we've all had the experience of electronics dying right after the warranty expires! ☺

85.

a. Let D_1 = detection on 1st fixation, D_2 = detection on 2nd fixation.
P(detection in at most 2 fixations) = $P(D_1) + P(D_1' \cap D_2)$; since the fixations are independent,
$P(D_1) + P(D_1' \cap D_2) = P(D_1) + P(D_1')\,P(D_2) = p + (1-p)p = p(2-p)$.

b. Define $D_1, D_2, \ldots, D_n$ as in **a**. Then P(at most n fixations) =
$P(D_1) + P(D_1' \cap D_2) + P(D_1' \cap D_2' \cap D_3) + \ldots + P(D_1' \cap D_2' \cap \cdots \cap D_{n-1}' \cap D_n) =$
$p + (1-p)p + (1-p)^2 p + \ldots + (1-p)^{n-1}p = p[1 + (1-p) + (1-p)^2 + \ldots + (1-p)^{n-1}] =$

$$p \cdot \frac{1-(1-p)^n}{1-(1-p)} = 1-(1-p)^n .$$

Alternatively, P(at most n fixations) = 1 – P(at least n+1 fixations are required) =
1 – P(no detection in 1st n fixations) = $1 - P(D_1' \cap D_2' \cap \cdots \cap D_n') = 1 - (1-p)^n$.

c. P(no detection in 3 fixations) = $(1-p)^3$.

d. P(passes inspection) = P({not flawed} ∪ {flawed and passes})
= P(not flawed) + P(flawed and passes)
= .9 + P(flawed) P(passes | flawed) = .9 + (.1)$(1-p)^3$.

e. Borrowing from **d**, P(flawed | passed) = $\dfrac{P(\text{flawed} \cap \text{passed})}{P(\text{passed})} = \dfrac{.1(1-p)^3}{.9+.1(1-p)^3}$. For p = .5,

P(flawed | passed) = $\dfrac{.1(1-.5)^3}{.9+.1(1-.5)^3} = .0137$.

86.

a. $P(A) = \frac{2,000}{10,000}$ = .2. Using the law of total probability, $P(B) = P(A)P(B \mid A) + P(A')P(B \mid A')$ = $(.2)\frac{1,999}{9,999} + (.8)\frac{2,000}{9,999}$ = **.2** exactly. That is, $P(B)$ = $P(A)$ = .2. Finally, use the multiplication rule: $P(A \cap B) = P(A) \times P(B \mid A) = (.2)\frac{1,999}{9,999}$ = .039984. Events A and B are *not* independent, since $P(B)$ = .2 while $P(B \mid A) = \frac{1,999}{9,999} = .19992$, and these are not equal.

b. If A and B were independent, we'd have $P(A \cap B) = P(A) \times P(B) = (.2)(.2) = .04$. This is very close to the answer .039984 from part **a**. This suggests that, for most practical purposes, we could treat events A and B in this example as if they were independent.

c. Repeating the steps in part **a**, you again get $P(A)$ = $P(B)$ = .2. However, using the multiplication rule, $P(A \cap B) = P(A) \times P(B \mid A) = \frac{2}{10} \times \frac{1}{9}$ =.0222. This is very different from the value of .04 that we'd get if A and B were independent!

The critical difference is that the population size in parts **a-b** is huge, and so the probability a second board is green *almost* equals .2 (i.e., 1,999/9,999 = .19992 ≈ .2). But in part **c**, the conditional probability of a green board shifts a lot: 2/10 = .2, but 1/9 = .1111.

87.

a. Use the information provided and the addition rule:
$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \Rightarrow P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2) = .55 + .65 - .80 = .40$.

b. By definition, $P(A_2 \mid A_3) = \frac{P(A_2 \cap A_3)}{P(A_3)} = \frac{.40}{.70}$ = .5714. If a person likes vehicle #3, there's a 57.14% chance s/he will also like vehicle #2.

c. No. From **b**, $P(A_2 \mid A_3) = .5714 \neq P(A_2) = .65$. Therefore, A_2 and A_3 are not independent. Alternatively, $P(A_2 \cap A_3) = .40 \neq P(A_2)P(A_3) = (.65)(.70) = .455$.

d. The goal is to find $P(A_2 \cup A_3 \mid A_1')$, i.e. $\frac{P([A_2 \cup A_3] \cap A_1')}{P(A_1')}$. The denominator is simply 1 – .55 = .45. There are several ways to calculate the numerator; the simplest approach using the information provided is to draw a Venn diagram and observe that $P([A_2 \cup A_3] \cap A_1') = P(A_1 \cup A_2 \cup A_3) - P(A_1)$ = .88 – .55 = .33. Hence, $P(A_2 \cup A_3 \mid A_1')$ = $\frac{.33}{.45}$ = .7333.

88. Let *D* = patient has disease, so *P*(*D*) = .05. Let ++ denote the event that the patient gets two independent, positive tests. Given the sensitivity and specificity of the test, *P*(++ | *D*) = (.98)(.98) = .9604, while *P*(++ | *D*′) = (1 – .99)(1 – .99) = .0001. (That is, there's a 1-in-10,000 chance of a healthy person being mis-diagnosed with the disease <u>twice</u>.) Apply Bayes' Theorem:

$$P(D\,|\,{+}{+}) = \frac{P(D)P({+}{+}\,|\,D)}{P(D)P({+}{+}\,|\,D)+P(D')P({+}{+}\,|\,D')} = \frac{(.05)(.9604)}{(.05)(.9604)+(.95)(.0001)} = .9980$$

89. The question asks for *P*(<u>exactly</u> one tag lost | at <u>most</u> one tag lost) = $P((C_1\cap C_2')\cup(C_1'\cap C_2)\,|\,(C_1\cap C_2)')$. Since the first event is contained in (a subset of) the second event, this equals $\frac{P((C_1\cap C_2')\cup(C_1'\cap C_2))}{P((C_1\cap C_2)')} = \frac{P(C_1\cap C_2')+P(C_1'\cap C_2)}{1-P(C_1\cap C_2)} = \frac{P(C_1)P(C_2')+P(C_1')P(C_2)}{1-P(C_1)P(C_2)}$ by independence = $\frac{\pi(1-\pi)+(1-\pi)\pi}{1-\pi^2} = \frac{2\pi(1-\pi)}{1-\pi^2} = \frac{2\pi}{1+\pi}$.

Supplementary Exercises

90.

a. $\binom{10}{3}$ = 120.

b. There are 9 other senators from whom to choose the other two subcommittee members, so the answer is 1 × $\binom{9}{2}$ = 36.

c. There are 120 possible subcommittees. Among those, the number which would include none of the 5 most senior senators (i.e., all 3 members are chosen from the 5 most junior senators) is $\binom{5}{3}$ = 10. Hence, the number of subcommittees with at least one senior senator is 120 – 10 = 110, and the chance of this randomly occurring is 110/120 = .9167.

d. The number of subcommittees that can form from the 8 "other" senators is $\binom{8}{3}$ = 56, so the probability of this event is 56/120 = .4667.

91.

a. $P(\text{line 1}) = \frac{500}{1500} = .333$;

$P(\text{crack}) = \frac{.50(500)+.44(400)+.40(600)}{1500} = \frac{666}{1500} = .444$.

b. This is one of the percentages provided: $P(\text{blemish} \mid \text{line 1}) = .15$.

c. $P(\text{surface defect}) = \frac{.10(500)+.08(400)+.15(600)}{1500} = \frac{172}{1500}$;

$P(\text{line 1} \cap \text{surface defect}) = \frac{.10(500)}{1500} = \frac{50}{1500}$;

so, $P(\text{line 1} \mid \text{surface defect}) = \frac{50/1500}{172/1500} = \frac{50}{172} = .291$.

92.

a. He will have one type of form left if either 4 withdrawals or 4 course substitutions remain. This means the first six were either 2 withdrawals and 4 subs or 6 withdrawals and 0 subs; the desired probability is $\frac{\binom{6}{2}\binom{4}{4}+\binom{6}{6}\binom{4}{0}}{\binom{10}{6}} = \frac{16}{210} = .0762$.

b. He can start with the withdrawal forms or the course substitution forms, allowing two sequences: W-C-W-C or C-W-C-W. The number of ways the first sequence could arise is (6)(4)(5)(3) = 360, and the number of ways the second sequence could arise is (4)(6)(3)(5) = 360, for a total of 720 such possibilities. The total number of ways he could select four forms one at a time is $P_{4,10}$ = (10)(9)(8)(7) = 5040. So, the probability of a perfectly alternating sequence is 720/5040 = .143.

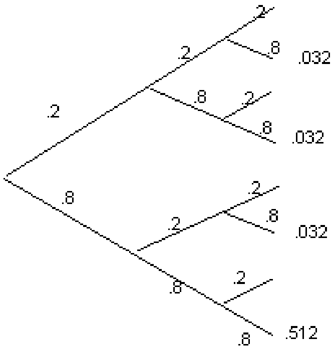
93. Apply the addition rule: $P(A\cup B) = P(A) + P(B) - P(A\cap B) \Rightarrow .626 = P(A) + P(B) - .144$. Apply independence: $P(A\cap B) = P(A)P(B) = .144$.
So, $P(A) + P(B) = .770$ and $P(A)P(B) = .144$.
Let $x = P(A)$ and $y = P(B)$. Using the first equation, $y = .77 - x$, and substituting this into the second equation yields $x(.77 - x) = .144$ or $x^2 - .77x + .144 = 0$. Use the quadratic formula to solve:

$$x = \frac{.77 \pm \sqrt{(-.77)^2 - (4)(1)(.144)}}{2(1)} = \frac{.77 \pm .13}{2}$$ = .32 or .45. Since $x = P(A)$ is assumed to be the larger probability, $x = P(A) = .45$ and $y = P(B) = .32$.

94. The probability of a bit reversal is .2, so the probability of maintaining a bit is .8.

a. Using independence, P(all three relays correctly send 1) = (.8)(.8)(.8) = .512.

b. In the accompanying tree diagram, each .2 indicates a bit reversal (and each .8 its opposite). There are several paths that maintain the original bit: no reversals or exactly two reversals (e.g., 1 → 1 → 0 → 1, which has reversals at relays 2 and 3). The total probability of these options is .512 + (.8)(.2)(.2) + (.2)(.8)(.2) + (.2)(.2)(.8) = .512 + 3(.032) = .608.



c. Using the answer from **b**, $P(1 \text{ sent} \mid 1 \text{ received}) = \dfrac{P(1 \text{ sent} \cap 1 \text{ received})}{P(1 \text{ received})} =$

$$\frac{P(1 \text{ sent})P(1 \text{ received} \mid 1 \text{ sent})}{P(1 \text{ sent})P(1 \text{ received} \mid 1 \text{ sent}) + P(0 \text{ sent})P(1 \text{ received} \mid 0 \text{ sent})} = \frac{(.7)(.608)}{(.7)(.608) + (.3)(.392)} = \frac{.4256}{.5432} =$$

.7835.
In the denominator, $P(1 \text{ received} \mid 0 \text{ sent}) = 1 - P(0 \text{ received} \mid 0 \text{ sent}) = 1 - .608$, since the answer from **b** also applies to a 0 being relayed as a 0.

95.

a. There are 5! = 120 possible orderings, so P(BCDEF) = $\frac{1}{120}$ = .0833.

b. The number of orderings in which F is third equals 4×3×1*×2×1 = 24 (*because F must be here), so P(F is third) = $\frac{24}{120}$ = .2. Or more simply, since the five friends are ordered completely at random, there is a ⅕ chance F is specifically in position three.

c. Similarly, P(F last) = $\frac{4\times 3\times 2\times 1\times 1}{120}$ = .2.

d. P(F hasn't heard after 10 times) = P(not on #1 $\cap$ not on #2 $\cap$ … $\cap$ not on #10) = $\frac{4}{5}\times\cdots\times\frac{4}{5}=\left(\frac{4}{5}\right)^{10}$ = .1074.

96. Palmberg equation: $P_d(c)=\frac{(c/c^*)^\beta}{1+(c/c^*)^\beta}$

a. $P_d(c^*)=\frac{(c^*/c^*)^\beta}{1+(c^*/c^*)^\beta}=\frac{1^\beta}{1+1^\beta}=\frac{1}{1+1}=.5$.

b. The probability of detecting a crack that is twice the size of the "50-50" size c^* equals $P_d(2c^*)=\frac{(2c^*/c^*)^\beta}{1+(2c^*/c^*)^\beta}=\frac{2^\beta}{1+2^\beta}$. When $\beta = 4$, $P_d(2c^*)=\frac{2^4}{1+2^4}=\frac{16}{17}$ = .9412.

c. Using the answers from **a** and **b**, P(exactly one of two detected) = P(first is, second isn't) + P(first isn't, second is) = (.5)(1 – .9412) + (1 – .5)(.9412) = .5.

d. If $c = c^*$, then $P_d(c)$ = .5 irrespective of β. If $c < c^*$, then $c/c^* < 1$ and $P_d(c) \to \frac{0}{0+1} = 0$ as $\beta \to \infty$. Finally, if $c > c^*$ then $c/c^* > 1$ and, from calculus, $P_d(c) \to 1$ as $\beta \to \infty$.

97. When three experiments are performed, there are 3 different ways in which detection can occur on exactly 2 of the experiments: (i) #1 and #2 and not #3; (ii) #1 and not #2 and #3; and (iii) not #1 and #2 and #3. If the impurity is present, the probability of exactly 2 detections in three (independent) experiments is (.8)(.8)(.2) + (.8)(.2)(.8) + (.2)(.8)(.8) = .384. If the impurity is absent, the analogous probability is 3(.1)(.1)(.9) = .027. Thus, applying Bayes' theorem, P(impurity is present | detected in exactly 2 out of 3) $=\frac{P(\text{detected in exactly 2}\cap\text{present})}{P(\text{detected in exactly 2})}=\frac{(.384)(.4)}{(.384)(.4)+(.027)(.6)}$ = .905.

98. Our goal is to find $P(A \cup B \cup C \cup D \cup E)$. We'll need all of the following probabilities:

$P(A) = P$(Allison gets her calculator back) = 1/5. This is intuitively obvious; you can also see it by writing out the 5! = 120 orderings in which the friends could get calculators (ABCDE, ABCED, …, EDCBA) and observe that 24 of the 120 have A in the first position. So, $P(A) = 24/120 = 1/5$. By the same reasoning, $P(B) = P(C) = P(D) = P(E) = 1/5$.

$P(A \cap B) = P$(Allison and Beth get their calculators back) = 1/20. This can be computed by considering all 120 orderings and noticing that six — those of the form ABxyz — have A and B in the correct positions. Or, you can use the multiplication rule: $P(A \cap B) = P(A)P(B \mid A) = (1/5)(1/4) = 1/20$. All other pairwise intersection probabilities are also 1/20.

$P(A \cap B \cap C) = P$(Allison and Beth and Carol get their calculators back) = 1/60, since this can only occur if two ways — ABCDE and ABCED — and 2/120 = 1/60. So, all three-wise intersections have probability 1/60.

$P(A \cap B \cap C \cap D) = 1/120$, since this can only occur if all 5 girls get their own calculators back. In fact, all four-wise intersections have probability 1/120, as does $P(A \cap B \cap C \cap D \cap E)$ — they're the same event.

Finally, put all the parts together, using a general inclusion-exclusion rule for unions:

$$\begin{aligned} P(A\cup B\cup C\cup D\cup E) &= P(A)+P(B)+P(C)+P(D)+P(E) \\ &\quad -P(A\cap B)-P(A\cap C)-\cdots-P(D\cap E) \\ &\quad +P(A\cap B\cap C)+\cdots+P(C\cap D\cap E) \\ &\quad -P(A\cap B\cap C\cap D)-\cdots-P(B\cap C\cap D\cap E) \\ &\quad +P(A\cap B\cap C\cap D\cap E) \\ &= 5\cdot\frac{1}{5}-10\cdot\frac{1}{20}+10\cdot\frac{1}{60}-5\cdot\frac{1}{120}+\frac{1}{120}=1-\frac{1}{2}+\frac{1}{6}-\frac{1}{24}+\frac{1}{120}=\frac{76}{120}=.633 \end{aligned}$$

The final answer has the form $1-\frac{1}{2}+\frac{1}{6}-\frac{1}{24}=\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{5!}$. Generalizing to n friends, the probability at least one will get her own calculator back is $\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\cdots+(-1)^{n-1}\frac{1}{n!}$.

When n is large, we can relate this to the power series for e^x evaluated at $x = -1$:

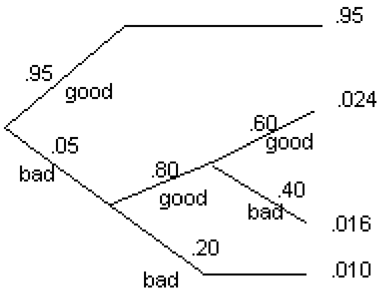
$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots \Rightarrow$$

$$e^{-1} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots = 1-\left[\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\cdots\right]\Rightarrow$$

$$1-e^{-1} = \frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\cdots$$

So, for large n, P(at least one friend gets her own calculator back) ≈ $1 - e^{-1} = .632$. Contrary to intuition, the chance of this event does not converge to 1 (because "someone is bound to get hers back") or to 0 (because "there are just too many possible arrangements"). Rather, in a large group, there's about a 63.2% chance someone will get her own item back (a match), and about a 36.8% chance that nobody will get her own item back (no match).

99. Refer to the tree diagram below.



a. P(pass inspection) = P(pass initially $\cup$ passes after recrimping) =
P(pass initially) + P(fails initially $\cap$ goes to recrimping $\cap$ is corrected after recrimping) =
.95 + (.05)(.80)(.60) (following path "bad-good-good" on tree diagram) = .974.

b. P(needed no recrimping | passed inspection) = $\dfrac{P(\text{passed initially})}{P(\text{passed inspection})} = \dfrac{.95}{.974} = .9754$.

100.

a. First, the probabilities of the A_i are $P(A_1) = P(JJ) = (.6)^2 = .36$; $P(A_2) = P(MM) = (.4)^2 = .16$; and $P(A_3) = P(JM \text{ or } MJ) = (.6)(.4) + (.4)(.6) = .48$.
Second, $P(\text{Jay wins} \mid A_1) = 1$, since Jay is two points ahead and, thus has won; $P(\text{Jay wins} \mid A_2) = 0$, since Maurice is two points ahead and, thus, Jay has lost; and $P(\text{Jay wins} \mid A_3) = p$, since at that moment the score has returned to deuce and the game has effectively started over. Apply the law of total probability:

$$P(\text{Jay wins}) = P(A_1)P(\text{Jay wins} \mid A_1) + P(A_2)P(\text{Jay wins} \mid A_2) + P(A_3)P(\text{Jay wins} \mid A_3)$$
$$p = (.36)(1) + (.16)(0) + (.48)(p)$$

Therefore, $p = .36 + .48p$; solving for p gives $p = \dfrac{.36}{1-.48} = .6923$.

b. Apply Bayes' rule: $P(JJ \mid \text{Jay wins}) = \dfrac{P(JJ)P(\text{Jay wins} \mid JJ)}{P(\text{Jay wins})} = \dfrac{(.36)(1)}{.6923} = .52$.

101. Let A = 1st functions, B = 2nd functions, so $P(B) = .9$, $P(A \cup B) = .96$, $P(A \cap B)=.75$. Use the addition rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow .96 = P(A) + .9 - .75 \Rightarrow P(A) = .81$.

Therefore, $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{.75}{.81} = .926$.

102.

a. $P(F) = 919/2026 = .4536$. $P(C) = 308/2026 = .1520$.

b. $P(F \cap C) = 110/2026 = .0543$. Since $P(F) \times P(C) = .4536 \times .1520 = .0690 \neq .0543$, we find that events F and C are <u>not</u> independent.

c. $P(F \mid C) = P(F \cap C)/P(C) = 110/308 = .3571$.

d. $P(C \mid F) = P(C \cap F)/P(F) = 110/919 = .1197$.

e. Divide each of the two rows, Male and Female, by its row total.

	Blue	**Brown**	**Green**	**Hazel**
Male	.3342	.3180	.1789	.1689
Female	.3906	.3156	.1197	.1741

According to the data, brown and hazel eyes have similar likelihoods for males and females. However, females are much more likely to have blue eyes than males (39% versus 33%) and, conversely, males have a much greater propensity for green eyes than do females (18% versus 12%).

103. A tree diagram can help here.

a. $P(E_1 \cap L) = P(E_1)P(L \mid E_1) = (.40)(.02) = .008$.

b. The law of total probability gives $P(L) = \sum P(E_i)P(L \mid E_i) = (.40)(.02) + (.50)(.01) + (.10)(.05) = .018$.

c. $P(E_1' \mid L') = 1 - P(E_1 \mid L') = 1 - \dfrac{P(E_1 \cap L')}{P(L')} = 1 - \dfrac{P(E_1)P(L' \mid E_1)}{1 - P(L)} = 1 - \dfrac{(.40)(.98)}{1 - .018} = .601$.

104. Let B denote the event that a component needs rework. By the law of total probability,
$P(B) = \sum P(A_i)P(B \mid A_i) = (.50)(.05) + (.30)(.08) + (.20)(.10) = .069$.
Thus, $P(A_1 \mid B) = \dfrac{(.50)(.05)}{.069} = .362$, $P(A_2 \mid B) = \dfrac{(.30)(.08)}{.069} = .348$, and $P(A_3 \mid B) = .290$.

105. This is the famous “Birthday Problem” in probability.

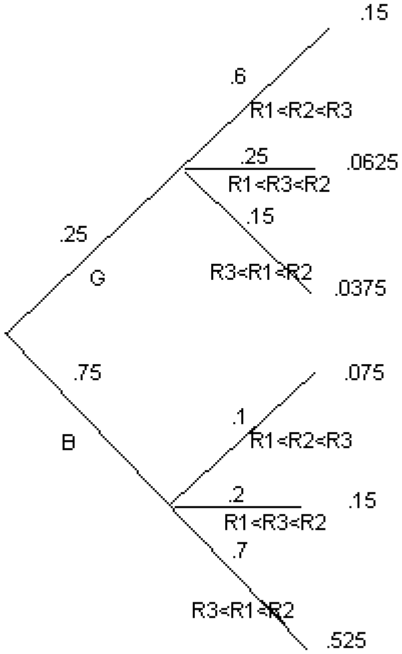
a. There are 365^{10} possible lists of birthdays, e.g. (Dec 10, Sep 27, Apr 1, …). Among those, the number with zero matching birthdays is $P_{10,365}$ (sampling ten birthdays without replacement from 365 days. So, $P(\text{all different}) = \dfrac{P_{10,365}}{365^{10}} = \dfrac{(365)(364)\cdots(356)}{(365)^{10}} = .883$. $P(\text{at least two the same}) = 1 - .883 = .117$.

b. The general formula is P(at least two the same) = 1 – $\frac{P_{k,365}}{365^k}$. By trial and error, this probability equals .476 for k = 22 and equals .507 for k = 23. Therefore, the smallest k for which k people have at least a 50-50 chance of a birthday match is 23.

c. There are 1000 possible 3-digit sequences to end a SS number (000 through 999). Using the idea from **a**, P(at least two have the same SS ending) = 1 – $\frac{P_{10,1000}}{1000^{10}}$ = 1 – .956 = .044.

Assuming birthdays and SS endings are independent, P(at least one “coincidence”) = P(birthday coincidence $\cup$ SS coincidence) = .117 + .044 – (.117)(.044) = .156.

106. See the accompanying tree diagram.



a. $P(G \mid R_1 < R_2 < R_3) = \frac{.15}{.15+.075} = .67$ while $P(B \mid R_1 < R_2 < R_3) = .33$, so classify the specimen as granite. Equivalently, $P(G \mid R_1 < R_2 < R_3) = .67 > ½$ so granite is more likely.

b. $P(G \mid R_1 < R_3 < R_2) = \frac{.0625}{.2125} = .2941 < ½$, so classify the specimen as basalt.

$P(G \mid R_3 < R_1 < R_2) = \frac{.0375}{.5625} = .0667 < ½$, so classify the specimen as basalt.

c. P(erroneous classification) = P(B classified as G) + P(G classified as B) =
$P(B)P$(classified as G | B) + $P(G)P$(classified as B | G) =
$(.75)P(R_1 < R_2 < R_3 \mid B) + (.25)P(R_1 < R_3 < R_2 \text{ or } R_3 < R_1 < R_2 \mid G) =$
(.75)(.10) + (.25)(.25 + .15) = .175.

d. For what values of p will $P(G \mid R_1 < R_2 < R_3)$, $P(G \mid R_1 < R_3 < R_2)$, and $P(G \mid R_3 < R_1 < R_2)$ all exceed ½? Replacing .25 and .75 with p and $1 - p$ in the tree diagram,

$P(G \mid R_1 < R_2 < R_3) = \frac{.6p}{.6p+.1(1-p)} = \frac{.6p}{.1+.5p} > .5 \text{ iff } p > \frac{1}{7}$;

$P(G \mid R_1 < R_3 < R_2) = \frac{.25p}{.25p+.2(1-p)} > .5 \text{ iff } p > \frac{4}{9}$;

$P(G \mid R_3 < R_1 < R_2) = \frac{.15p}{.15p+.7(1-p)} > .5 \text{ iff } p > \frac{14}{17}$ (most restrictive). Therefore, one would always classify a rock as granite iff $p > \frac{14}{17}$.

107. P(detection by the end of the nth glimpse) = 1 – P(not detected in first n glimpses) =

$1 - P(G_1' \cap G_2' \cap \cdots \cap G_n') = 1 - P(G_1')P(G_2')\cdots P(G_n') = 1 - (1 - p_1)(1 - p_2) \ldots (1 - p_n) = 1 - \prod_{i=1}^{n}(1 - p_i)$.

108.

a. P(walks on 4th pitch) = P(first 4 pitches are balls) = $(.5)^4$ = .0625.

b. P(walks on 6th pitch) = P(2 of the first 5 are strikes $\cap$ #6 is a ball) =

P(2 of the first 5 are strikes)P(#6 is a ball) = $\binom{5}{2}(.5)^2(.5)^3(.5)$ = .15625.

c. Following the pattern from **b**, P(walks on 5th pitch) = $\binom{4}{1}(.5)^1(.5)^3(.5)$ = .125. Therefore, P(batter walks) = P(walks on 4th) + P(walks on 5th) + P(walks on 6th) =
.0625 + .125 + .15625 = .34375.

d. P(first batter scores while no one is out) = P(first four batters all walk) = $(.34375)^4$ = .014.

109.

a. P(all in correct room) = $\frac{1}{4!} = \frac{1}{24}$ = .0417.

b. The 9 outcomes which yield completely incorrect assignments are: 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4321, and 4312, so P(all incorrect) = $\frac{9}{24}$ = .375.

110.

a. P(all full) = $P(A \cap B \cap C)$ = (.9)(.7)(.8) = .504.
P(at least one isn't full) = 1 – P(all full) = 1 – .504 = .496.

b. P(only NY is full) = $P(A \cap B' \cap C') = P(A)P(B')P(C')$ = (.9)(1–.7)(1–.8) = .054.
Similarly, P(only Atlanta is full) = .014 and P(only LA is full) = .024.
So, P(exactly one full) = .054 + .014 + .024 = .092.

111. Note: $s = 0$ means that the very first candidate interviewed is hired. Each entry below is the candidate hired for the given policy and outcome.

Outcome	$s = 0$	$s = 1$	$s = 2$	$s = 3$	**Outcome**	$s = 0$	$s = 1$	$s = 2$	$s = 3$
1234	1	4	4	4	3124	3	1	4	4
1243	1	3	3	3	3142	3	1	4	2
1324	1	4	4	4	3214	3	2	1	4
1342	1	2	2	2	3241	3	2	1	1
1423	1	3	3	3	3412	3	1	1	2
1432	1	2	2	2	3421	3	2	2	1
2134	2	1	4	4	4123	4	1	3	3
2143	2	1	3	3	4132	4	1	2	2
2314	2	1	1	4	4213	4	2	1	3
2341	2	1	1	1	4231	4	2	1	1
2413	2	1	1	3	4312	4	3	1	2
2431	2	1	1	1	4321	4	3	2	1

From the table, we derive the following probability distribution based on s:

s	0	1	2	3
P(hire #1)	$\frac{6}{24}$	$\frac{11}{24}$	$\frac{10}{24}$	$\frac{6}{24}$

Therefore $s = 1$ is the best policy.

112. P(at least one occurs) = $1 - P$(none occur) = $1 - (1 - p_1)(1 - p_2)(1 - p_3)(1 - p_4)$.
P(at least two occur) = $1 - P$(none or exactly one occur) =
$1 - [(1 - p_1)(1 - p_2)(1 - p_3)(1 - p_4) + p_1(1 - p_2)(1 - p_3)(1 - p_4) + (1 - p_1)\, p_2(1 - p_3)(1 - p_4) +$
$(1 - p_1)(1 - p_2)p_3(1 - p_4) + (1 - p_1)(1 - p_2)(1 - p_3)p_4]$.

113. $P(A_1) = P$(draw slip 1 or 4) = ½; $P(A_2) = P$(draw slip 2 or 4) = ½;
$P(A_3) = P$(draw slip 3 or 4) = ½; $P(A_1 \cap A_2) = P$(draw slip 4) = ¼;
$P(A_2 \cap A_3) = P$(draw slip 4) = ¼; $P(A_1 \cap A_3) = P$(draw slip 4) = ¼.
Hence $P(A_1 \cap A_2) = P(A_1)P(A_2)$ = ¼; $P(A_2 \cap A_3) = P(A_2)P(A_3)$ = ¼; and
$P(A_1 \cap A_3) = P(A_1)P(A_3)$ = ¼. Thus, there exists pairwise independence. However,
$P(A_1 \cap A_2 \cap A_3) = P$(draw slip 4) = ¼ ≠ ⅛ = $P(A_1)P(A_2)P(A_3)$, so the events are not mutually independent.

114. $P(A_1 | A_2 \cap A_3) = \dfrac{P(A_1 \cap A_2 \cap A_3)}{P(A_2 \cap A_3)} = \dfrac{P(A_1)P(A_2)P(A_3)}{P(A_2)P(A_3)} = P(A_1)$.

CHAPTER 3

Section 3.1

1.

$\mathcal{S}$:	*FFF*	*SFF*	*FSF*	*FFS*	*FSS*	*SFS*	*SSF*	*SSS*
X:	0	1	1	1	2	2	2	3

2. X = 1 if a randomly selected book is non-fiction and X = 0 otherwise;
X = 1 if a randomly selected executive is a female and X = 0 otherwise;
X = 1 if a randomly selected driver has automobile insurance and X = 0 otherwise.

3. Examples include: M = the difference between the large and the smaller outcome with possible values 0, 1, 2, 3, 4, or 5; T = 1 if the sum of the two resulting numbers is even and T = 0 otherwise, a Bernoulli random variable. See the back of the book for other examples.

4. Since a 4-digit code can have between zero and four 0s, the possible values of X are 0, 1, 2, 3, 4. As examples, the PIN 9876 yields X = 0 (no 0s), 1006 corresponds to X = 2 (two 0s), and the very poor PIN choice of 0000 implies that X = 4.

5. No. In the experiment in which a coin is tossed repeatedly until a H results, let Y = 1 if the experiment terminates with at most 5 tosses and Y = 0 otherwise. The sample space is infinite, yet Y has only two possible values. See the back of the book for another example.

6. The possible X values are1, 2, 3, 4, … (all positive integers). Some examples are:

Outcome:	*RL*	*AL*	*RAARL*	*RRRRL*	*AARRL*
X:	2	2	5	5	5

7.

- **a.** Possible values of X are 0, 1, 2, …, 12; discrete.
- **b.** With n = # on the list, values of Y are 0, 1, 2, … , N; discrete.
- **c.** Possible values of U are 1, 2, 3, 4, … ; discrete.
- **d.** Possible values of X are $(0, \infty)$ if we assume that a rattlesnake can be arbitrarily short or long; not discrete.
- **e.** Possible values of Z are all possible sales tax percentages for online purchases, but there are only finitely-many of these. Since we could list these different percentages $\{z_1, z_2, \ldots, z_N\}$, Z is discrete.
- **f.** Since 0 is the smallest possible pH and 14 is the largest possible pH, possible values of Y are [0, 14]; not discrete.
- **g.** With m and M denoting the minimum and maximum possible tension, respectively, possible values of X are $[m, M]$; not discrete.
- **h.** The number of possible tries is 1, 2, 3, …; each try involves 3 racket spins, so possible values of X are 3, 6, 9, 12, 15, …; discrete.

8. The least possible value of Y is 3; all possible values of Y are 3, 4, 5, 6, ….
$Y = 3$: *SSS*; $Y = 4$: *FSSS*; $Y = 5$: *FFSSS*, *SFSSS*;
$Y = 6$: *SSFSSS*, *SFFSSS*, *FSFSSS*, *FFFSSS*;
$Y = 7$: *SSFFSSS, SFSFSSS, SFFFSSS, FSSFSSS, FSFFSSS, FFSFSSS, FFFFSSS*

9.

- **a.** Returns to 0 can occur only after an even number of tosses, so possible X values are 2, 4, 6, 8, …. Because the values of X are enumerable, X is discrete.
- **b.** Now a return to 0 is possible after any number of tosses greater than 1, so possible values are 2, 3, 4, 5, …. Again, X is discrete.

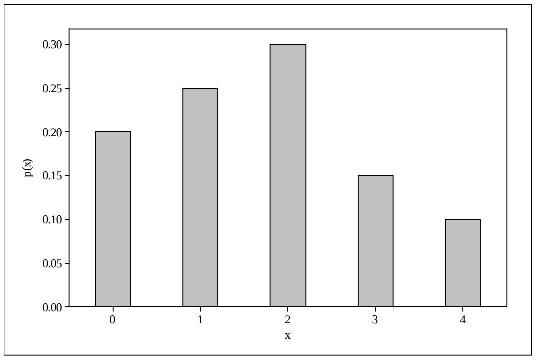
10.

- **a.** Possible values of T are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
- **b.** Possible values of X are: –4, –3, –2, –1, 0, 1, 2, 3, 4, 5, 6.
- **c.** Possible values of U are: 0, 1, 2, 3, 4, 5, 6.
- **d.** Possible values of Z are: 0, 1, 2.

Section 3.2

11.

a.



b. $P(X \geq 2) = p(2) + p(3) + p(4) = .30 + .15 + .10 = .55$, while $P(X > 2) = .15 + .10 = .25$.

c. $P(1 \leq X \leq 3) = p(1) + p(2) + p(3) = .25 + .30 + .15 = .70$.

d. Who knows? (This is just a little joke by the author.)

12.

a. Since there are 50 seats, the flight will accommodate all ticketed passengers who show up as long as there are no more than 50. $P(Y \leq 50) = .05 + .10 + .12 + .14 + .25 + .17 = .83$.

b. This is the complement of part **a**: $P(Y > 50) = 1 - P(Y \leq 50) = 1 - .83 = .17$.

c. If you're the first standby passenger, you need no more than 49 people to show up (so that there's space left for you). $P(Y \leq 49) = .05 + .10 + .12 + .14 + .25 = .66$. On the other hand, if you're third on the standby list, you need no more than 47 people to show up (so that, even with the two standby passengers ahead of you, there's still room). $P(Y \leq 47) = .05 + .10 + .12 = .27$.

13.

a. $P(X \le 3) = p(0) + p(1) + p(2) + p(3) = .10+.15+.20+.25 = .70$.

b. $P(X < 3) = P(X \le 2) = p(0) + p(1) + p(2) = .45$.

c. $P(X \ge 3) = p(3) + p(4) + p(5) + p(6) = .55$.

d. $P(2 \le X \le 5) = p(2) + p(3) + p(4) + p(5) = .71$.

e. The number of lines <u>not</u> in use is $6 – X$, and $P(2 \le 6 – X \le 4) = P(–4 \le –X \le –2) =$
$P(2 \le X \le 4) = p(2) + p(3) + p(4) = .65$.

f. $P(6 – X \ge 4) = P(X \le 2) = .10 + .15 + .20 = .45$.

14.

a. As the hint indicates, the sum of the probabilities must equal 1. Applied here, we get $\sum_{y=1}^{5} p(y) = k[1 + 2 + 3 + 4 + 5] = 15k = 1 \Rightarrow k = \frac{1}{15}$. In other words, the probabilities of the five y-values are $\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{4}{15}, \frac{5}{15}$.

b. $P(Y \le 3) = P(Y = 1, 2, 3) = \frac{1}{15} + \frac{2}{15} + \frac{3}{15} = \frac{6}{15} = .4$.

c. $P(2 \le Y \le 4) = P(Y = 2, 3, 4) = \frac{2}{15} + \frac{3}{15} + \frac{4}{15} = \frac{9}{15} = .6$.

d. Do the probabilities total 1? Let's check: $\sum_{y=1}^{5}\left(\frac{y^2}{50}\right) = \frac{1}{50}[1+4+9+16+25] = \frac{55}{50} \ne 1$. No, that formula cannot be a pmf.

15.

a. (1,2) (1,3) (1,4) (1,5) (2,3) (2,4) (2,5) (3,4) (3,5) (4,5)

b. X can only take on the values 0, 1, 2. $p(0) = P(X = 0) = P(\{(3,4)\ (3,5)\ (4,5)\}) = 3/10 = .3$;
$p(2) = P(X = 2) = P(\{(1,2)\}) = 1/10 = .1$; $p(1) = P(X = 1) = 1 – [p(0) + p(2)] = .60$; and otherwise $p(x) = 0$.

c. $F(0) = P(X \le 0) = P(X = 0) = .30$;
$F(1) = P(X \le 1) = P(X = 0 \text{ or } 1) = .30 + .60 = .90$;
$F(2) = P(X \le 2) = 1$.
Therefore, the complete cdf of X is

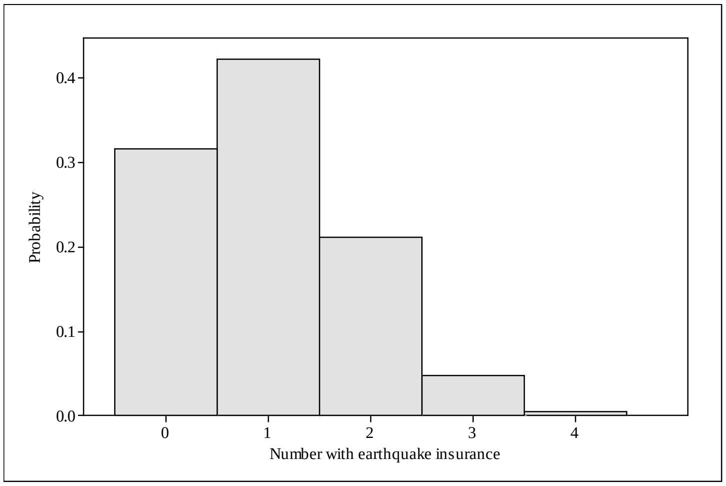
$$F(x) = \begin{cases} 0 & x < 0 \\ .30 & 0 \le x < 1 \\ .90 & 1 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

16.

a.

x	**Outcomes**	$p(x)$
0	*FFFF*	$.3164 = (.75)^4$
1	*FFFS,FFSF,FSFF,SFFF*	$.4219 = 4[(.75)^3(.25)]$
2	*FFSS,FSFS,SFFS,FSSF,SFSF,SSFF*	$.2109 = 6[(.75)^2(.25)^2]$
3	*FSSS, SFSS,SSFS,SSSF*	$.0469 = 4[(.75)(.25)^3]$
4	*SSSS*	$.0039 = (.25)^4$

b.



c. $p(x)$ is largest for $X = 1$.

d. $P(X \geq 2) = p(2) + p(3) + p(4) = .2109 + .0469 + .0039 = .2614$.

17.

a. $p(2) = P(Y = 2) = P(\text{first 2 batteries are acceptable}) = P(AA) = (.9)(.9) = .81$.

b. $p(3) = P(Y = 3) = P(UAA \text{ or } AUA) = (.1)(.9)^2 + (.1)(.9)^2 = 2[(.1)(.9)^2] = .162$.

c. The fifth battery must be an A, and exactly one of the first four must also be an A. Thus, $p(5) = P(AUUUA \text{ or } UAUUA \text{ or } UUAUA \text{ or } UUUAA) = 4[(.1)^3(.9)^2] = .00324$.

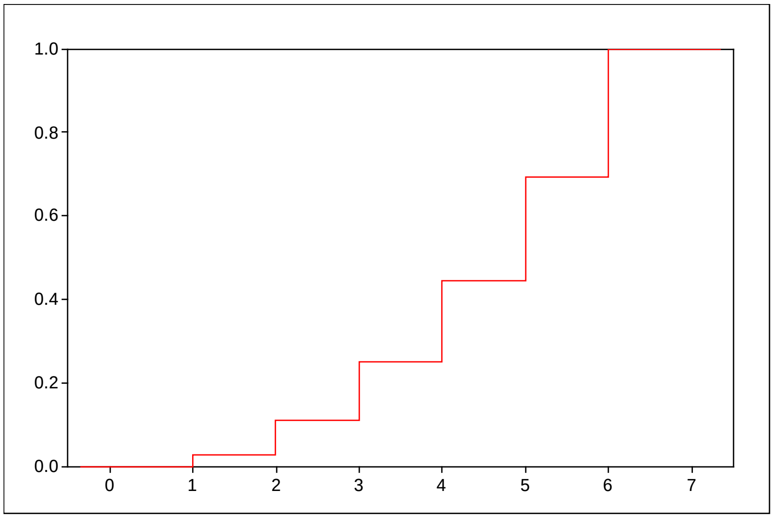
d. $p(y) = P(\text{the } y^{\text{th}} \text{ is an } A \text{ and so is exactly one of the first } y - 1) = (y - 1)(.1)^{y-2}(.9)^2$, for $y = 2, 3, 4, 5, \ldots$.

18.

a. $p(1) = P(M = 1) = P(\{(1,1)\}) = \frac{1}{36}$; $p(2) = P(M = 2) = P(\{(1,2)(2,1)(2,2)\}) = \frac{3}{36}$; $p(3) = P(M = 3) = P(\{(1,3)(2,3)(3,1)(3,2)(3,3)\}) = \frac{5}{36}$. Continuing the pattern, $p(4) = \frac{7}{36}$, $p(5) = \frac{9}{36}$, and $p(6) = \frac{11}{36}$.

b. Using the values in **a**,

$$F(m) = \begin{cases} 0 & m < 1 \\ \frac{1}{36} & 1 \le m < 2 \\ \frac{4}{36} & 2 \le m < 3 \\ \frac{9}{36} & 3 \le m < 4 \\ \frac{16}{36} & 4 \le m < 5 \\ \frac{25}{36} & 5 \le m < 6 \\ 1 & m \ge 6 \end{cases}$$



19. $p(0) = P(Y = 0) = P(\text{both arrive on Wed}) = (.3)(.3) = .09$;
$p(1) = P(Y = 1) = P((\text{W,Th}) \text{ or } (\text{Th,W}) \text{ or } (\text{Th,Th})) = (.3)(.4) + (.4)(.3) + (.4)(.4) = .40$;
$p(2) = P(Y = 2) = P((\text{W,F}) \text{ or } (\text{Th,F}) \text{ or } (\text{F,W}) \text{ or } (\text{F,Th}) \text{ or } (\text{F,F})) = .32$;
$p(3) = 1 – [.09 + .40 + .32] = .19$.

20.

a. $P(X = 0) = P(\text{none are late}) = (.6)^5 = .07776$; $P(X = 1) = P(\text{one single is late}) = 2(.4)(.6)^4 = .10368$. $P(X = 2) = P(\text{both singles are late or one couple is late}) = (.4)^2(.6)^3 + 3(.4)(.6)^4 = .19008$. $P(X = 3) = P(\text{one single and one couple is late}) = 2(.4)3(.4)(.6)^3 = .20736$. Continuing in this manner, $P(X = 4) = .17280$, $P(X = 5) = .13824$, $P(X = 6) = .06912$, $P(X = 7) = .03072$, and $P(X = 8) = (.4)^5 = .01024$.

b. The jumps in $F(x)$ occur at 0, … , 8. We only display the cumulative probabilities here and not the entire cdf: $F(0) = .07776$, $F(1) = .18144$, $F(2) = .37152$, $F(3) = .57888$, $F(4) = .75168$, $F(5) = .88992$, $F(6) = .95904$, $F(7) = .98976$, $F(8) = 1$.

And so, $P(2 \le X \le 6) = F(6) – F(2 – 1) = F(6) – F(1) = .95904 – .18144 = .77760$.

21.

a. First, $1 + 1/x > 1$ for all $x = 1, …, 9$, so $\log(1 + 1/x) > 0$. Next, check that the probabilities sum to 1: $\sum_{x=1}^{9}\log_{10}(1+1/x) = \sum_{x=1}^{9}\log_{10}\left(\frac{x+1}{x}\right) = \log_{10}\left(\frac{2}{1}\right)+\log_{10}\left(\frac{3}{2}\right)+\cdots+\log_{10}\left(\frac{10}{9}\right)$; using properties of logs, this equals $\log_{10}\left(\frac{2}{1}\times\frac{3}{2}\times\cdots\times\frac{10}{9}\right) = \log_{10}(10) = 1$.

b. Using the formula $p(x) = \log_{10}(1 + 1/x)$ gives the following values: $p(1) = .301$, $p(2) = .176$, $p(3) = .125$, $p(4) = .097$, $p(5) = .079$, $p(6) = .067$, $p(7) = .058$, $p(8) = .051$, $p(9) = .046$. The distribution specified by *B*enford’s Law is <u>not</u> uniform on these nine digits; rather, lower digits (such as 1 and 2) are much more likely to be the lead digit of a number than higher digits (such as 8 and 9).

c. The jumps in $F(x)$ occur at 0, … , 8. We display the cumulative probabilities here: $F(1) = .301$, $F(2) = .477$, $F(3) = .602$, $F(4) = .699$, $F(5) = .778$, $F(6) = .845$, $F(7) = .903$, $F(8) = .954$, $F(9) = 1$. So, $F(x) = 0$ for $x < 1$; $F(x) = .301$ for $1 \le x < 2$;
$F(x) = .477$ for $2 \le x < 3$; etc.

d. $P(X \le 3) = F(3) = .602$; $P(X \ge 5) = 1 – P(X < 5) = 1 – P(X \le 4) = 1 – F(4) = 1 – .699 = .301$.

22. The jumps in $F(x)$ occur at x = 0, 1, 2, 3, 4, 5, and 6, so we first calculate $F(\ )$ at each of these values: $F(0) = P(X \le 0) = P(X = 0) = p(0) = .10$, $F(1) = P(X \le 1) = p(0) + p(1) = .25$,
$F(2) = P(X \le 2) = p(0) + p(1) + p(2) = .45$, $F(3) = .70$, $F(4) = .90$, $F(5) = .96$, and $F(6) = 1$.
The complete cdf of X is

$$F(x) = \begin{cases} .00 & x < 0 \\ .10 & 0 \le x < 1 \\ .25 & 1 \le x < 2 \\ .45 & 2 \le x < 3 \\ .70 & 3 \le x < 4 \\ .90 & 4 \le x < 5 \\ .96 & 5 \le x < 6 \\ 1.00 & 6 \le x \end{cases}$$

Then **a.** $P(X \le 3) = F(3) = .70$, **b.** $P(X < 3) = P(X \le 2) = F(2) = .45$, **c.** $P(X \ge 3) = 1 – P(X \le 2) = 1 – F(2) = 1 – .45 = .55$, **d.** $P(2 \le X \le 5) = F(5) – F(1) = .96 – .25 = .71$.

23.

a. $p(2) = P(X = 2) = F(3) – F(2) = .39 – .19 = .20$.

b. $P(X > 3) = 1 – P(X \leq 3) = 1 – F(3) = 1 – .67 = .33$.

c. $P(2 \leq X \leq 5) = F(5) – F(2–1) = F(5) – F(1) = .92 – .19 = .78$.

d. $P(2 < X < 5) = P(2 < X \leq 4) = F(4) – F(2) = .92 – .39 = .53$.

24.

a. Possible X values are those values at which $F(x)$ jumps, and the probability of any particular value is the size of the jump at that value. Thus we have:

x	1	3	4	6	12
$p(x)$	.30	.10	.05	.15	.40

b. $P(3 \leq X \leq 6) = F(6) – F(3–) = .60 – .30 = .30$; $P(4 \leq X) = 1 – P(X < 4) = 1 – F(4–) = 1 – .40 = .60$.

25. $p(0) = P(Y = 0) = P(B \text{ first}) = p$;
$p(1) = P(Y = 1) = P(G \text{ first, then } B) = (1 – p)p$;
$p(2) = P(Y = 2) = P(GGB) = (1 – p)^2p$;
Continuing, $p(y) = P(y\ G\text{s and then a } B) = (1 – p)^yp$ for $y = 0,1,2,3,…$.

26.

a. Possible X values are 1, 2, 3, …
$p(1) = P(X = 1) = P(\text{return home after just one visit}) = \frac{1}{3}$;
$p(2) = P(X = 2) = P(\text{visit a second friend, and then return home}) = \frac{2}{3}\cdot\frac{1}{3}$;
$p(3) = P(X = 3) = P(\text{three friend visits, and then return home}) = \left(\frac{2}{3}\right)^2\cdot\frac{1}{3}$;
and in general, $p(x) = \left(\frac{2}{3}\right)^{x-1}\cdot\frac{1}{3}$ for $x = 1, 2, 3, …$

b. The number of straight line segments is $Y = 1 + X$ (since the last segment traversed returns Alvie to 0). Borrow the answer from **a**, and $p(y) = \left(\frac{2}{3}\right)^{y-2}\cdot\frac{1}{3}$ for $y = 2, 3, …$.

c. Possible Z values are 0, 1, 2, 3 , …. In what follows, notice that Alvie can't visit two female friends in a row or two male friends in a row.
$p(0) = P(\text{male first and then home}) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$;
$p(1) = P(\text{exactly one visit to a female}) = P(F \text{ first, then } 0) + P(F, M, 0) + P(M, F, 0) + P(M, F, M, 0)$ $= \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{2}{3}\frac{1}{3} + \frac{1}{2}\frac{2}{3}\frac{1}{3} + \frac{1}{2}\frac{2}{3}\frac{2}{3}\frac{1}{3} = \frac{25}{54}$; for the event $Z = 2$, two additional visits occur, and the probability of those is $\frac{2}{3}\frac{2}{3} = \frac{4}{9}$, so $p(2) = \frac{4}{9}p(1) = \frac{4}{9}\cdot\frac{25}{54}$; similarly, $p(3) = \frac{4}{9}p(2) = \left(\frac{4}{9}\right)^2\cdot\frac{25}{54}$; and so on. Therefore, $p(0) = \frac{1}{6}$ and $p(z) = \left(\frac{4}{9}\right)^{z-1}\cdot\frac{25}{54}$ for $z = 1, 2, 3, …$.

27.

a. The sample space consists of all possible permutations of the four numbers 1, 2, 3, 4:

outcome	***x* value**	**outcome**	***x* value**	**outcome**	***x* value**
1234	4	2314	1	3412	0
1243	2	2341	0	3421	0
1324	2	2413	0	4132	1
1342	1	2431	1	4123	0
1423	1	3124	1	4213	1
1432	2	3142	0	4231	2
2134	2	3214	2	4312	0
2143	0	3241	1	4321	0

b. From the table in **a**, $p(0) = P(X = 0) = \frac{9}{24}$, $p(1) = P(X = 1) = \frac{8}{24}$, $p(2) = P(Y = 2) = \frac{6}{24}$, $p(3) = P(X = 3) = 0$, and $p(4) = P(Y = 4) = \frac{1}{24}$.

28. If $x_1 < x_2$, then $F(x_2) = P(X \le x_2) = P(\{X \le x_1\} \cup \{x_1 < X \le x_2\}) = P(X \le x_1) + P(x_1 < X \le x_2)$. Since all probabilities are non-negative, this is $\ge P(X \le x_1) + 0 = P(X \le x_1) = F(x_1)$. That is, $x_1 < x_2$ implies $F(x_1) \le F(x_2)$, QED.

Looking at the proof above, $F(x_1) = F(x_2)$ iff $P(x_1 < X \le x_2) = 0$.

Section 3.3

29.

a. $E(X) = \sum_{\text{all } x} xp(x) = 1(.05) + 2(.10) + 4(.35) + 8(.40) + 16(.10) = 6.45$ GB.

b. $V(X) = \sum_{\text{all } x} (x - \mu)^2\, p(x) = (1 - 6.45)^2(.05) + (2 - 6.45)^2(.10) + \ldots + (16 - 6.45)^2(.10) = 15.6475$.

c. $\sigma = \sqrt{V(X)} = \sqrt{15.6475} = 3.956$ GB.

d. $E(X^2) = \sum_{\text{all } x} x^2 p(x) = 1^2(.05) + 2^2(.10) + 4^2(.35) + 8^2(.40) + 16^2(.10) = 57.25$. Using the shortcut formula, $V(X) = E(X^2) - \mu^2 = 57.25 - (6.45)^2 = 15.6475$.

30.

a. $E(Y) = \sum_{y=0}^{3} y \cdot p(y)$ = 0(.60) + 1(.25) + 2(.10) + 3(.05) = .60.

b. $E(100Y^2) = \sum_{y=0}^{3} 100y^2 \cdot p(y)$ = 0(.60) + 100(.25) + 400(.10) + 900(.05) = $110.

31. From the table in Exercise 12, $E(Y)$ = 45(.05) + 46(.10) + … + 55(.01) = 48.84; similarly, $E(Y^2) = 45^2(.05) + 46^2(.10) + \ldots + 55^2(.01)$ = 2389.84; thus $V(Y) = E(Y^2) - [E(Y)]^2 = 2389.84 - (48.84)^2 = 4.4944$ and $\sigma_Y = \sqrt{4.4944} = 2.12$.
One standard deviation from the mean value of Y gives 48.84 ± 2.12 = 46.72 to 50.96. So, the probability Y is within one standard deviation of its mean value equals $P(46.72 < Y < 50.96) = P(Y = 47, 48, 49, 50)$ = .12 + .14 + .25 + .17 = .68.

32.

a. $E(X) = (16)(.2) + (18)(.5) + (20)(.3) = 18.2$ ft^3; $E(X^2) = (16)^2(.2) + (18)^2(.5) + (20)^2(.3) = 333.2$. Put these together, and $V(X) = E(X^2) - [E(X)]^2 = 333.2 - (18.2)^2 = 1.96$.

b. Use the linearity/rescaling property: $E(70X - 650) = 70\mu - 650 = 70(18.2) - 650$ = $624. Alternatively, you can figure out the price for each of the three freezer types and take the weighted average.

c. Use the linearity/rescaling property again: $V(70X - 650) = 70^2\sigma^2 = 70^2(1.96) = 9604$. (The 70 gets squared because variance is itself a square quantity.)

d. We cannot use the rescaling properties for $E(X - .008X^2)$, since this isn't a linear function of X. However, since we've already found both $E(X)$ and $E(X^2)$, we may as well use them: the expected actual capacity of a freezer is $E(X - .008X^2) = E(X) - .008E(X^2) = 18.2 - .008(333.2) = 15.5344$ ft^3. Alternatively, you can figure out the actual capacity for each of the three freezer types and take the weighted average.

33.

a. $E(X^2) = \sum_{x=0}^{1} x^2 \cdot p(x) = 0^2(1 - p) + 1^2(p) = p$.

b. $V(X) = E(X^2) - [E(X)]^2 = p - [p]^2 = p(1 - p)$.

c. $E(X^{79}) = 0^{79}(1 - p) + 1^{79}(p) = p$. In fact, $E(X^n) = p$ for any non-negative power n.

34. Yes, the expectation is finite. $E(X) = \sum_{x=1}^{\infty} x \cdot p(x) = \sum_{x=1}^{\infty} x \cdot \frac{c}{x^3} = c\sum_{x=1}^{\infty} \frac{1}{x^2}$; it is a well-known result from the theory of infinite series that $\sum_{x=1}^{\infty} \frac{1}{x^2} < \infty$, so $E(X)$ is finite.

35. Let $h_3(X)$ and $h_4(X)$ equal the net revenue (sales revenue minus order cost) for 3 and 4 copies purchased, respectively. If 3 magazines are ordered ($6 spent), net revenue is \$4 – \$6 = –\$2 if $X = 1$, 2(\$4) – \$6 = \$2 if $X = 2$, 3(\$4) – \$6 = \$6 if $X = 3$, and also \$6 if $X = 4$, 5, or 6 (since that additional demand simply isn't met. The values of $h_4(X)$ can be deduced similarly. *B*oth distributions are summarized below.

x	1	2	3	4	5	6
$h_3(x)$	–2	2	6	6	6	6
$h_4(x)$	–4	0	4	8	8	8
$p(x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{2}{15}$

Using the table, $E[h_3(X)] = \sum_{x=1}^{6} h_3(x)\cdot p(x) = (-2)(\frac{1}{15}) + \ldots + (6)(\frac{2}{15}) = \$4.93$.

Similarly, $E[h_4(X)] = \sum_{x=1}^{6} h_4(x)\cdot p(x) = (-4)(\frac{1}{15}) + \ldots + (8)(\frac{2}{15}) = \$5.33$.

Therefore, ordering 4 copies gives slightly higher revenue, on the average.

36. You have to be careful here: if \$0 damage is incurred, then there's no deductible for the insured driver to pay! Here's one approach: let $h(X)$ = the amount paid by the insurance company on an accident claim, which is \$0 for a "no damage" event and \$500 less than actual damages (X – 500) otherwise. The pmf of $h(X)$ looks like this:

x	0	1000	5000	10000
$h(x)$	0	500	4500	9500
$p(x)$	.8	.1	.08	.02

Based on the pmf, the average payout across these types of accidents is $E(h(X))$ = 0(.8) + 500(.1) + 4500(.08) + 9500(.02) = \$600. If the insurance company charged \$600 per client, they'd break even (a bad idea!). To have an expected profit of \$100 — that is, to have a *mean* profit of \$100 per client — they should charge \$600 + \$100 = \$700.

37. Using the hint, $E(X) = \sum_{x=1}^{n} x\cdot\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{x=1}^{n} x = \frac{1}{n}\left[\frac{n(n+1)}{2}\right] = \frac{n+1}{2}$. Similarly,

$$E(X^2) = \sum_{x=1}^{n} x^2\cdot\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{x=1}^{n} x^2 = \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{(n+1)(2n+1)}{6}, \text{ so}$$

$$V(X) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12}.$$

38.

a. $E(X)$ = 1(.15) + 2(.35) + 3(.35) + 4(.15) = 2.5. By linearity, $E(5 - X) = 5 - E(X)$ = 5 – 2.5 = 2.5 as well.

b. Since $150/(5 - X)$ is not a linear function of X, we cannot use the results from **a**. Instead, we must create a new weighted average:
$E[150/(5 - X)]$ = [150/(5 – 1)](.15) + [150/(5 – 2)](.35) + [150/(5 – 3)](.35) + [150/(5 – 4)](.15) = 71.875. Since \$71.875 < \$75, they're better off in the long run charging a flat fee of \$75.

39. From the table, $E(X) = \sum xp(x) = 2.3$, $E(X^2) = 6.1$, and $V(X) = 6.1 - (2.3)^2 = .81$. Each lot weighs 5 lbs, so the number of pounds left = $100 - 5X$. Thus the expected weight left is $E(100 - 5X) = 100 - 5E(X) =$ 88.5 lbs, and the variance of the weight left is $V(100 - 5X) = V(-5X) = (-5)^2V(X) = 25V(X) = 20.25$.

40.

a. The line graph of the pmf of $-X$ is just the line graph of the pmf of X reflected about zero, so both have the same degree of spread about their respective means, suggesting $V(-X) = V(X)$.

b. With $a = -1$ and $b = 0$, $V(-X) = V(aX + b) = a^2V(X) = (-1)^2V(X) = V(X)$.

41. Use the hint: $V(aX + b) = E[((aX+b)-E(aX+b))^2] = \sum[ax+b-E(aX+b)]^2\,p(x) =$

$$\sum[ax+b-(a\mu+b)]^2\,p(x) = \sum[ax-a\mu]^2\,p(x) = a^2\sum(x-\mu)^2\,p(x) = a^2V(X).$$

42.

a. $E[X(X-1)] = E(X^2) - E(X) \Rightarrow E(X^2) = E[X(X-1)] + E(X) = 27.5 + 5 = 32.5$.

b. $V(X) = E(X^2) - [E(X)]^2 = 32.5 - (5)^2 = 7.5$.

c. Substituting **a** into **b**, $V(X) = E[X(X-1)] + E(X) - [E(X)]^2$.

43. With $a = 1$ and $b = -c$, $E(X - c) = E(aX + b) = a\,E(X) + b = E(X) - c$.
When $c = \mu$, $E(X - \mu) = E(X) - \mu = \mu - \mu = 0$; i.e., the expected deviation from the mean is zero.

44.

a. See the table below.

k	2	3	4	5	10
$1/k^2$	.25	.11	.06	.04	.01

b. From the table in Exercise 13, $\mu = 2.64$ and $\sigma^2 = 2.3704 \Rightarrow \sigma = 1.54$.
For $k = 2$, $P(|X - \mu| \geq 2\sigma) = P(|X - 2.64| \geq 2(1.54)) = P(X \geq 2.64 + 2(1.54)$ or $X \leq 2.64 - 2(1.54)) = P(X \geq 5.72$ or $X \leq -.44) = P(X = 6) = .04$. Chebyshev's bound of .25 is much too conservative.

For $k = 3, 4, 5$, or 10, $P(|X - \mu| \geq k\sigma)$ turns out to be zero, whereas Chebyshev's bound is positive. Again, this points to the conservative nature of the bound $1/k^2$.

c. $\mu = 0$ and $\sigma = 1/3$, so $P(|X - \mu| \geq 3\sigma) = P(|X| \geq 1) = P(X = -1 \text{ or } +1) = \frac{1}{18} + \frac{1}{18} = \frac{1}{9}$, identical to Chebyshev's upper bound of $1/k^2 = 1/3^2 = 1/9$.

d. There are many. For example, let $p(-1) = p(1) = \frac{1}{50}$ and $p(0) = \frac{24}{25}$.

45. $a \le X \le b$ means that $a \le x \le b$ for all x in the range of X. Hence $ap(x) \le xp(x) \le bp(x)$ for all x, and

$$\begin{aligned}\sum ap(x) \le \sum xp(x) \le \sum bp(x)\\ a\sum p(x) \le \sum xp(x) \le b\sum p(x)\\ a\cdot 1 \le E(X) \le b\cdot 1\\ a \le E(X) \le b\end{aligned}$$

Section 3.4

46.

a. $b(3;8,.35) = \binom{8}{3}(.35)^3(.65)^5 = .279$.

b. $b(5;8,.6) = \binom{8}{5}(.6)^5(.4)^3 = .279$.

c. $P(3 \le X \le 5) = b(3;7,.6) + b(4;7,.6) + \mathrm{b}(5;7,.6) = .745$.

d. $P(1 \le X) = 1 - P(X = 0) = 1 - \binom{9}{0}(.1)^0(.9)^9 = 1 - (.9)^9 = .613$.

47.

a. $B(4;15,.7) = .001$.

b. $b(4;15,.7) = B(4;15,.7) - B(3;15,.7) = .001 - .000 = .001$.

c. Now $p = .3$ (multiple vehicles). $b(6;15,.3) = B(6;15,.3) - B(5;15,.3) = .869 - .722 = .147$.

d. $P(2 \le X \le 4) = B(4;15,.7) - B(1;15,.7) = .001$.

e. $P(2 \le X) = 1 - P(X \le 1) = 1 - B(1;15,.7) = 1 - .000 = 1$.

f. The information that 11 accidents involved multiple vehicles is redundant (since $n = 15$ and $x = 4$). So, this is actually identical to **b**, and the answer is .001.

48. $X \sim \text{Bin}(25, .05)$

a. $P(X \le 3) = B(3;25,.05) = .966$, while $P(X < 3) = P(X \le 2) = B(2;25,.05) = .873$.

b. $P(X \ge 4) = 1 - P(X \le 3) = 1 - B(3;25,.05) = .1 - .966 = .034$.

c. $P(1 \le X \le 3) = P(X \le 3) - P(X \le 0) = .966 - .277 = .689$.

d. $E(X) = np = (25)(.05) = 1.25$, $\sigma_X = \sqrt{np(1-p)} = \sqrt{25(.05)(.95)} = 1.09$.

e. With $n = 50$, $P(X = 0) = \binom{50}{0}(.05)^0(.95)^{50} = (.95)^{50} = .077$.

49. Let X be the number of "seconds," so $X \sim \text{Bin}(6, .10)$.

a. $P(X = 1) = \binom{n}{x}p^x(1-p)^{n-x} = \binom{6}{1}(.1)^1(.9)^5 = .3543$.

b. $P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)] = 1 - \left[\binom{6}{0}(.1)^0(.9)^6 + \binom{6}{1}(.1)^1(.9)^5\right] = 1 - [.5314 + .3543] =$ .1143.

c. Either 4 or 5 goblets must be selected.

Select 4 goblets with zero defects: $P(X = 0) = \binom{4}{0}(.1)^0(.9)^4 = .6561$.

Select 4 goblets, one of which has a defect, and the 5th is good: $\left[\binom{4}{1}(.1)^1(.9)^3\right] \times .9 = .26244$

So, the desired probability is .6561 + .26244 = .91854.

50. Let X be the number of faxes, so $X \sim \text{Bin}(25, .25)$.

a. $P(X \leq 6) = B(6;25,.25) = .561$.

b. $P(X = 6) = b(6;25,.25) = .183$.

c. $P(X \geq 6) = 1 - P(X \leq 5) = 1 - B(5;25,.25) = .622$.

d. $P(X > 6) = 1 - P(X \leq 6) = 1 - .561 = .439$.

51. Let X be the number of faxes, so $X \sim \text{Bin}(25, .25)$.

a. $E(X) = np = 25(.25) = 6.25$.

b. $V(X) = np(1–p) = 25(.25)(.75) = 4.6875$, so $SD(X) = 2.165$.

c. $P(X > 6.25 + 2(2.165)) = P(X > 10.58) = 1 - P(X \leq 10.58) = 1 - P(X \leq 10) = 1 - B(10;25,.25) = .030$.

52. Let X be the number of students who want a new copy, so $X \sim \text{Bin}(n = 25, p = .3)$.

a. $E(X) = np = 25(.3) = 7.5$ and $SD(X) = \sqrt{np(1-p)} = \sqrt{25(.3)(.7)} = 2.29$.

b. Two standard deviations from the mean converts to 7.5 ± 2(2.29) = 2.92 & 12.08. For X to be more than two standard deviations from the means requires $X < 2.92$ or $X > 12.08$. Since X must be a non-negative integer, $P(X < 2.92 \text{ or } X > 12.08) = 1 - P(2.92 \leq X \leq 12.08) = 1 - P(3 \leq X \leq 12) =$

$1 - \sum_{x=3}^{12}\binom{25}{x}(.3)^x(.7)^{25-x} = 1 - .9736 = .0264$.

c. If $X > 15$, then more people want new copies than the bookstore carries. At the other end, though, there are $25 – X$ students wanting used copies; if $25 – X > 15$, then there aren't enough used copies to meet demand.

The inequality $25 – X > 15$ is the same as $X < 10$, so the bookstore can't meet demand if either $X > 15$ or $X < 10$. All 25 students get the type they want iff $10 \le X \le 15$:
$P(10 \le X \le 15) = \sum_{x=10}^{15} \binom{25}{x} (.3)^x (.7)^{25-x} = .1890$.

d. The bookstore sells X new books and $25 – X$ used books, so total revenue from these 25 sales is given by $h(X) = 100(X) + 70(25 – X) = 30X + 1750$. Using linearity/rescaling properties, expected revenue equals $E(h(X)) = E(30X + 1750) = 30\mu + 1750 = 30(7.5) + 1750 = \$1975$.

53. Let "success" = has at least one citation and define X = number of individuals with at least one citation. Then $X \sim \text{Bin}(n = 15, p = .4)$.

a. If at least 10 have no citations (failure), then at most 5 have had at least one (success):
$P(X \le 5) = B(5;15,.40) = .403$.

b. Half of 15 is 7.5, so less than half means 7 or fewer: $P(X \le 7) = B(7;15,.40) = .787$.

c. $P(5 \le X \le 10) = P(X \le 10) – P(X \le 4) = .991 – .217 = .774$.

54. Let X equal the number of customers who choose an oversize racket, so $X \sim \text{Bin}(10, .60)$.

a. $P(X \ge 6) = 1 – P(X \le 5) = 1 – B(5;20,.60) = 1 – .367 = .633$.

b. $\mu = np = 10(.6) = 6$ and $\sigma = \sqrt{10(.6)(.4)} = 1.55$, so $\mu \pm \sigma = (4.45, 7.55)$.
$P(4.45 < X < 7.55) = P(5 \le X \le 7) = P(X \le 7) – P(X \le 4) = .833 – .166 = .667$.

c. This occurs iff between 3 and 7 customers want the oversize racket (otherwise, one type will run out early). $P(3 \le X \le 7) = P(X \le 7) – P(X \le 2) = .833 – .012 = .821$.

55. Let "success" correspond to a telephone that is submitted for service while under warranty and must be replaced. Then $p = P(\text{success}) = P(\text{replaced} \mid \text{submitted})\cdot P(\text{submitted}) = (.40)(.20) = .08$. Thus X, the number among the company's 10 phones that must be replaced, has a binomial distribution with $n = 10$ and $p = .08$, so $P(X = 2) = \binom{10}{2}(.08)^2(.92)^8 = .1478$.

56. Let X = the number of students in the sample needing accommodation, so $X \sim \text{Bin}(25, .02)$.

a. $P(X = 1) = 25(.02)(.98)^{24} = .308$.

b. $P(X \ge 1) = 1 – P(X=0) = 1 – (.98)^{25} = 1 – .603 = .397$.

c. $P(X \ge 2) = 1 – P(X \le 1) = 1 – [.603 + .308] = .089$.

d. $\mu = 25(.02) = .5$ and $\sigma = \sqrt{25(.02)(.98)} = \sqrt{.49} = .7$, so $\mu \pm \sigma = (-.9, 1.9)$. $P(-.9 \le X \le 1.9) = P(X = 0 \text{ or } 1) = .911$.

e. $\dfrac{.5(4.5)+24.5(3)}{25} = 3.03$ hours. Notice the sample size of 25 is actually irrelevant.

57. Let X = the number of flashlights that work, and let event B = {battery has acceptable voltage}.
Then P(flashlight works) = P(both batteries work) = $P(B)P(B) = (.9)(.9) = .81$. We have assumed here that the batteries' voltage levels are independent.
Finally, $X \sim \text{Bin}(10, .81)$, so $P(X \ge 9) = P(X = 9) + P(X = 10) = .285 + .122 = .407$.

58. Let p denote the actual proportion of defectives in the batch, and X denote the number of defectives in the sample.

a. If the actual proportion of defectives is p, then $X \sim \text{Bin}(10, p)$, and the batch is accepted iff $X \le 2$.
Using the binomial formula, $P(X \le 2) = \binom{10}{0}p^0(1-p)^{10} + \binom{10}{1}p^1(1-p)^9 + \binom{10}{2}p^2(1-p)^8 =$
$[(1-p)^2 + 10p(1-p) + 45p^2](1-p)^8$. Values for this expression are tabulated below.

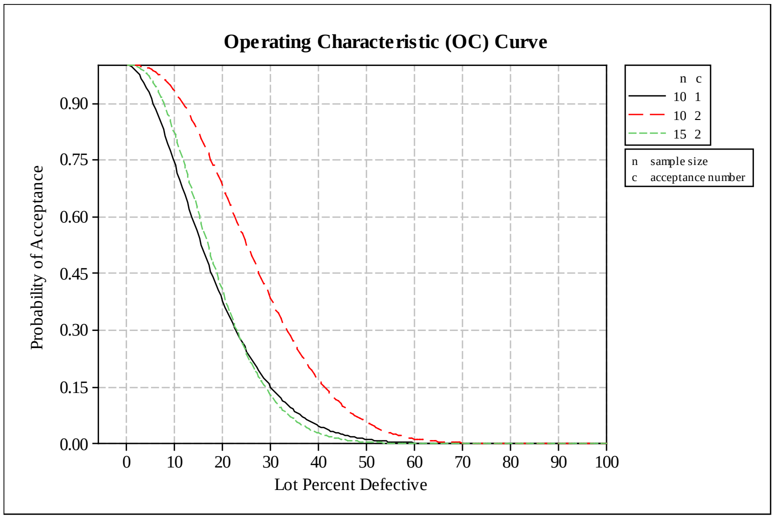
p:	.01	.05	.10	.20	.25
$P(X \le 2)$:	.9999	.9885	.9298	.6778	.5256

b. The polynomial function listed in part **a** is graphed below.

c. Replace "2" with "1," and the shipment is accepted iff $X \le 1$ and the probability of this event is given by $P(X \le 1) = \binom{10}{0}p^0(1-p)^{10} + \binom{10}{1}p^1(1-p)^9 = (1+9p)(1-p)^9$. Values for this new expression are tabulated below.

p:	.01	.05	.10	.20	.25
$P(X \le 1)$:	.9957	.9139	.7361	.3758	.2440

This operating characteristic (OC) curve is also graphed below.



d. Now $n = 15$, and $P(X \le 2) = \binom{15}{0} p^0 (1-p)^{15} + \binom{15}{1} p^1 (1-p)^{14} + \binom{15}{2} p^2 (1-p)^{13}$. Values for this function are tabulated below. The corresponding OC curve is also presented above.

p:	.01	.05	.10	.20	.25
$P(X \le 2)$:	.9996	.9638	.8159	.3980	.2361

e. The exercise says the batch is acceptable iff $p \le 10$, so we want P(accept) to be high when p is less than .10 and low when p is greater than .10. The plan in **d** seems most satisfactory in these respects.

59. In this example, $X \sim \text{Bin}(25, p)$ with p unknown.

a. P(rejecting claim when $p = .8$) = $P(X \le 15$ when $p = .8) = B(15; 25, .8) = .017$.

b. P(not rejecting claim when p = .7) = $P(X > 15$ when $p = .7) = 1 - P(X \le 15$ when $p = .7) =$
$= 1 - B(15; 25, .7) = 1 - .189 = .811$.
For $p = .6$, this probability is $= 1 - B(15; 25, .6) = 1 - .575 = .425$.

c. The probability of rejecting the claim when $p = .8$ becomes $B(14; 25, .8) = .006$, smaller than in **a** above. However, the probabilities of **b** above increase to .902 and .586, respectively. So, by changing 15 to 14, we're making it less likely that we will reject the claim when it's true (p really is $\ge .8$), but more likely that we'll "fail" to reject the claim when it's false (p really is $< .8$).

60. Using the hint, $h(X) = 1 \cdot X + 2.25(25 - X) = 62.5 - 1.5X$, which is a linear function. Since the mean of X is $E(X) = np = (25)(.6) = 15$, $E(h(X)) = 62.5 - 1.5E(X) = 62.5 - 1.5(15) = \$40$.

61. If topic A is chosen, then $n = 2$. When $n = 2$, P(at least half received) = $P(X \geq 1) = 1 - P(X = 0)$ =
$1-\binom{2}{0}(.9)^0(.1)^2$ = .99.
If topic B is chosen, then $n = 4$. When $n = 4$, P(at least half received) = $P(X \geq 2) = 1 - P(X \leq 1)$ =
$1-\left[\binom{4}{0}(.9)^0(.1)^4+\binom{4}{1}(.9)^1(.1)^3\right]$ = .9963.
Thus topic B should be chosen if $p = .9$.
However, if $p = .5$, then the probabilities are .75 for A and .6875 for B (using the same method as above), so now A should be chosen.

62.

a. $np(1 - p) = 0$ if either $p = 0$ (whence every trial is a failure, so there is no variability in X) or if $p = 1$ (whence every trial is a success and again there is no variability in X).

b. $\frac{d}{dp}\left[np(1-p)\right] = n[(1)(1 - p) + p(-1)] = n[1 - 2p] = 0 \Rightarrow p = .5$, which is easily seen to correspond to a maximum value of $V(X)$.

63.

a. $b(x; n, 1 - p) = \binom{n}{x}(1-p)^x(p)^{n-x} = \binom{n}{n-x}(p)^{n-x}(1-p)^x = b(n-x; n, p)$.
Conceptually, P(x S's when $P(\text{S}) = 1 - p$) = P(n–x F's when $P(\text{F}) = p$), since the two events are identical, but the labels S and F are arbitrary and so can be interchanged (if P(S) and P(F) are also interchanged), yielding P(n–x S's when $P(\text{S}) = 1 - p$) as desired.

b. Use the conceptual idea from **a:** $B(x; n, 1 - p)$ = P(<u>at most</u> x S's when $P(\text{S}) = 1 - p$) =
P(<u>at least</u> n–x F's when $P(\text{F}) = p$), since these are the same event
= P(at least n–x S's when $P(\text{S}) = p$), since the S and F labels are arbitrary
= 1 – P(at most n–x–1 S's when $P(\text{S}) = p$) = $1 - B(n-x-1; n, p)$.

c. Whenever $p > .5$, $(1 - p) < .5$ so probabilities involving X can be calculated using the results **a** and **b** in combination with tables giving probabilities only for $p \leq .5$.

64. Proof of $E(X) = np$:

$$\begin{aligned} E(X) &= \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x \cdot \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\ &= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} = np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} \\ &= np \sum_{y=0}^{n-1} \frac{(n-1)!}{(y)!(n-1-y)!} p^y (1-p)^{n-1-y} \quad (y \text{ replaces } x-1) \\ &= np \left[\sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{n-1-y} \right] \end{aligned}$$

The expression in brackets is the sum over all possible values $y = 0, 1, 2, \ldots, n-1$ of a binomial pmf based on $n - 1$ trials. Thus, the sum in brackets equals 1, leaving only np, as desired.

65.

a. Although there are three payment methods, we are only concerned with S = uses a debit card and F = does not use a debit card. Thus we can use the binomial distribution. So, if X = the number of customers who use a debit card, $X \sim \text{Bin}(n = 100, p = .2)$. From this,
$E(X) = np = 100(.2) = 20$, and $V(X) = npq = 100(.2)(1–.2) = 16$.

b. With S = doesn't pay with cash, $n = 100$ and $p = .7$, so $\mu = np = 100(.7) = 70$, and $V = 21$.

66.

a. Let Y = the number with reservations who show up, a binomial rv with $n = 6$ and $p = .8$. Since there are only four spaces available, at least one individual cannot be accommodated if Y is more than 4. The desired probability is $P(Y = 5 \text{ or } 6) = b(5; 6, .8) + b(6;6,.8) = .3932 + .2621 = .6553$.

b. Let $h(Y)$ = the number of available spaces. Then

when y is:	0	1	2	3	4	5	6
$h(y)$ is:	4	3	2	1	0	0	0

The expected number of available spaces when the limousine departs equals

$E[h(Y)] = \sum_{y=0}^{6} h(y) \cdot b(y;6,.8)$ = 4(.000) + 3(.002) = 2(.015) + 1(.082) + 0 + 0 + 0 = 0.118.

c. Possible X values are 0, 1, 2, 3, and 4. $X = 0$ if there are 3 reservations and 0 show or 4 reservations and 0 show or 5 reservations and 0 show or 6 reservations and none show, so
$P(X = 0) = b(0; 3, .8)(.1) + b(0; 4, .8)(.2) + b(0; 5, .8)(.3) + b(0; 6, .8)(.4)$
$= .0080(.1) + .0016(.2) + .0003(.3) + .0001(.4) = .0013$.
Similarly, $P(X = 1) = b(1; 3, .8)(.1) + \ldots + b(1; 6, .8)(.4) = .0172$; $P(X = 2) = .0906$;
$P(X = 3) = .2273$; and $P(X = 4) = .6636$.

These values are displayed below.

x	0	1	2	3	4
$p(x)$	.0013	.0172	.0906	.2273	.6636

67. When $n = 20$ and $p = .5$, $\mu = 10$ and $\sigma = 2.236$, so $2\sigma = 4.472$ and $3\sigma = 6.708$.
The inequality $|X - 10| \geq 4.472$ is satisfied if either $X \leq 5$ or $X \geq 15$, or
$P(|X - \mu| \geq 2\sigma) = P(X \leq 5 \text{ or } X \geq 15) = .021 + .021 = .042$. The inequality $|X - 10| \geq 6.708$ is satisfied if either $X \leq 3$ or $X \geq 17$, so $P(|X - \mu| \geq 3\sigma) = P(X \leq 3 \text{ or } X \geq 17) = .001 + .001 = .002$.

In the case $p = .75$, $\mu = 15$ and $\sigma = 1.937$, so $2\sigma = 3.874$ and $3\sigma = 5.811$.
$P(|X - 15| \geq 3.874) = P(X \leq 11 \text{ or } X \geq 19) = .041 + .024 = .065$, whereas
$P(|X - 15| \geq 5.811) = P(X \leq 9) = .004$.

All these probabilities are considerably less than the upper bounds given by Chebyshev:
for $k = 2$, Chebyshev's bound is $1/2^2 = .25$; for $k = 3$, the bound is $1/3^2 = .11$.

Section 3.5

68.

a. There are 18 items (people) total, 8 of which are "successes" (first-time examinees). Among these 18 items, 6 have been randomly assigned to this particular examiner. So, the random variable X is hypergeometric, with $N = 18$, $M = 8$, and $n = 6$.

b. $$P(X = 2) = \frac{\binom{8}{2}\binom{18-8}{6-2}}{\binom{18}{6}} = \frac{\binom{8}{2}\binom{10}{4}}{\binom{18}{6}} = \frac{(28)(210)}{18564} = .3167.$$

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{\binom{8}{0}\binom{10}{6}}{\binom{18}{6}} + \frac{\binom{8}{1}\binom{10}{5}}{\binom{18}{6}} + .3167 =$$

$.0113 + .1086 + .3167 = .4366$.
$P(X \geq 2) = 1 - P(X \leq 1) = 1 - [P(X = 0) + P(X = 1)] = 1 - [.0113 + .1086] = .8801$.

c. $$E(X) = n \cdot \frac{M}{N} = 6 \cdot \frac{8}{18} = 2.67;\ V(X) = \left(\frac{18-6}{18-1}\right) \cdot 6\left(\frac{8}{18}\right)\left(1 - \frac{8}{18}\right) = 1.04575;\ \sigma = 1.023.$$

69. According to the problem description, X is hypergeometric with $n = 6$, $N = 12$, and $M = 7$.

a. $$P(X = 4) = \frac{\binom{7}{4}\binom{5}{2}}{\binom{12}{6}} = \frac{350}{924} = .379\ .\ P(X \leq 4) = 1 - P(X > 4) = 1 - [P(X = 5) + P(X = 6)] =$$

$$1 - \left[\frac{\binom{7}{5}\binom{5}{1}}{\binom{12}{6}} + \frac{\binom{7}{6}\binom{5}{0}}{\binom{12}{6}}\right] = 1 - [.114 + .007] = 1 - .121 = .879.$$

b. $E(X) = n \cdot \frac{M}{N} = 6 \cdot \frac{7}{12} = 3.5$; $V(X) = \left(\frac{12-6}{12-1}\right)6\left(\frac{7}{12}\right)\left(1-\frac{7}{12}\right) = 0.795$; $\sigma = 0.892$. So,
$P(X > \mu + \sigma) = P(X > 3.5 + 0.892) = P(X > 4.392) = P(X = 5 \text{ or } 6) = .121$ (from part **a**).

c. We can approximate the hypergeometric distribution with the binomial if the population size and the number of successes are large. Here, $n = 15$ and $M/N = 40/400 = .1$, so
$h(x;15, 40, 400) \approx b(x;15, .10)$. Using this approximation, $P(X \leq 5) \approx B(5; 15, .10) = .998$ from the binomial tables. (This agrees with the exact answer to 3 decimal places.)

70.

a. $P(X = 10) = h(10; 15, 30, 50) = \dfrac{\binom{30}{10}\binom{20}{5}}{\binom{50}{15}} = .2070$.

b. $P(X \geq 10) = h(10; 15, 30, 50) + h(11; 15, 30, 50) + \ldots + h(15; 15, 30, 50)$
$= .2070 + .1176 + .0438 + .0101 + .0013 + .0001 = .3799$.

c. P(at least 10 from the same class) = P(at least 10 from second class [answer from **b**]) + P(at least 10 from first class). *B*ut "at least 10 from 1st class" is the same as "at most 5 from the second" or $P(X \leq 5)$.
$P(X \leq 5) = h(5; 15, 30, 50) + h(4; 15, 30, 50) + \ldots + h(0; 15, 30, 50)$
$= .011697 + .002045 + .000227 + .000015 + .000001 + .000000 = .01398$.
So the desired probability is $P(X \geq 10) + P(X \leq 5) = .3799 + .01398 = .39388$.

d. $E(X) = n \cdot \frac{M}{N} = 15 \cdot \frac{30}{50} = 9$; $V(X) = \left(\frac{50-15}{50-1}\right)9\left(\frac{30}{50}\right)\left(1-\frac{30}{50}\right) = 1.543$; $\sigma_X = 1.242$.

e. Let Y = the number not among these first 15 that are from the second section. Since there are 30 students total in that section, $Y = 30 – X$. Then $E(Y) = 30 – E(X) = 30 – 9 = 21$, and
$V(Y) = V(30 – X) = V(X) = 1.543$ and $\sigma_Y = \sigma_X = 1.242$.

71.

a. Possible values of X are 5, 6, 7, 8, 9, 10. (In order to have less than 5 of the granite, there would have to be more than 10 of the basaltic). X is hypergeometric, with $n = 15$, $N = 20$, and $M = 10$. So, the pmf of X is

$$p(x) = h(x; 15, 10, 20) = \frac{\binom{10}{x}\binom{10}{15-x}}{\binom{20}{15}}.$$

The pmf is also provided in table form below.

x	5	6	7	8	9	10
$p(x)$	.0163	.1354	.3483	.3483	.1354	.0163

b. P(all 10 of one kind or the other) = $P(X = 5) + P(X = 10) = .0163 + .0163 = .0326$.

c. $\mu = n\cdot\frac{M}{N} = 15\cdot\frac{10}{20} = 7.5$; $V(X) = \left(\frac{20-15}{20-1}\right)15\left(\frac{10}{20}\right)\left(1-\frac{10}{20}\right) = .9868$; $\sigma = .9934$.

$\mu \pm \sigma = 7.5 \pm .9934 = (6.5066, 8.4934)$, so we want $P(6.5066 < X < 8.4934)$. That equals $P(X = 7) + P(X = 8) = .3483 + .3483 = .6966$.

72.

a. There are $N = 11$ candidates, $M = 4$ in the "top four" (obviously), and $n = 6$ selected for the first day's interviews. So, the probability x of the "top four" are interviewed on the first day equals $h(x; 6, 4, 11) =$

$$\frac{\binom{4}{x}\binom{7}{6-x}}{\binom{11}{6}}.$$

b. With X = the number of "top four" interview candidates on the first day, $E(X) = n\cdot\frac{M}{N} = 6\cdot\frac{4}{11} = 2.18$.

73.

a. The successes here are the top $M = 10$ pairs, and a sample of $n = 10$ pairs is drawn from among the $N = 20$. The probability is therefore $h(x; 10, 10, 20) = \frac{\binom{10}{x}\binom{10}{10-x}}{\binom{20}{10}}$.

b. Let X = the number among the top 5 who play east-west. (Now, $M = 5$.)
Then P(all of top 5 play the same direction) = $P(X = 5) + P(X = 0) =$

$$h(5; 10, 5, 20) + h(5; 10, 5, 20) = \frac{\binom{5}{5}\binom{15}{5}}{\binom{20}{10}} + \frac{\binom{5}{0}\binom{15}{10}}{\binom{20}{10}} = .033.$$

c. Generalizing from earlier parts, we now have $N = 2n$; $M = n$. The probability distribution of X is hypergeometric: $p(x) = h(x; n, n, 2n) = \frac{\binom{n}{x}\binom{n}{n-x}}{\binom{2n}{n}}$ for $x = 0, 1, \ldots, n$. Also,

$$E(X) = n\cdot\frac{n}{2n} = \frac{1}{2}n \text{ and } V(X) = \left(\frac{2n-n}{2n-1}\right)\cdot n\cdot\frac{n}{2n}\cdot\left(1-\frac{n}{2n}\right) = \frac{n^2}{4(2n-1)}.$$

74.

a. Hypergeometric, with $n = 10$, $M = 15$, and $N = 50$: $p(x) = h(x;\, 10, 15, 50)$.

b. When N is large relative to n, The hypergeometric model can be approximated by an appropriate binomial model: $h(x\,;n\,,M\,,N) \doteq b\left(x;n,\frac{M}{N}\right)$. So, $h(x;\, 10, 150, 500) \doteq b(x;10,.3)$.

c. Using the hypergeometric model, $E(X) = 10\cdot\left(\frac{150}{500}\right) = 3$ and

$V(X) = \frac{490}{499}(10)(.3)(.7) = .982(2.1) = 2.06$. Using the binomial model, $E(X) = (10)(.3) = 3$, the same as the mean of the exact model, and $V(X) = 10(.3)(.7) = 2.1$, slightly more than the exact variance.

75. Let X = the number of boxes that do <u>not</u> contain a prize until you find 2 prizes. Then $X \sim \text{NB}(2, .2)$.

a. With S = a female child and F = a male child, let X = the number of F's before the 2nd S. Then $P(X = x) = nb(x;\, 2, .2) = \binom{x+2-1}{2-1}(.2)^2(1-.2)^x = (x + 1)(.2)^2(.8)^x$.

b. $P(4 \text{ boxes purchased}) = P(2 \text{ boxes without prizes}) = P(X = 2) = nb(2;\, 2, .2) = (2 + 1)(.2)^2(.8)^2 = .0768$.

c. $P(\text{at most 4 boxes purchased}) = P(X \le 2) = \sum_{x=0}^{2} nb(x;2,.8) = .04 + .064 + .0768 = .1808$.

d. $E(X) = \frac{r(1-p)}{p} = \frac{2(1-.2)}{.2} = 8$. The total number of boxes you expect to buy is 8 + 2 = 10.

76. This question relates to the negative binomial distribution, but we can't use it directly (X, as it's defined, doesn't fit the negative binomial description). Instead, let's reason it out.
Clearly, X can't be 0, 1, or 2.
$P(X = 3) = P(BBB \text{ or } GGG) = (.5)^3 + (.5)^3 = .25$.
$P(X = 4) = P(BBGB \text{ or } BGBB \text{ or } GBBB \text{ or } GGBG \text{ or } GBGG \text{ or } BGGG) = 6(.5)^4 = .375$.
$P(X = 5)$ look scary until we realize that 5 is the maximum possible value of X!
(If you have 5 kids, you must have at least 3 of one gender.) So, $P(X = 5) = 1 – .25 – .375 = .375$, and that completes the pmf of X.

x	3	4	5
$p(x)$	.25	.375	.375

77. This is identical to an experiment in which a single family has children until exactly 6 females have been born (since p = .5 for each of the three families). So, $p(x) = nb(x; 6, .5) = \binom{x+5}{5}(.5)^6(1-.5)^x = \binom{x+5}{5}(.5)^{6+x}$. Also, $E(X) = \frac{r(1-p)}{p} = \frac{6(1-.5)}{.5}$ = 6; notice this is just 2 + 2 + 2, the sum of the expected number of males born to each family.

78. The geometric pmf is given by $p(y) = (1-p)^y p = (.591)^y(.409)$ for y = 0, 1, 2, ….

a. $P(Y = 3) = (.591)^3(.409) = .0844$. $P(Y \le 3) = P(Y = 0, 1, 2, 3) = (.409)[1 + (.591) + (.591)^2 + (.591)^3] = .878$.

b. Using the negative binomial formulas with r = 1, $E(Y) = \frac{1-p}{p} = \frac{.591}{.409}$ = 1.445 and $V(Y) = \frac{1-p}{p^2} = \frac{.591}{(.409)^2}$ = 3.533, so $SD(Y)$ = 1.88. The probability a drought length exceeds its mean by at least one standard deviation is $P(Y \ge \mu + \sigma) = P(Y \ge 1.445 + 1.88) = P(Y \ge 3.325) = 1 - P(Y < 3.325) = 1 - P(Y \le 3) = 1 - .878$ from part **a** = .122.

Section 3.6

79. All these solutions are found using the cumulative Poisson table, $F(x; \mu) = F(x; 1)$.

a. $P(X \le 5) = F(5; 1) = .999$.

b. $P(X = 2) = \frac{e^{-1}1^2}{2!} = .184$. Or, $P(X = 2) = F(2; 1) - F(1; 1) = .920 - .736 = .184$.

c. $P(2 \le X \le 4) = P(X \le 4) - P(X \le 1) = F(4; 1) - F(1; 1) = .260$.

d. For X Poisson, $\sigma = \sqrt{\mu} = 1$, so $P(X > \mu + \sigma) = P(X > 2) = 1 - P(X \le 2) = 1 - F(2; 1) = 1 - .920 = .080$.

80. Solutions are found using the cumulative Poisson table, $F(x; \mu) = F(x; 4)$.

a. $P(X \le 4) = F(4; 4) = .629$, while $P(X < 4) = P(X \le 3) = F(3; 4) = .434$.

b. $P(4 \le X \le 8) = F(8; 4) - F(3; 4) = .545$.

c. $P(X \ge 8) = 1 - P(X < 8) = 1 - P(X \le 7) = 1 - F(7; 4) = .051$.

d. For this Poisson model, μ = 4 and so $\sigma = \sqrt{4} = 2$. The desired probability is $P(X \le \mu + \sigma) = P(X \le 4 + 2) = P(X \le 6) = F(6; 4) = .889$.

81. Let $X \sim \text{Poisson}(\mu = 20)$.

a. $P(X \le 10) = F(10; 20) = .011$.

b. $P(X > 20) = 1 - F(20; 20) = 1 - .559 = .441$.

c. $P(10 \le X \le 20) = F(20; 20) - F(9; 20) = .559 - .005 = .554$;
$P(10 < X < 20) = F(19; 20) - F(10; 20) = .470 - .011 = .459$.

d. $E(X) = \mu = 20$, so $\sigma = \sqrt{20} = 4.472$. Therefore, $P(\mu - 2\sigma < X < \mu + 2\sigma) =$ $P(20 - 8.944 < X < 20 + 8.944) = P(11.056 < X < 28.944) = P(X \le 28) - P(X \le 11) =$ $F(28; 20) - F(11; 20) = .966 - .021 = .945$.

82.

a. $P(X = 1) = F(1; .2) - F(0; .2) = .982 - .819 = .163$.

b. $P(X \ge 2) = 1 - P(X \le 1) = 1 - F(1; .2) = 1 - .982 = .018$.

c. The probability a disk contains zero missing pulses is $P(X = 0) = \dfrac{e^{-.2}(.2)^0}{0!} = .819$. Since the two disks are independent, $P(1^{st}$ doesn't $\cap$ 2^{nd} doesn't$) = P(1^{st}$ doesn't$) \cdot P(2^{nd}$ doesn't$)$ $= (.819)(.819) = .671$.

83. The exact distribution of X is binomial with $n = 1000$ and $p = 1/200$; we can approximate this distribution by the Poisson distribution with $\mu = np = 5$.

a. $P(5 \le X \le 8) = F(8; 5) - F(4; 5) = .492$.

b. $P(X \ge 8) = 1 - P(X \le 7) = 1 - F(7; 5) = 1 - .867 = .133$.

84.

a. The experiment is binomial with $n = 200$ and $p = 1/88$, so $\mu = np = 2.27$ and $\sigma = \sqrt{npq} = \sqrt{2.247} = 1.50$.

b. X has approximately a Poisson distribution with $\mu = 2.27$, so $P(X \ge 2) = 1 - P(X = 0, 1) \approx$ $1 - \left[\dfrac{e^{-2.27}2.27^0}{0!} + \dfrac{e^{-2.27}2.27^1}{1!}\right] = 1 - .3378 = .6622$. (The exact binomial answer is .6645.)

c. Now $\mu = 352(1/88) = 4$, so $P(X < 5) = P(X \le 4) \approx F(4; 4) = .629$.

85.

a. $\mu = 8$ when $t = 1$, so $P(X = 6) = \dfrac{e^{-8}8^6}{6!} = .122$; $P(X \ge 6) = 1 - F(5; 8) = .809$; and $P(X \ge 10) = 1 - F(9; 8) = .283$.

b. $t = 90$ min $= 1.5$ hours, so $\mu = 12$; thus the expected number of arrivals is 12 and the standard deviation is $\sigma = \sqrt{12} = 3.464$.

c. $t = 2.5$ hours implies that $\mu = 20$. So, $P(X \ge 20) = 1 - F(19; 20) = .530$ and $P(X \le 10) = F(10; 20) = .011$.

86.

a. The expected number of organisms in 1 m^3 of water is 10, so $X \sim$ Poisson(10).
$P(X \geq 8) = 1 - P(X \leq 7) = 1 - F(7; 10) = 1 - .220 = .780$.

b. The expected number of organisms in 1.5 m^3 of water is 10(1.5) = 15, so $X \sim$ Poisson(15). Since X is Poisson, $\sigma = \sqrt{\mu} = \sqrt{15} = 3.87$.
$P(X > \mu + \sigma) = P(X > 15 + 3.87) = P(X > 18.87) = 1 - P(X \leq 18.87) = 1 - P(X \leq 18) = 1 - F(18; 15) = 1 - .993 = .007$.

c. Let d equal the amount of discharge, so $X \sim$ Poisson($10d$). Set $.999 = P(X \geq 1)$ and solve for d:

$$.999 = P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{e^{-10d}(10d)^0}{0!} = e^{-10d} \Rightarrow e^{-10d} = .001 \Rightarrow d = -0.1 \ln(.001) = 0.69 \text{ m}^3.$$

87.

a. For a two hour period the parameter of the distribution is $\mu = \alpha t = (4)(2) = 8$,

so $P(X = 10) = \dfrac{e^{-8}8^{10}}{10!} = .099$.

b. For a 30-minute period, $\alpha t = (4)(.5) = 2$, so $P(X = 0) = \dfrac{e^{-2}2^0}{0!} = .135$.

c. The expected value is simply $E(X) = \alpha t = 2$.

88. Let X = the number of diodes on a board that fail. Then $X \sim \text{Bin}(n = 200, p = .01)$.

a. $E(X) = np = (200)(.01) = 2$; $V(X) = npq = (200)(.01)(.99) = 1.98$, so $\sigma = 1.407$.

b. X has approximately a Poisson distribution with $\mu = np = 2$, so $P(X \geq 4) = 1 - P(X \leq 3) = 1 - F(3; 2) = 1 - .857 = .143$.

c. For any one board, P(board works properly) = P(all diodes work) = $P(X = 0) = \dfrac{e^{-2}2^0}{0!} = .135$.

Let Y = the number among the five boards that work, a binomial rv with $n = 5$ and $p = .135$.

Then $P(Y \geq 4) = P(Y = 4) + P(Y = 5) = \binom{5}{4}(.135)^4(.865) + \binom{5}{5}(.135)^5(.865)^0 = .00148$.

89. In this example, α = rate of occurrence = 1/(mean time between occurrences) = 1/.5 = 2.

a. For a two-year period, $\mu = \alpha t = (2)(2) = 4$ loads.

b. Apply a Poisson model with $\mu = 4$: $P(X > 5) = 1 - P(X \leq 5) = 1 - F(5; 4) = 1 - .785 = .215$.

c. For $\alpha = 2$ and the value of t unknown, P(no loads occur during the period of length t) =

$P(X = 0) = \dfrac{e^{-2t}(2t)^0}{0!} = e^{-2t}$. Solve for t: $e^{-2t} \leq .1 \Rightarrow -2t \leq \ln(.1) \Rightarrow t \geq 1.1513$ years.

90. $E(X) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\mu}\mu^x}{x!} = \sum_{x=1}^{\infty} x \cdot \frac{e^{-\mu}\mu^x}{x!} = \mu \sum_{x=1}^{\infty} \frac{e^{-\mu}\mu^{x-1}}{(x-1)!} = \mu \sum_{y=0}^{\infty} \frac{e^{-\mu}\mu^y}{y!}$. Since the summation now represents the sum of a Poisson pmf across <u>all</u> its values, that summation is 1 and the result is indeed μ, as desired.

91.

a. For a quarter-acre (.25 acre) plot, the mean parameter is $\mu = (80)(.25) = 20$, so $P(X \le 16) = F(16; 20) = .221$.

b. The expected number of trees is $\alpha \cdot$(area) = 80 trees/acre (85,000 acres) = 6,800,000 trees.

c. The area of the circle is $\pi r^2 = \pi(.1)^2 = .01\pi = .031416$ square miles, which is equivalent to $.031416(640) = 20.106$ acres. Thus X has a Poisson distribution with parameter $\mu = \alpha(20.106) = 80(20.106) = 1608.5$. That is, the pmf of X is the function $p(x; 1608.5)$.

92.

a. Let Y denote the number of cars that arrive in the hour, so $Y \sim$ Poisson(10). Then
$P(Y = 10$ and no violations$) = P(Y = 10) \cdot P($no violations $\mid Y = 10) =$
$\frac{e^{-10}10^{10}}{10!} \cdot (.5)^{10}$, assuming the violation statuses of the 10 cars are mutually independent. This expression equals .000122.

b. Following the method from **a**, $P(y$ arrive and exactly 10 have no violations$) =$
$P(y$ arrive$) \cdot P($exactly 10 have no violations $\mid y$ arrive$) =$
$\frac{e^{-10}10^y}{y!} \cdot P($exactly 10 "successes" in y trials when $p = .5) =$

$$\frac{e^{-10}10^y}{y!} \cdot \binom{y}{10}(.5)^{10}(.5)^{y-10} = \frac{e^{-10}10^y}{y!}\frac{y!}{10!(y-10)!}(.5)^y = \frac{e^{-10}5^y}{10!(y-10)!}$$

c. $P(\text{exactly 10 without a violation}) = \sum_{y=10}^{\infty} \frac{e^{-10}5^y}{10!(y-10)!} = \frac{e^{-10}\cdot 5^{10}}{10!}\sum_{y=10}^{\infty}\frac{5^{y-10}}{(y-10)!} = \frac{e^{-10}\cdot 5^{10}}{10!}\sum_{u=0}^{\infty}\frac{5^u}{u!} =$

$\frac{e^{-10}\cdot 5^{10}}{10!}\cdot e^5 = \frac{e^{-5}\cdot 5^{10}}{10!} = p(10; 5)$.

In fact, generalizing this argument shows that the number of "no-violation" arrivals within the hour has a Poisson distribution with mean parameter equal to $\mu = \alpha p = 10(.5) = 5$.

93.

a. No events occur in the time interval $(0, t + \Delta t)$ if and only if no events occur in $(0, t)$ <u>and</u> no events occur in $(t, t + \Delta t)$. Since it's assumed the numbers of events in non-overlapping intervals are independent (Assumption 3),
$P($no events in $(0, t + \Delta t)) = P($no events in $(0, t)) \cdot P($no events in $(t, t + \Delta t)) \Rightarrow$
$P_0(t + \Delta t) = P_0(t) \cdot P($no events in $(t, t + \Delta t)) = P_0(t) \cdot [1 - \alpha\Delta t - o(\Delta t)]$ by Assumption 2.

b. Rewrite **a** as $P_0(t + \Delta t) = P_0(t) - P_0(t)[\alpha\Delta t + o(\Delta t)]$, so $P_0(t + \Delta t) - P_0(t) = -P_0(t)[\alpha\Delta t + o(\Delta t)]$ and $\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\alpha P_0(t) - P_0(t)\cdot\frac{o(\Delta t)}{\Delta t}$. Since $\frac{o(\Delta t)}{\Delta t} \to 0$ as $\Delta t \to 0$ and the left-hand side of the equation converges to $\frac{dP_0(t)}{dt}$ as $\Delta t \to 0$, we find that $\frac{dP_0(t)}{dt} = -\alpha P_0(t)$.

c. Let $P_0(t) = e^{-\alpha t}$. Then $\frac{dP_0(t)}{dt} = \frac{d}{dt}[e^{-\alpha t}] = -\alpha e^{-\alpha t} = -\alpha P_0(t)$, as desired. (This suggests that the probability of zero events in (0, t) for a process defined by Assumptions 1-3 is equal to $e^{-\alpha t}$.)

d. Similarly, the product rule implies $\frac{d}{dt}\left[\frac{e^{-\alpha t}(\alpha t)^k}{k!}\right] = \frac{-\alpha e^{-\alpha t}(\alpha t)^k}{k!} + \frac{k\alpha e^{-\alpha t}(\alpha t)^{k-1}}{k!} =$

$-\alpha\frac{e^{-\alpha t}(\alpha t)^k}{k!} + \alpha\frac{e^{-\alpha t}(\alpha t)^{k-1}}{(k-1)!} = -\alpha P_k(t) + \alpha P_{k-1}(t)$, as desired.

Supplementary Exercises

94. Outcomes are (1,2,3) (1,2,4) (1,2,5) … (5,6,7); there are 35 such outcomes, each having probability $\frac{1}{35}$. The W values for these outcomes are 6 (= 1 + 2 + 3), 7, 8, …, 18. Since there is just one outcome with W value 6, $p(6) = P(W = 6) = \frac{1}{35}$. Similarly, there are three outcomes with W value 9 [(1,2,6) (1,3,5) and (2,3,4)], so $p(9) = \frac{3}{35}$. Continuing in this manner yields the following distribution:

w	6	7	8	9	10	11	12	13	14	15	16	17	18
$p(w)$	$\frac{1}{35}$	$\frac{1}{35}$	$\frac{2}{35}$	$\frac{3}{35}$	$\frac{4}{35}$	$\frac{4}{35}$	$\frac{5}{35}$	$\frac{4}{35}$	$\frac{4}{35}$	$\frac{3}{35}$	$\frac{2}{35}$	$\frac{1}{35}$	$\frac{1}{35}$

Since the distribution is symmetric about 12, $\mu = 12$, and $\sigma^2 = \sum_{w=6}^{18}(w-12)^2\,p(w) =$

$\frac{1}{35}[(-6)^2(1) + (-5)^2(1) + \ldots + (5)^2(1) + (6)^2(1)] = 8$.

95.

a. We'll find $p(1)$ and $p(4)$ first, since they're easiest, then $p(2)$. We can then find $p(3)$ by subtracting the others from 1.

$p(1) = P(\text{exactly one suit}) = P(\text{all }\spadesuit) + P(\text{all }\heartsuit) + P(\text{all }\diamondsuit) + P(\text{all }\clubsuit) =$

$4 \cdot P(\text{all }\spadesuit) = 4\cdot\frac{\binom{13}{5}\binom{39}{0}}{\binom{52}{5}} = .00198$, since there are 13 ♠s and 39 other cards.

$p(4) = 4 \cdot P(2\ \spadesuit, 1\ \heartsuit, 1\ \diamondsuit, 1\ \clubsuit) = 4\cdot\frac{\binom{13}{2}\binom{13}{1}\binom{13}{1}\binom{13}{1}}{\binom{52}{5}} = .26375$.

$p(2) = P(\text{all }\heartsuit\text{s and }\spadesuit\text{s, with} \geq \text{one of each}) + \ldots + P(\text{all }\diamondsuit\text{s and }\clubsuit\text{s with} \geq \text{one of each}) =$

$\binom{4}{2}\cdot P$(all ♥s and ♠s, with $\geq$ one of each) =

$6 \cdot [P(1$ ♥ and 4 ♠) + $P(2$ ♥ and 3 ♠) + $P(3$ ♥ and 2 ♠) + $P(4$ ♥ and 1 ♠)] =

$$6\cdot\left[2\cdot\frac{\binom{13}{4}\binom{13}{1}}{\binom{52}{5}}+2\cdot\frac{\binom{13}{3}\binom{13}{2}}{\binom{52}{5}}\right]=6\left[\frac{18,590+44,616}{2,598,960}\right]=.14592\,.$$

Finally, $p(3) = 1 - [p(1) + p(2) + p(4)] = .58835$.

b. $\mu = \sum_{x=1}^{4} x\cdot p(x) = 3.114$; $\sigma^2 = \left[\sum_{x=1}^{4} x^2\cdot p(x)\right] - (3.114)^2 = .405 \Rightarrow \sigma = .636$.

96. $p(y) = P(Y = y) = P$(exactly y trials to achieve r S's) = P(exactly $r - 1$ S's in the first $y - 1$ trials, then a S) = P(exactly $r - 1$ S's in the first $y - 1$ trials) $\cdot P(S)$ =

$\binom{y-1}{r-1}p^{r-1}(1-p)^{(y-1)-(r-1)}\cdot p = \binom{y-1}{r-1}p^{r}(1-p)^{y-r}$ for $y = r, r + 1, r + 2, \ldots$

97.

a. From the description, $X \sim \text{Bin}(15, .75)$. So, the pmf of X is $b(x; 15, .75)$.

b. $P(X > 10) = 1 - P(X \leq 10) = 1 - B(10;15, .75) = 1 - .314 = .686$.

c. $P(6 \leq X \leq 10) = B(10; 15, .75) - B(5; 15, .75) = .314 - .001 = .313$.

d. $\mu = (15)(.75) = 11.75$, $\sigma^2 = (15)(.75)(.25) = 2.81$.

e. Requests can all be met if and only if $X \leq 10$, and $15 - X \leq 8$, i.e. iff $7 \leq X \leq 10$. So,
P(all requests met) = $P(7 \leq X \leq 10) = B(10; 15, .75) - B(6; 15, .75) = .310$.

98.

a. First, consider a Poisson distribution with $\mu = \theta$. Since the sum of the pmf across all x-values (0, 1, 2, 3, …) must equal 1,

$$1 = \sum_{x=0}^{\infty}\frac{e^{-\theta}\theta^x}{x!} = \frac{e^{-\theta}\theta^0}{0!}+\sum_{x=1}^{\infty}\frac{e^{-\theta}\theta^x}{x!} = e^{-\theta}+\sum_{x=1}^{\infty}\frac{e^{-\theta}\theta^x}{x!} \Rightarrow \sum_{x=1}^{\infty}\frac{e^{-\theta}\theta^x}{x!} = 1 - e^{-\theta}$$

Also, the sum of the specified pmf across $x = 1, 2, 3, \ldots$ must equal 1, so

$1 = \sum_{x=1}^{\infty}k\frac{e^{-\theta}\theta^x}{x!} = k\sum_{x=1}^{\infty}\frac{e^{-\theta}\theta^x}{x!} = k[1 - e^{-\theta}]$ from above. Therefore, $k = \frac{1}{1-e^{-\theta}}$.

b. Again, first consider a Poisson distribution with $\mu = \theta$. Since the expected value is θ,

$\theta = \sum_{x=0}^{\infty}x\cdot p(x;\theta) = 0 + \sum_{x=1}^{\infty}x\cdot p(x;\theta) = \sum_{x=1}^{\infty}x\cdot\frac{e^{-\theta}\theta^x}{x!}$. Multiply both sides by k:

$k\theta = k\sum_{x=1}^{\infty}x\cdot\frac{e^{-\theta}\theta^x}{x!} = \sum_{x=1}^{\infty}x\cdot k\frac{e^{-\theta}\theta^x}{x!}$; the right-hand side is the expected value of the specified distribution. So, the mean of a "zero-truncated" Poisson distribution is $k\theta$, i.e. $\frac{\theta}{1-e^{-\theta}}$.

The mean value 2.313035 corresponds to $\theta = 2$: $\dfrac{2}{1-e^{-2}} = 2.313035$. And so, finally,

$$P(X \le 5) = \sum_{x=1}^{5} k\frac{e^{-\theta}\theta^x}{x!} = \frac{e^{-2}}{1-e^{-2}}\sum_{x=1}^{5}\frac{2^x}{x!} = .9808.$$

c. Using the same trick as in part **b**, the mean-square value of our distribution is

$$\sum_{x=1}^{\infty} x^2 \cdot k\frac{e^{-\theta}\theta^x}{x!} = k\sum_{x=1}^{\infty} x^2 \cdot \frac{e^{-\theta}\theta^x}{x!} = k\sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\theta}\theta^x}{x!} = k \cdot E(Y^2)\text{, where } Y \sim \text{Poisson}(\theta).$$

For any rv, $V(Y) = E(Y^2) - \mu^2 \Rightarrow E(Y^2) = V(Y) + \mu^2$; for the Poisson($\theta$) rv, $E(Y^2) = \theta + \theta^2$. Therefore, the mean-square value of our distribution is $k \cdot (\theta + \theta^2)$, and the variance is $V(X) = E(X^2) - [E(X)]^2 = k \cdot (\theta + \theta^2) - (k\theta)^2 = k\theta + k(1-k)\theta^2$. Substituting $\theta = 2$ gives $V(X) \approx 1.58897$, so $\sigma_X \approx 1.2605$.

99. Let X = the number of components out of 5 that function, so $X \sim \text{Bin}(5, .9)$. Then a 3-out-of 5 system works when X is at least 3, and $P(X \ge 3) = 1 - P(X \le 2) = 1 - B(2; 5, .9) = .991$.

100.

a. Let X denote the number of defective chips in the sample of 25, so $X \sim \text{Bin}(25, .05)$. Then the batch will be rejected with probability $P(X \ge 5) = 1 - P(X \le 4) = 1 - B(4; 25, .05) = .007$.

b. Now $X \sim \text{Bin}(25, .10)$, so $P(X \ge 5) = 1 - B(4; 25, .10) = .098$.

c. Now $X \sim \text{Bin}(25, .20)$, $P(X \ge 5) = 1 - B(4;25,.20) = .579$.

d. All of the probabilities would decrease, since the criterion for rejecting a batch is now more lax. That’s bad if the proportion of defective chips is large and good if the proportion of defective chips is small.

101.

a. $X \sim \text{Bin}(n = 500, p = .005)$. Since n is large and p is small, X can be approximated by a Poisson distribution with $\mu = np = 2.5$. The approximate pmf of X is $p(x; 2.5) = \dfrac{e^{-2.5}2.5^x}{x!}$.

b. $P(X = 5) = \dfrac{e^{-2.5}2.5^5}{5!} = .0668$.

c. $P(X \ge 5) = 1 - P(X \le 4) = 1 - p(4; 2.5) = 1 - .8912 = .1088$.

102. $X \sim \text{Bin}(25, p)$.

a. $P(7 \le X \le 18) = B(18; 25, .5) - B(6; 25, .5) = .986$.

b. $P(7 \le X \le 18) = B(18; 25, .8) - B(6; 25, .8) = .220$.

c. With $p = .5$, $P(\text{rejecting the claim}) = P(X \le 7) + P(X \ge 18) = .022 + [1 - .978] = .022 + .022 = .044$.

d. The claim will not be rejected when $7 < X < 18$, i.e. when $8 \le X \le 17$.
With $p = .6$, $P(8 \le X \le 17) = B(17; 25, .6) - B(7; 25, .6) = .846 - .001 = .845$.
With $p = .8$, $P(8 \le X \le 17) = B(17; 25, .8) - B(7; 25, .8) = .109 - .000 = .109$.

e. We want P(rejecting the claim when $p = .5) \leq .01$.
Using the decision rule "reject if $X \leq 6$ or $X \geq 19$" gives the probability .014, which is too large. We should use "reject if $X \leq 5$ or $X \geq 20$," which yields P(rejecting the claim) = .002 + .002 = .004.

103. Let Y denote the number of tests carried out.
For $n = 3$, possible Y values are 1 and 4. $P(Y = 1) = P$(no one has the disease) = $(.9)^3 = .729$ and $P(Y = 4) = 1 - .729 = .271$, so $E(Y) = (1)(.729) + (4)(.271) = 1.813$, as contrasted with the 3 tests necessary without group testing.
For $n = 5$, possible values of Y are 1 and 6. $P(Y = 1) = P$(no one has the disease) = $(.9)^5 = .5905$, so $P(Y = 6) = 1 - .5905 = .4095$ and $E(Y) = (1)(.5905) + (6)(.4095) = 3.0475$, less than the 5 tests necessary without group testing.

104. Regard any particular symbol being received as constituting a trial. Then P(success) = P(symbol is sent correctly, or is sent incorrectly and subsequently corrected) = $(1 - p_1) + p_1p_2$. The block of n symbols gives a binomial experiment with n trials and $p = 1 - p_1 + p_1p_2$.

105. $p(2) = P(X = 2) = P(SS) = p^2$, and $p(3) = P(FSS) = (1 - p)p^2$.

For $x \geq 4$, consider the first $x - 3$ trials and the last 3 trials separately. To have $X = x$, it must be the case that the last three trials were FSS, and that two-successes-in-a-row was not already seen in the first $x - 3$ tries.

The probability of the first event is simply $(1 - p)p^2$.
The second event occurs if two-in-a-row hadn't occurred after 2 or 3 or … or $x - 3$ tries. The probability of this second event equals $1 - [p(2) + p(3) + \ldots + p(x - 3)]$. (For $x = 4$, the probability in brackets is empty; for $x = 5$, it's $p(2)$; for $x = 6$, it's $p(2) + p(3)$; and so on.)

Finally, since trials are independent, $P(X = x) = (1 - [p(2) + \ldots + p(x - 3)]) \cdot (1 - p)p^2$.

For $p = .9$, the pmf of X up to $x = 8$ is shown below.

x	2	3	4	5	6	7	8
$p(x)$	.81	.081	.081	.0154	.0088	.0023	.0010

So, $P(X \leq 8) = p(2) + \ldots + p(8) = .9995$.

106.

a. With $X \sim \text{Bin}(25, .1)$, $P(2 \leq X \leq 6) = B(6; 25, .1) - B(1; 25, .1) = .991 - .271 = 720$.

b. $E(X) = np = 25(.1) = 2.5$, $SD(X) = \sqrt{npq} = \sqrt{25(.1)(.9)} = \sqrt{2.25} = 1.5$.

c. $P(X \geq 7$ when $p = .1) = 1 - B(6; 25, .1) = 1 - .991 = .009$.

d. $P(X \leq 6$ when $p = .2) = B(6; 25, .2) = .780$, which is quite large.

107.

a. Let event A = seed carries single spikelets, and event B = seed produces ears with single spikelets. Then $P(A \cap B) = P(A) \cdot P(B \mid A) = (.40)(.29) = .116$.
Next, let X = the number of seeds out of the 10 selected that meet the condition $A \cap B$. Then $X \sim \text{Bin}(10, .116)$. So, $P(X = 5) = \binom{10}{5}(.116)^5(.884)^5 = .002857$.

b. For any one seed, the event of interest is B = seed produces ears with single spikelets. Using the law of total probability, $P(B) = P(A \cap B) + P(A' \cap B) = (.40)(.29) + (.60)(.26) = .272$.
Next, let Y = the number out of the 10 seeds that meet condition B. Then $Y \sim \text{Bin}(10, .272)$. $P(Y = 5) = \binom{10}{5}(.272)^5(1-.272)^5 = .0767$, while

$$P(Y \le 5) = \sum_{y=0}^{5}\binom{10}{y}(.272)^y(1-.272)^{10-y} = .041813 + \ldots + .076719 = .97024.$$

108. With S = favored acquittal, the population size is $N = 12$, the number of population S's is $M = 4$, the sample size is $n = 4$, and the pmf of the number of interviewed jurors who favor acquittal is the hypergeometric pmf: $h(x; 4, 4, 12)$. $E(X) = 4 \cdot \left(\frac{4}{12}\right) = 1.33$.

109.

a. $P(X = 0) = F(0; 2)$ or $\frac{e^{-2}2^0}{0!} = 0.135$.

b. Let S = an operator who receives no requests. Then the number of operators that receive no requests follows a $\text{Bin}(n = 5, p = .135)$ distribution. So, $P(4\ S\text{'s in 5 trials}) = b(4; 5, .135) = \binom{5}{4}(.135)^4(.865)^1 = .00144$.

c. For any non-negative integer x, P(all operators receive exactly x requests) = P(first operator receives x) · … · P(fifth operator receives x) = $[p(x; 2)]^5 = \left[\frac{e^{-2}2^x}{x!}\right]^5 = \frac{e^{-10}2^{5x}}{(x!)^5}$.
Then, P(all receive the same number) = P(all receive 0 requests) + P(all receive 1 request) + P(all receive 2 requests) + … = $\sum_{x=0}^{\infty}\frac{e^{-10}2^{5x}}{(x!)^5}$.

110. The number of grasshoppers within a circular region of radius R follows a Poisson distribution with $\mu = \alpha \cdot \text{area} = \alpha\pi R^2$.

P(at least one) = 1 – P(none) = $1 - \frac{e^{-\alpha\pi R^2}(\alpha\pi R^2)^0}{0!} = 1 - e^{-\alpha\pi R^2} = .99 \Rightarrow e^{-\alpha\pi R^2} = .01 \Rightarrow R^2 = \frac{-\ln(.01)}{\alpha\pi} = .7329 \Rightarrow R = .8561$ yards.

111. The number of magazine copies sold is X so long as X is no more than five; otherwise, all five copies are sold. So, mathematically, the number sold is min(X, 5), and E[min(x, 5)] = $\sum_{x=0}^{\infty}\min(x,5)\,p(x;4)$ = 0p(0; 4) + 1p(1; 4) + 2p(2; 4) + 3p(3; 4) + 4p(4; 4) + $\sum_{x=5}^{\infty}5\,p(x;4) =$

1.735 + $5\sum_{x=5}^{\infty}p(x;4)$ = 1.735 + $5\left[1-\sum_{x=0}^{4}p(x;4)\right]$ = 1.735 + 5[1 – F(4; 4)] = 3.59.

112.

a. $P(X = x) = P$(A wins in x games) + P(B wins in x games) =
P(9 S's in 1st x – 1 ∩ S on the xth) + P(9 F's in 1st x – 1 ∩ F on the xth) =

$$\binom{x-1}{9}p^9(1-p)^{(x-1)-9}\cdot p + \binom{x-1}{9}(1-p)^9p^{(x-1)-9}\cdot(1-p) =$$

$$\binom{x-1}{9}\left[p^{10}(1-p)^{x-10}+(1-p)^{10}p^{x-10}\right].$$

b. Possible values of X are now all positive integers ≥ 10: 10, 11, 12, …. Similar to **a**,
$P(X = x) = P$(A wins in x games) + P(B wins in x games) =
P(9 S's in 1st x – 1 ∩ S on the xth) + P(9 F's in 1st x – 1 ∩ F on the xth) =

$$\binom{x-1}{9}p^9(1-p)^{(x-1)-9}\cdot p + \binom{x-1}{9}q^9(1-q)^{(x-1)-9}\cdot q =$$

$$\binom{x-1}{9}\left[p^{10}(1-p)^{x-10}+q^{10}(1-q)^{x-10}\right].$$ Finally,

$$P(X \ge 20) = 1 - P(X < 20) = \sum_{x=10}^{19}\binom{x-1}{9}\left[p^{10}(1-p)^{x-10}+q^{10}(1-q)^{x-10}\right].$$

113.

a. No, since the probability of a "success" is not the same for all tests.

b. There are four ways exactly three could have positive results. Let *D* represent those with the disease and *D*′ represent those without the disease.

Combination		**Probability**
D	*D*′	
0	3	$\left[\binom{5}{0}(.2)^0(.8)^5\right]\cdot\left[\binom{5}{3}(.9)^3(.1)^2\right]$ =(.32768)(.0729) = .02389
1	2	$\left[\binom{5}{1}(.2)^1(.8)^4\right]\cdot\left[\binom{5}{2}(.9)^2(.1)^3\right]$ =(.4096)(.0081) = .00332
2	1	$\left[\binom{5}{2}(.2)^2(.8)^3\right]\cdot\left[\binom{5}{1}(.9)^1(.1)^4\right]$ =(.2048)(.00045) = .00009216
3	0	$\left[\binom{5}{3}(.2)^3(.8)^2\right]\cdot\left[\binom{5}{0}(.9)^0(.1)^5\right]$ =(.0512)(.00001) = .000000512

Adding up the probabilities associated with the four combinations yields 0.0273.

114. The coefficient $k(r, x)$ is the generalized combination formula $\dfrac{(x+r-1)(x+r-2)\cdots(r)}{x!}$.

With $r = 2.5$ and $p = .3$, $P(X = 4) = \dfrac{(5.5)(4.5)(3.5)(2.5)}{4!}(.3)^{2.5}(.7)^4 = .1068$;
using $k(r, 0) = 1$, $P(X \geq 1) = 1 - P(X = 0) = 1 - (.3)^{2.5} = .9507$.

115.

a. Notice that $p(x; \mu_1, \mu_2) = .5\ p(x; \mu_1) + .5\ p(x; \mu_2)$, where both terms $p(x; \mu_i)$ are Poisson pmfs. Since both pmfs are ≥ 0, so is $p(x; \mu_1, \mu_2)$. That verifies the first requirement.

Next, $\sum_{x=0}^{\infty} p(x;\mu_1,\mu_2) = .5\sum_{x=0}^{\infty} p(x;\mu_1) + .5\sum_{x=0}^{\infty} p(x;\mu_2) = .5 + .5 = 1$, so the second requirement for a pmf is met. Therefore, $p(x; \mu_1, \mu_2)$ is a valid pmf.

b. $E(X) = \sum_{x=0}^{\infty} x\cdot p(x;\mu_1,\mu_2) = \sum_{x=0}^{\infty} x[.5p(x;\mu_1)+.5p(x;\mu_2)] = .5\sum_{x=0}^{\infty} x\cdot p(x;\mu_1) + .5\sum_{x=0}^{\infty} x\cdot p(x;\mu_2) = .5E(X_1) + .5E(X_2)$, where $X_i \sim \text{Poisson}(\mu_i)$. Therefore, $E(X) = .5\mu_1 + .5\mu_2$.

c. This requires using the variance shortcut. Using the same method as in **b,**

$E(X^2)$ = $.5\sum_{x=0}^{\infty} x^2 \cdot p(x;\mu_1) + .5\sum_{x=0}^{\infty} x^2 \cdot p(x;\mu_2)$ = $.5E(X_1^2) + .5E(X_2^2)$. For any Poisson rv, $E(X^2) = V(X) + [E(X)]^2 = \mu + \mu^2$, so $E(X^2)$ = $.5(\mu_1 + \mu_1^2) + .5(\mu_2 + \mu_2^2)$.
Finally, $V(X)$ = $.5(\mu_1 + \mu_1^2) + .5(\mu_2 + \mu_2^2)$ – $[.5\mu_1 + .5\mu_2]^2$, which can be simplified to equal $.5\mu_1 + .5\mu_2 + .25(\mu_1 - \mu_2)^2$.

d. Simply replace the weights .5 and .5 with .6 and .4, so $p(x; \mu_1, \mu_2) = .6\ p(x; \mu_1) + .4\ p(x; \mu_2)$.

116.

a. $\dfrac{b(x+1;n,p)}{b(x;n,p)} = \cdots = \dfrac{(n-x)}{(x+1)} \cdot \dfrac{p}{(1-p)} > 1$ if $np - (1-p) > x$, from which the stated conclusion follows.

b. $\dfrac{p(x+1;\mu)}{p(x;\mu)} = \cdots = \dfrac{\mu}{(x+1)} > 1$ if $x < \mu - 1$, from which the stated conclusion follows. If μ is an integer, then $\mu - 1$ is a mode, but $p(\mu; \mu) = p(\mu - 1; \mu)$ so μ is also a mode.

117. $P(X = j)$ = $\sum_{i=1}^{10} P$ (arm on track $i \cap X = j$) = $\sum_{i=1}^{10} P$ ($X = j$ | arm on i) $\cdot\ p_i$ = $\sum_{i=1}^{10} P$ (next seek at $i + j + 1$ or $i - j - 1$) $\cdot\ p_i$ = $\sum_{i=1}^{10}(p_{i+j+1} + p_{i-j-1})p_i$, where in the summation we take $p_k = 0$ if $k < 0$ or $k > 10$.

118.

$$E(X) = \sum_{x=0}^{n} x \cdot \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} = \sum_{x=1}^{n} \frac{\frac{M!}{(x-1)!(M-x)!} \cdot \binom{N-M}{n-x}}{\binom{N}{n}} =$$

$$n \cdot \frac{M}{N} \sum_{x=1}^{n} \binom{M-1}{x-1} \frac{\binom{N-M}{n-x}}{\binom{N-1}{n-1}} = n \cdot \frac{M}{N} \sum_{y=0}^{n-1} \binom{M-1}{y} \frac{\binom{N-1-(M-1)}{n-1-y}}{\binom{N-1}{n-1}} =$$

$$n \cdot \frac{M}{N} \sum_{y=0}^{n-1} h(y; n-1, M-1, N-1) = n \cdot \frac{M}{N}.$$

119. Using the hint, $\sum_{\text{all } x}(x-\mu)^2 p(x) \ge \sum_{x:|x-\mu|\ge k\sigma}(x-\mu)^2 p(x) \ge \sum_{x:|x-\mu|\ge k\sigma}(k\sigma)^2 p(x) = k^2\sigma^2 \sum_{x:|x-\mu|\ge k\sigma} p(x)$.
The left-hand side is, by definition, σ^2. On the other hand, the summation on the right-hand side represents $P(|X - \mu| \ge k\sigma)$.
So $\sigma^2 \ge k^2\, \sigma^2 \cdot P(|X - \mu| \ge k\sigma)$, whence $P(|X - \mu| \ge k\sigma) \le 1/k^2$.

120.

a. For [0, 4], $\mu = \int_0^4 e^{2+.6t}dt$ = 123.44; for [2, 6], $\mu = \int_2^6 e^{2+.6t}dt$ = 409.82.

b. $\mu = \int_0^{0.9907} e^{2+.6t}dt$ = 9.9996 ≈ 10, so the desired probability is $F(15; 10) = .951$.

121.

a. Let A_1 = {voice}, A_2 = {data}, and X = duration of a call. Then $E(X) = E(X|A_1)P(A_1) + E(X|A_2)P(A_2)$ = 3(.75) + 1(.25) = 2.5 minutes.

b. Let X = the number of chips in a cookie. Then $E(X) = E(X|i = 1)P(i = 1) + E(X|\ i = 2)P(i = 2) + E(X|\ i = 3)P(i = 3)$. If X is Poisson, then its mean is the specified μ — that is, $E(X|i) = i + 1$. Therefore, $E(X)$ = 2(.20) + 3(.50) + 4(.30) = 3.1 chips.

122. For $x = 1, 2, \ldots, 9$, $p(x) = (1 - p)^{x-1}p$, representing $x - 1$ failed transmissions followed by a success. Otherwise, if the first 9 transmissions fail, then $X = 10$ regardless of the 10th outcome, so $p(10) = (1 - p)^9$. (Obviously, $p(x) = 0$ for x other than 1, 2, … , 10.)

$E(X) = \left[\sum_{x=1}^{9} x(1-p)^{x-1}p\right] + 10\cdot(1-p)^9$, which can be simplified to $\dfrac{1-(1-p)^{10}}{p}$.

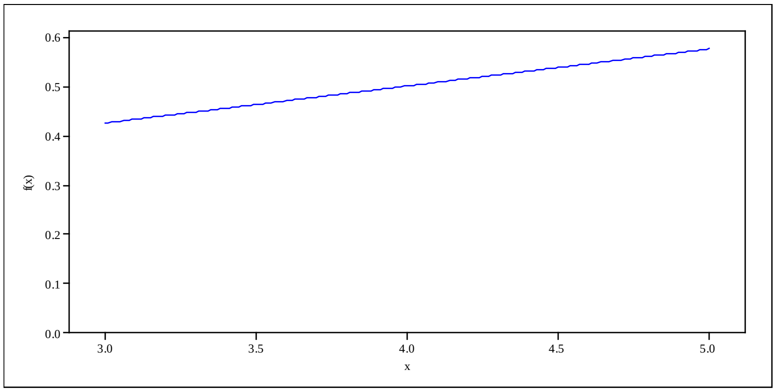
CHAPTER 4

Section 4.1

1.

a. The pdf is the straight-line function graphed below on [3, 5]. The function is clearly non-negative; to verify its integral equals 1, compute:

$$\int_3^5 (.075x + .2)\,dx = .0375x^2 + .2x\Big]_3^5 = (.0375(5)^2 + .2(5)) - (.0375(3)^2 + .2(3))$$
$$= 1.9375 – .9375 = 1$$



b. $P(X \le 4) = \int_3^4 (.075x + .2)\,dx = .0375x^2 + .2x\Big]_3^4 = (.0375(4)^2 + .2(4)) - (.0375(3)^2 + .2(3))$ = 1.4 – .9375 = .4625. Since X is a continuous rv, $P(X < 4) = P(X \le 4) = .4625$ as well.

c. $P(3.5 \le X \le 4.5) = \int_{3.5}^{4.5} (.075x + .2)\,dx = .0375x^2 + .2x\Big]_{3.5}^{4.5} = \cdots = .5$.

$P(4.5 < X) = P(4.5 \le X) = \int_{4.5}^{5} (.075x + .2)\,dx = .0375x^2 + .2x\Big]_{4.5}^{5} = \cdots = .278125$.

2. $f(x) = \frac{1}{10}$ for $-5 \le x \le 5$ and = 0 otherwise

a. $P(X < 0) = \int_{-5}^{0} \frac{1}{10}\,dx = .5$.

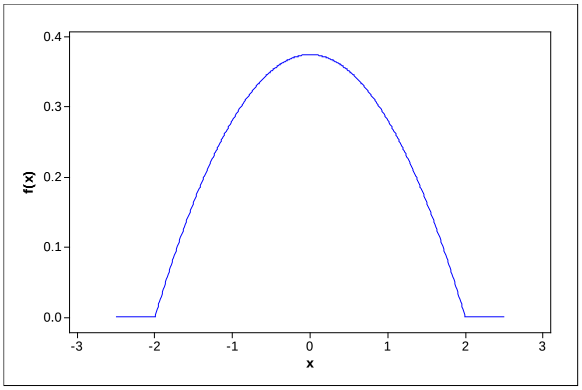
b. $P(-2.5 < X < 2.5) = \int_{-2.5}^{2.5} \frac{1}{10}\,dx = .5$.

c. $P(-2 \le X \le 3) = \int_{-2}^{3} \frac{1}{10}\,dx = .5$.

d. $P(k < X < k + 4) = \int_{k}^{k+4} \frac{1}{10}\,dx = \frac{1}{10}x\Big]_k^{k+4} = \frac{1}{10}[(k+4) - k] = .4$.

3.

a.



b. $P(X > 0) = \int_0^2 .09375(4-x^2)dx = .09375\left(4x - \frac{x^3}{3}\right)\Big]_0^2 = .5$.

This matches the symmetry of the pdf about $x = 0$.

c. $P(-1 < X < 1) = \int_{-1}^{1} .09375(4-x^2)dx = .6875$.

d. $P(X < -.5 \text{ or } X > .5) = 1 - P(-.5 \le X \le .5) = 1 - \int_{-.5}^{.5} .09375(4-x^2)dx = 1 - .3672 = .6328$.

4.

a. $\int_{-\infty}^{\infty} f(x;\theta)dx = \int_0^{\infty} \frac{x}{\theta^2} e^{-x^2/2\theta^2} dx = -e^{-x^2/2\theta^2}\Big]_0^{\infty} = 0 - (-1) = 1$

b. $P(X \le 200) = \int_{-\infty}^{200} f(x;\theta)dx = \int_0^{200} \frac{x}{\theta^2} e^{-x^2/2\theta^2} dx = -e^{-x^2/2\theta^2}\Big]_0^{200} \approx -.1353 + 1 = .8647$.

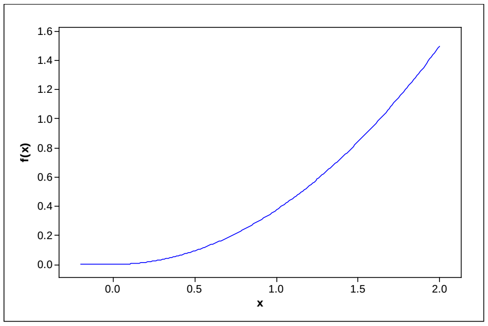
$P(X < 200) = P(X \le 200) \approx .8647$, since X is continuous.
$P(X \ge 200) = 1 - P(X < 200) \approx .1353$.

c. $P(100 \le X \le 200) = \int_{100}^{200} f(x;\theta)dx = -e^{-x^2/20{,}000}\Big]_{100}^{200} \approx .4712$.

d. For $x > 0$, $P(X \le x) = \int_{-\infty}^{x} f(y;\theta)dy = \int_0^x \frac{y}{\theta^2} e^{-y^2/2\theta^2} dx = -e^{-y^2/2\theta^2}\Big]_0^x = 1 - e^{-x^2/2\theta^2}$.

5.

a. $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^2 kx^2 dx = \left.\frac{kx^3}{3}\right]_0^2 = \frac{8k}{3} \Rightarrow k = \frac{3}{8}$.



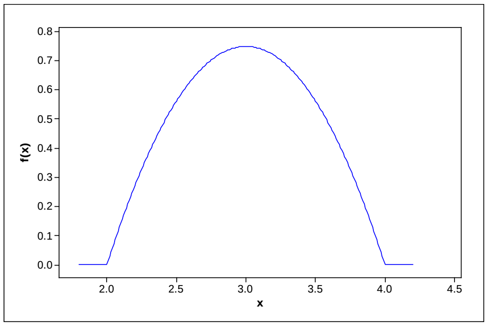
b. $P(0 \le X \le 1) = \int_0^1 \frac{3}{8}x^2 dx = \frac{1}{8}x^3\Big]_0^1 = \frac{1}{8} = .125$.

c. $P(1 \le X \le 1.5) = \int_1^{1.5} \frac{3}{8}x^2 dx = \frac{1}{8}x^3\Big]_1^{1.5} = \frac{1}{8}\left(\frac{3}{2}\right)^3 - \frac{1}{8}(1)^3 = \frac{19}{64} = .296875$.

d. $P(X \ge 1.5) = 1 - \int_{1.5}^2 \frac{3}{8}x^2 dx = \frac{1}{8}x^3\Big]_{1.5}^2 = \frac{1}{8}(2)^3 - \frac{1}{8}(1.5)^3 = .578125$.

6.

a.



b. $1 = \int_2^4 k[1-(x-3)^2]dx = \int_{-1}^1 k[1-u^2]du = \cdots = \frac{4k}{3} \Rightarrow k = \frac{3}{4}$.

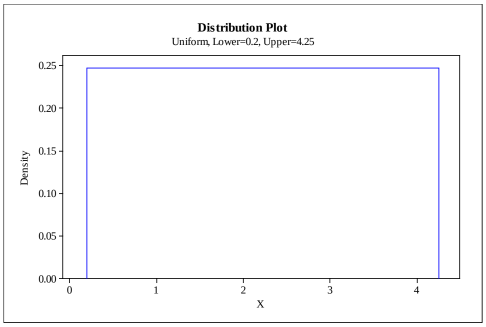
c. $P(X > 3) = \int_3^4 \frac{3}{4}[1-(x-3)^2]dx = .5$. This matches the symmetry of the pdf about $x = 3$.

d. $P\left(\frac{11}{4} \le X \le \frac{13}{4}\right) = \int_{11/4}^{13/4} \frac{3}{4}[1-(x-3)^2]dx = \frac{3}{4}\int_{-1/4}^{1/4}[1-u^2]du = \frac{47}{128} \approx .367$.

e. $P(|X - 3| > .5) = 1 - P(|X - 3| \le .5) = 1 - P(2.5 \le X \le 3.5) =$
$1 - \int_{-.5}^{.5} \frac{3}{4}[1-u^2]du = \cdots = 1 - .6875 = .3125$.

7.

a. $f(x) = \dfrac{1}{B-A} = \dfrac{1}{4.25-.20} = \dfrac{1}{4.05}$ for $.20 \le x \le 4.25$ and = 0 otherwise.



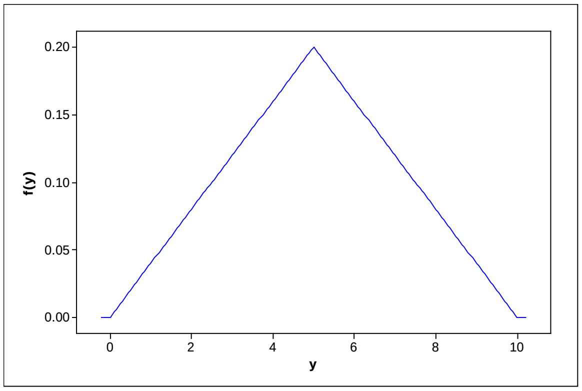
b. $P(X > 3) = \int_3^{4.25} \frac{1}{4.05}\,dx = \frac{1.25}{4.05} = .309.$

c. $P(\mu - 1 \le X \le \mu + 1) = \int_{\mu-1}^{\mu+1} \frac{1}{4.05}\,dx = \frac{2}{4.05}$ = .494. (We don't actually need to know μ here, but it's clearly the midpoint of 2.225 mm by symmetry.)

d. $P(a \le X \le a + 1) = \int_a^{a+1} \frac{1}{4.05}\,dx = \frac{1}{4.05} = .247.$

8.

a.



b. $\int_{-\infty}^{\infty} f(y)dy = \int_0^5 \frac{1}{25}y dy + \int_5^{10} (\frac{2}{5} - \frac{1}{25}y)dy = \left.\frac{y^2}{50}\right]_0^5 + \left.\left(\frac{2}{5}y - \frac{1}{50}y^2\right)\right]_5^{10} =$

$\frac{25}{50} + \left[(4-2) - \left(2 - \frac{1}{2}\right)\right] = \frac{1}{2} + \frac{1}{2} = 1$

c. $P(Y \le 3) = \int_0^3 \frac{1}{25}y\,dy = \left.\frac{y^2}{50}\right]_0^3 = \frac{9}{50} = .18.$

d. $P(Y \le 8) = \int_0^5 \frac{1}{25}y\,dy + \int_5^8 (\frac{2}{5} - \frac{1}{25}y)dy = \frac{23}{25} = .92.$

e. Use parts **c** and **d:** $P(3 \le Y \le 8) = P(Y \le 8) - P(Y < 3) = .92 - .18 = .74$.

f. $P(Y < 2 \text{ or } Y > 6) = \; = \int_0^2 \frac{1}{25} y\, dy + \int_6^{10} (\frac{2}{5} - \frac{1}{25} y) dy = \cdots = .4$.

9.

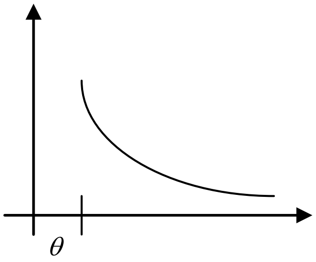
a. $P(X \le 5) = \int_1^5 .15 e^{-.15(x-1)} dx = .15 \int_0^4 e^{-.15u} du$ (after the substitution $u = x - 1$)

$= -e^{-.15u}\Big]_0^4 = 1 - e^{-.6} \approx .451$. $P(X > 5) = 1 - P(X \le 5) = 1 - .451 = .549$.

b. $P(2 \le X \le 5) = \int_2^5 .15 e^{-.15(x-1)}\, dx = \int_1^4 .15 e^{-.15u}\, du = -e^{-.15u}\Big]_1^4 = .312$.

10.

a. The pdf is a decreasing function of x, beginning at $x = \theta$.



b. $\int_{-\infty}^{\infty} f(x;k,\theta)\, dx = \int_\theta^\infty \frac{k\theta^k}{x^{k+1}} dx = k\theta^k \int_\theta^\infty x^{-k-1}\, dx = \theta^k \cdot (-x^{-k})\Big]_\theta^\infty = 0 - \theta^k \cdot (-\theta^{-k}) = 1$.

c. $P(X \le b) = \int_\theta^b \frac{k\theta^k}{x^{k+1}} dx = -\frac{\theta^k}{x^k}\Bigg]_\theta^b = 1 - \left(\frac{\theta}{b}\right)^k$.

d. $P(a \le X \le b) = \int_a^b \frac{k\theta^k}{x^{k+1}} dx = -\frac{\theta^k}{x^k}\Bigg]_a^b = \left(\frac{\theta}{a}\right)^k - \left(\frac{\theta}{b}\right)^k$.

Section 4.2

11.

a. $P(X \le 1) = F(1) = \frac{1^2}{4} = .25$.

b. $P(.5 \le X \le 1) = F(1) - F(.5) = \frac{1^2}{4} - \frac{.5^2}{4}$ = .1875.

c. $P(X > 1.5) = 1 - P(X \le 1.5) = 1 - F(1.5) = 1 - \frac{1.5^2}{4} =$.4375.

d. $.5 = F(\tilde{\mu}) = \frac{\tilde{\mu}^2}{4} \Rightarrow \tilde{\mu}^2 = 2 \Rightarrow \tilde{\mu} = \sqrt{2} \approx 1.414$.

e. $f(x) = F'(x) = \frac{x}{2}$ for $0 \le x < 2$, and = 0 otherwise.

f. $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^2 x \cdot \frac{x}{2}dx = \frac{1}{2}\int_0^2 x^2 dx = \left.\frac{x^3}{6}\right]_0^2 = \frac{8}{6} \approx 1.333$.

g. $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^2 x^2 \frac{x}{2}dx = \frac{1}{2}\int_0^2 x^3 dx = \left.\frac{x^4}{8}\right]_0^2 = 2$, so $V(X) = E(X^2) - [E(X)]^2 =$

$2 - \left(\frac{8}{6}\right)^2 = \frac{8}{36} \approx .222$, and $\sigma_X = \sqrt{.222}$ = .471.

h. From **g**, $E(X^2) = 2$.

12.

a. $P(X < 0) = F(0) = .5$.

b. $P(-1 \le X \le 1) = F(1) - F(-1) = .6875$.

c. $P(X > .5) = 1 - P(X \le .5) = 1 - F(.5) = 1 - .6836 = .3164$.

d. $f(x) = F'(x) = \frac{d}{dx}\left(\frac{1}{2} + \frac{3}{32}\left(4x - \frac{x^3}{3}\right)\right) = 0 + \frac{3}{32}\left(4 - \frac{3x^2}{3}\right) = .09375\left(4 - x^2\right)$.

e. By definition, $F(\tilde{\mu}) = .5$. $F(0) = .5$ from **a** above, which is as desired.

13.

a. $1 = \int_1^\infty \frac{k}{x^4}dx = k\int_1^\infty x^{-4}dx = \frac{k}{-3}x^{-3}\Big]_1^\infty = 0 - \left(\frac{k}{-3}\right)(1)^{-3} = \frac{k}{3} \Rightarrow k = 3$.

b. For $x \geq 1$, $F(x)$= $\int_{-\infty}^{x} f(y)dy = \int_1^x \frac{3}{y^4}dy = -y^{-3}\Big|_1^x = -x^{-3} + 1 = 1 - \frac{1}{x^3}$. For $x < 1$, $F(x) = 0$ since the distribution begins at 1. Put together, $F(x) = \begin{cases} 0 & x < 1 \\ 1 - \frac{1}{x^3} & 1 \leq x \end{cases}$.

c. $P(X > 2) = 1 - F(2) = 1 - \frac{7}{8} = \frac{1}{8}$ or .125;
$P(2 < X < 3) = F(3) - F(2) = \left(1 - \frac{1}{27}\right) - \left(1 - \frac{1}{8}\right) = .963 - .875 = .088$.

d. The mean is $E(X) = \int_1^\infty x\left(\frac{3}{x^4}\right)dx = \int_1^\infty \left(\frac{3}{x^3}\right)dx = -\frac{3}{2}x^{-2}\Big|_1^\infty = 0 + \frac{3}{2} = \frac{3}{2}$ = 1.5. Next,
$E(X^2) = \int_1^\infty x^2\left(\frac{3}{x^4}\right)dx = \int_1^\infty \left(\frac{3}{x^2}\right)dx = -3x^{-1}\Big|_1^\infty = 0 + 3 = 3$, so $V(X) = 3 - (1.5)^2 = .75$. Finally, the standard deviation of X is $\sigma = \sqrt{.75} = .866$.

e. $P(1.5 - .866 < X < 1.5 + .866) = P(.634 < X < 2.366) = F(2.366) - F(.634) = .9245 - 0 = .9245$.

14.

a. If X is uniformly distributed on the interval from A to B, then $E(X) = \int_A^B x \cdot \frac{1}{B-A}dx = \frac{A+B}{2}$, the midpoint of the interval. Also, $E(X^2) = \frac{A^2 + AB + B^2}{3}$, from which $V(X) = E(X^2) - [E(X)]^2 = \ldots = \frac{(B-A)^2}{12}$.
With $A = 7.5$ and $B = 20$, $E(X) = 13.75$ and $V(X) = 13.02$.

b. From Example 4.6, the complete cdf is $F(x) = \begin{cases} 0 & x < 7.5 \\ \frac{x - 7.5}{12.5} & 7.5 \leq x < 20 \\ 1 & 20 \leq x \end{cases}$.

c. $P(X \leq 10) = F(10) = .200$; $P(10 \leq X \leq 15) = F(15) - F(10) = .4$.

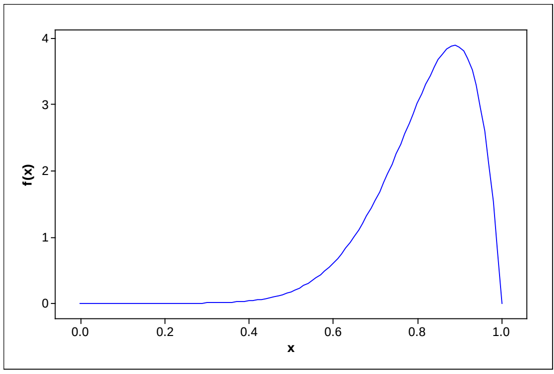
d. $\sigma = \sqrt{13.02} = 3.61$, so $\mu \pm \sigma = (10.14, 17.36)$. Thus, $P(\mu - \sigma \leq X \leq \mu + \sigma) =$
$P(10.14 \leq X \leq 17.36) = F(17.36) - F(10.14) = .5776$.
Similarly, $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = P(6.53 \leq X \leq 20.97) = 1$.

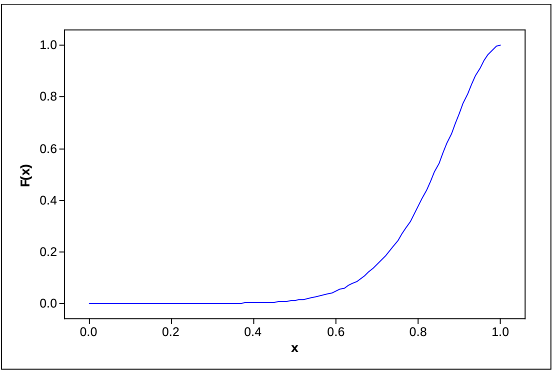
15.

a. Since X is limited to the interval (0, 1), $F(x) = 0$ for $x \le 0$ and $F(x) = 1$ for $x \ge 1$.
For $0 < x < 1$,

$$F(x) = \int_{-\infty}^{x} f(y)dy = \int_0^x 90y^8(1-y)dy = \int_0^x (90y^8 - 90y^9)dy = 10y^9 - 9y^{10}\Big]_0^x = 10x^9 - 9x^{10} \ .$$

The graphs of the pdf and cdf of X appear below.





b. $F(.5) = 10(.5)^9 - 9(.5)^{10} = .0107$.

c. $P(.25 < X \le .5) = F(.5) - F(.25) = .0107 - [10(.25)^9 - 9(.25)^{10}] = .0107 - .0000 = .0107$.
Since X is continuous, $P(.25 \le X \le .5) = P(.25 < X \le .5) = .0107$.

d. The 75th percentile is the value of x for which $F(x) = .75$: $10x^9 - 9x^{10} = .75 \Rightarrow x = .9036$ using software.

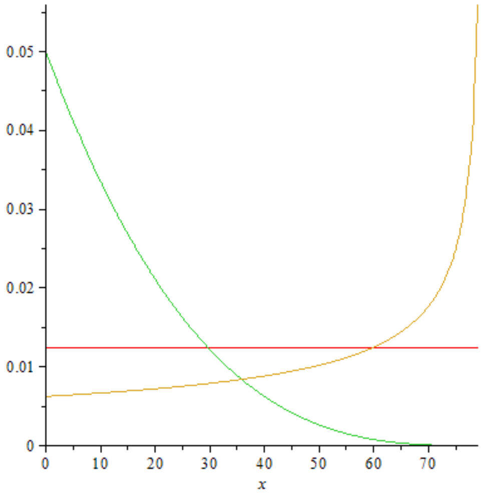
e. $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^1 x \cdot 90x^8(1-x)dx = \int_0^1 (90x^9 - 90x^{10})dx = 9x^{10} - \frac{90}{11}x^{11}\Big]_0^1 = 9 - \frac{90}{11} = \frac{9}{11} = .8182.$

Similarly, $E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx = \int_0^1 x^2 \cdot 90x^8(1-x)dx = \ldots = .6818$, from which $V(X) = .6818 - (.8182)^2 = .0124$ and $\sigma_X = .11134$.

f. $\mu \pm \sigma = (.7068, .9295)$. Thus, $P(\mu - \sigma \le X \le \mu + \sigma) = F(.9295) - F(.7068) = .8465 - .1602 = .6863$, and the probability X is <u>more</u> than 1 standard deviation from its mean value equals $1 - .6863 = 3137$.

16.

a. The graph below shows $f(x; \theta, 80)$ for $\theta = 4$ (green), $\theta = 1$ (red), and $\theta = .5$ (gold). For $\theta > 1$, X has a right-skewed distribution on [0, 80]; for $\theta = 1$, f is constant (i.e., $X \sim \text{Unif}[0, 80]$); and for $\theta < 1$, X has a left-skewed distribution and f has an asymptote as $x \to 80$.



b. For $0 < x < \tau$, $F(x) = \int_0^x \frac{\theta}{\tau}\left(1-\frac{y}{\tau}\right)^{\theta-1} dy$. Make the substitution $u = 1-\frac{y}{\tau}$, from which $dy = -\tau\, du$:

$F(x) = \int_1^{1-x/\tau} \frac{\theta}{\tau} u^{\theta-1} \cdot (-\tau) du = \int_{1-x/\tau}^1 \theta u^{\theta-1} du = u^\theta \Big|_{1-x/\tau}^1 = 1-\left(1-\frac{x}{\tau}\right)^\theta$. Also, $F(x) = 0$ for $x \le 0$ and $F(x) = 1$ for $x \ge \tau$.

c. Set $.5 = F(\eta)$ and solve for η: $.5 = 1-\left(1-\frac{\eta}{\tau}\right)^\theta \Rightarrow \left(1-\frac{\eta}{\tau}\right)^\theta = .5 \Rightarrow \frac{\eta}{\tau} = 1-.5^{1/\theta} \Rightarrow \eta = \tau(1-.5^{1/\theta})$.

d. $P(50 \le X \le 70) = F(70) - F(50) = 1-\left(1-\frac{70}{80}\right)^4 - \left[1-\left(1-\frac{50}{80}\right)^4\right] = \left(\frac{3}{8}\right)^4 - \left(\frac{1}{8}\right)^4 = .0195$.

17.

a. To find the $(100p)$th percentile, set $F(x) = p$ and solve for x:
$\frac{x-A}{B-A} = p \Rightarrow x = A + (B-A)p$.

b. $E(X) = \int_A^B x \cdot \frac{1}{B-A} dx = \frac{A+B}{2}$, the midpoint of the interval. Also, $E(X^2) = \frac{A^2+AB+B^2}{3}$, from which $V(X) = E(X^2) - [E(X)]^2 = \ldots = \frac{(B-A)^2}{12}$. Finally, $\sigma_X = \sqrt{V(X)} = \frac{B-A}{\sqrt{12}}$.

c. $E(X^n) = \int_A^B x^n \cdot \frac{1}{B-A} dx = \frac{1}{B-A} \frac{x^{n+1}}{n+1}\Big]_A^B = \frac{B^{n+1}-A^{n+1}}{(n+1)(B-A)}$.

18. $f(x) = \dfrac{1}{1-(-1)} = \dfrac{1}{2}$ for $-1 \le x \le 1$

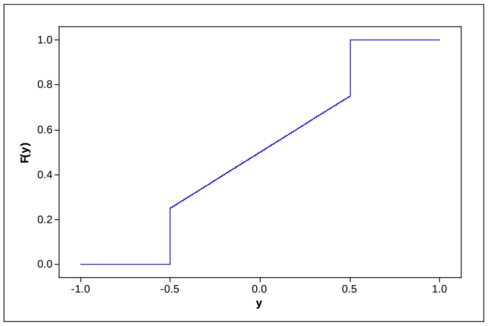
a. $P(Y = .5) = P(X \ge .5) = \int_{.5}^{1} \frac{1}{2}\,dx = .25$.

b. $P(Y = -.5) = .25$ as well, due to symmetry.

For $-.5 < y < .5$, $F(y) = .25 + \int_{-.5}^{y} \frac{1}{2}\,dx = .25 + .5(y + .5) = .5 + .5y$.

Since $Y \le .5$, $F(y) = 1$ for all $y \ge .5$. That is,

$$F(y) = \begin{cases} 0 & y < -.5 \\ .5 + .5y & -.5 \le y < .5 \\ 1 & .5 \le y \end{cases}$$



19.

a. $P(X \le 1) = F(1) = .25[1 + \ln(4)] = .597$.

b. $P(1 \le X \le 3) = F(3) - F(1) = .966 - .597 = .369$.

c. For $x < 0$ or $x > 4$, the pdf is $f(x) = 0$ since X is restricted to (0, 4). For $0 < x < 4$, take the first derivative of the cdf:

$$F(x) = \frac{x}{4}\left[1 + \ln\left(\frac{4}{x}\right)\right] = \frac{1}{4}x + \frac{\ln(4)}{4}x - \frac{1}{4}x\ln(x) \Rightarrow$$

$$f(x) = F'(x) = \frac{1}{4} + \frac{\ln(4)}{4} - \frac{1}{4}\ln(x) - \frac{1}{4}x\frac{1}{x} = \frac{\ln(4)}{4} - \frac{1}{4}\ln(x) = .3466 - .25\ln(x)$$

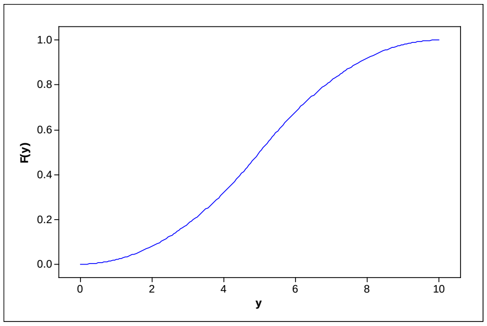
20.

a. For $0 \le y < 5$, $F(y) = \int_0^y \frac{u}{25}du = \frac{y^2}{50}$; for $5 \le y \le 10$,

$$F(y) = \int_0^y f(u)du = \int_0^5 f(u)du + \int_5^y f(u)du = \frac{5^2}{50} + \int_5^y \left(\frac{2}{5} - \frac{u}{25}\right)du = \cdots = -\frac{y^2}{50} + \frac{2}{5}y - 1$$

So, the complete cdf of Y is

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{y^2}{50} & 0 \le y < 5 \\ -\frac{y^2}{50} + \frac{2}{5}y - 1 & 5 \le y < 10 \\ 1 & y \ge 10 \end{cases}$$



b. In general, set $F(y) = p$ and solve for y.

For $0 < p < .5$, $p = F(y) = \frac{y^2}{50} \Rightarrow \eta(p) = y = \sqrt{50p}$; for $.5 \le p < 1$,

$p = -\frac{y^2}{50} + \frac{2}{5}y - 1 \Rightarrow \eta(p) = y = 10 - 5\sqrt{2(1-p)}$.

c. $E(Y) = 5$ by straightforward integration, or by the symmetry of $f(y)$ about $y = 5$.

Similarly, by symmetry $V(Y) = \int_0^{10}(y-5)^2 f(y)\,dy = 2\int_0^5 (y-5)^2 f(y)\,dy = 2\int_0^5 (y-5)^2 \frac{y^2}{50}dy = \ldots = \frac{50}{12} =$ 4.1667. For the waiting time X for a single bus, $E(X) = 2.5$ and $V(X) = \frac{25}{12}$; not coincidentally, the mean and variance of Y are exactly twice that of X.

21. $E(\text{area}) = E(\pi R^2) = \int_{-\infty}^{\infty} \pi r^2 f(r)dr = \int_9^{11} \pi r^2 \frac{3}{4}\left(1-(10-r)^2\right)dr = \cdots = \frac{501}{5}\pi = 314.79 \text{ m}^2$.

22.

a. For $1 \le x \le 2$, $F(x) = \int_1^x 2\left(1 - \frac{1}{y^2}\right)dy = 2\left(y + \frac{1}{y}\right)\Big]_1^x = 2\left(x + \frac{1}{x}\right) - 4$, so the cdf is

$$F(x) = \begin{cases} 0 & x < 1 \\ 2\left(x + \frac{1}{x}\right) - 4 & 1 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

b. Set $F(x) = p$ and solve for x: $2\left(x+\frac{1}{x}\right)-4 = p \Rightarrow 2x^2-(p+4)x+2=0 \Rightarrow$

$\eta(p) = x = \dfrac{(p+4)+\sqrt{(p+4)^2-4(2)(2)}}{2(2)} = \dfrac{p+4+\sqrt{p^2+8p}}{4}$. (The other root of the quadratic gives solutions outside the interval [1, 2].) To find the median $\tilde{\mu}$, set $p = .5$: $\tilde{\mu} = \eta(.5) = \ldots = 1.640$.

c. $E(X) = \int_1^2 x\cdot 2\left(1-\frac{1}{x^2}\right)dx = 2\int_1^2\left(x-\frac{1}{x}\right)dx = 2\left(\frac{x^2}{2}-\ln(x)\right)\Bigg]_1^2 = 1.614$. Similarly,

$E(X^2) = 2\int_1^2\left(x^2-1\right)dx = 2\left(\frac{x^3}{3}-x\right)\Bigg]_1^2 = \frac{8}{3} \Rightarrow V(X) = .0626.$

d. The amount left is given by $h(x) = \max(1.5 - x, 0)$, so

$E(h(X)) = \int_1^2 \max(1.5-x,0)\,f(x)dx = 2\int_1^{1.5}(1.5-x)\left(1-\frac{1}{x^2}\right)dx = .061$.

23. With X = temperature in °C, the temperature in °F equals $1.8X + 32$, so the mean and standard deviation in °F are $1.8\mu_X + 32 = 1.8(120) + 32 = 248$°F and $|1.8|\sigma_X = 1.8(2) = 3.6$°F. Notice that the additive constant, 32, affects the mean but does <u>not</u> affect the standard deviation.

24.

a. $E(X) = \int_\theta^\infty x\cdot\frac{k\theta^k}{x^{k+1}}dx = k\theta^k\int_\theta^\infty\frac{1}{x^k}dx = \frac{k\theta^k x^{-k+1}}{-k+1}\Bigg]_\theta^\infty = \frac{k\theta}{k-1}.$

b. If we attempt to substitute $k = 1$ into the previous answer, we get an undefined expression. More precisely, $\lim_{k\to 1^+} E(X) = \infty$.

c. $E(X^2) = k\theta^k\int_\theta^\infty\frac{1}{x^{k-1}}dx = \frac{k\theta^2}{k-2}$, so $V(X) = \left(\frac{k\theta^2}{k-2}\right)-\left(\frac{k\theta}{k-1}\right)^2 = \frac{k\theta^2}{(k-2)(k-1)^2}$.

d. Using the expression above, $V(X) = \infty$ since $E(X^2) = \infty$ if $k = 2$.

e. $E(X^n) = k\theta^k\int_\theta^\infty x^{n-(k+1)}dx$, which will be finite iff $n - (k+1) < -1$, i.e. if $n < k$.

25.

a. $P(Y \le 1.8\tilde{\mu} + 32) = P(1.8X + 32 \le 1.8\tilde{\mu} + 32) = P(X \le \tilde{\mu}) = .5$ since $\tilde{\mu}$ is the median of X. This shows that $1.8\tilde{\mu} + 32$ is the median of Y.

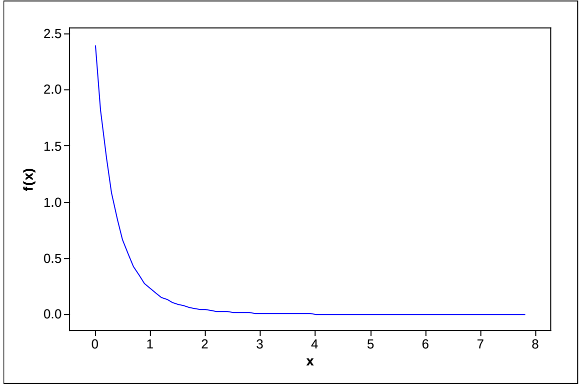
b. The 90th percentile for Y equals $1.8\eta(.9) + 32$, where $\eta(.9)$ is the 90th percentile for X. To see this, $P(Y \le 1.8\eta(.9) + 32) = P(1.8X + 32 \le 1.8\eta(.9) + 32) = P(X \le \eta(.9)) = .9$, since $\eta(.9)$ is the 90th percentile of X. This shows that $1.8\eta(.9) + 32$ is the 90th percentile of Y.

c. When $Y = aX + b$ (i.e. a linear transformation of X) and the (100p)th percentile of the X distribution is $\eta(p)$, then the corresponding (100p)th percentile of the Y distribution is $a\cdot\eta(p) + b$. This can be demonstrated using the same technique as in **a** and **b** above.

26.

a. $1 = \int_0^\infty k(1+x/2.5)^{-7}\,dx = \frac{2.5k}{-6}(1+x/2.5)^{-6}\Big|_0^\infty = \ldots = \frac{k}{2.4} \Rightarrow k = 2.4.$

b. The graph decreases at a pace comparable to x^{-7} as $x \to \infty$.



c. $E(X) = \int_0^\infty x\cdot 2.4(1+x/2.5)^{-7}\,dx$. Let $u = 1 + x/2.5$, so $x = 2.5(u - 1)$ and $dx = 2.5\,du$. The integral becomes $\int_1^\infty 2.5(u-1)\cdot 2.4u^{-7}\cdot 2.5du = 15\int_1^\infty (u^{-6} - u^{-7})\,du = \ldots = 0.5$, or \$500. Similarly, $E(X^2) =$ $\int_0^\infty x^2\cdot 2.4(1+x/2.5)^{-7}\,dx = \int_1^\infty (2.5(u-1))^2\cdot 2.4u^{-7}\cdot 2.5du = 0.625$, so $V(X) = 0.625 - (0.5)^2 = 0.375$ and $\sigma_X = \sqrt{0.375} = 0.612$, or \$612.

d. The maximum out-of-pocket expense, \$2500, occurs when \$500 + 20%(X – \$500) equals \$2500; this accounts for the \$500 deductible and the 20% of costs above \$500 not paid by the insurance plan. Solve: \$2,500 = \$500 + 20%(X – \$500) ⇒ X = \$10,500. At that point, the insurance plan has already paid \$8,000, and the plan will pay all expenses thereafter.

Recall that the units on X are <u>thousands</u> of dollars. If Y denotes the expenses paid by the company (also in \$1000s), $Y = 0$ for $X \le 0.5$; $Y = .8(X - 0.5)$ for $0.5 \le X \le 10.5$; and $Y = (X - 10.5) + 8$ for $X > 10.5$. From this,

$$E(Y) = \int_0^\infty y\cdot 2.4(1+x/2.5)^{-7}\,dx = \int_0^{0.5} 0\,dx + \int_{0.5}^{10.5} .8(x-0.5)\cdot 2.4(1+x/2.5)^{-7}\,dx +$$
$$\int_{10.5}^\infty (x-10.5)\cdot 2.4(1+x/2.5)^{-7}\,dx = 0 + .16024 + .00013 = .16037, \text{ or } \$160.37.$$

27. Since X is uniform on [0, 360], $E(X) = \frac{0+360}{2} = 180°$ and $\sigma_X = \frac{360-0}{\sqrt{12}} = 103.82°$. Using the suggested linear representation of Y, $E(Y) = (2\pi/360)\mu_X - \pi = (2\pi/360)(180) - \pi = 0$ radians, and $\sigma_Y = (2\pi/360)\sigma_X =$ 1.814 radians. (In fact, Y is uniform on $[-\pi, \pi]$.)

Section 4.3

28.

a. $P(0 \le Z \le 2.17) = \Phi(2.17) - \Phi(0) = .4850$.

b. $\Phi(1) - \Phi(0) = .3413$.

c. $\Phi(0) - \Phi(-2.50) = .4938$.

d. $\Phi(2.50) - \Phi(-2.50) = .9876$.

e. $\Phi(1.37) = .9147$.

f. $P(-1.75 < Z) + [1 - P(Z < -1.75)] = 1 - \Phi(-1.75) = .9599$.

g. $\Phi(2) - \Phi(-1.50) = .9104$.

h. $\Phi(2.50) - \Phi(1.37) = .0791$.

i. $1 - \Phi(1.50) = .0668$.

j. $P(|Z| \le 2.50) = P(-2.50 \le Z \le 2.50) = \Phi(2.50) - \Phi(-2.50) = .9876$.

29.

a. .9838 is found in the 2.1 row and the .04 column of the standard normal table so $c = 2.14$.

b. $P(0 \le Z \le c) = .291 \Rightarrow \Phi(c) - \Phi(0) = .2910 \Rightarrow \Phi(c) - .5 = .2910 \Rightarrow \Phi(c) = .7910 \Rightarrow$ from the standard normal table, $c = .81$.

c. $P(c \le Z) = .121 \Rightarrow 1 - P(Z < c) = .121 \Rightarrow 1 - \Phi(c) = .121 \Rightarrow \Phi(c) = .879 \Rightarrow c = 1.17$.

d. $P(-c \le Z \le c) = \Phi(c) - \Phi(-c) = \Phi(c) - (1 - \Phi(c)) = 2\Phi(c) - 1 = .668 \Rightarrow \Phi(c) = .834 \Rightarrow$ $c = 0.97$.

e. $P(c \le |Z|) = 1 - P(|Z| < c) = 1 - [\Phi(c) - \Phi(-c)] = 1 - [2\Phi(c) - 1] = 2 - 2\Phi(c) = .016 \Rightarrow \Phi(c) = .992 \Rightarrow$ $c = 2.41$.

30.

a. $\Phi(c) = .9100 \Rightarrow c \approx 1.34$, since .9099 is the entry in the 1.3 row, .04 column.

b. Since the standard normal distribution is symmetric about $z = 0$, the 9th percentile = –[the 91st percentile] = –1.34.

c. $\Phi(c) = .7500 \Rightarrow c \approx .675$, since .7486 and .7517 are in the .67 and .68 entries, respectively.

d. Since the standard normal distribution is symmetric about $z = 0$, the 25th percentile = –[the 75th percentile] = –.675.

e. $\Phi(c) = .06 \Rightarrow c \approx -1.555$, since .0594 and .0606 appear as the –1.56 and –1.55 entries, respectively.

31. By definition, z_α satisfies $\alpha = P(Z \geq z_\alpha) = 1 - P(Z < z_\alpha) = 1 - \Phi(z_\alpha)$, or $\Phi(z_\alpha) = 1 - \alpha$.

a. $\Phi(z_{.0055}) = 1 - .0055 = .9945 \Rightarrow z_{.0055} = 2.54$.

b. $\Phi(z_{.09}) = .91 \Rightarrow z_{.09} \approx 1.34$.

c. $\Phi(z_{.663}) = .337 \Rightarrow z_{.633} \approx -.42$.

32.

a. $P(X \leq 15) = P\left(Z \leq \frac{15-15.0}{1.25}\right) = P(Z \leq 0) = \Phi(0.00) = .5000$.

b. $P(X \leq 17.5) = P\left(Z \leq \frac{17.5-15.0}{1.25}\right) = P(Z \leq 2) = \Phi(2.00) = .9772$.

c. $P(X \geq 10) = P\left(Z \geq \frac{10-15.0}{1.25}\right) = P(Z \geq -4) = 1 - \Phi(-4.00) = 1 - .0000 = 1$.

d. $P(14 \leq X \leq 18) = P\left(\frac{14-15.0}{1.25} \leq Z \leq \frac{18-15.0}{1.25}\right) = P(-.8 \leq Z \leq 2.4) = \Phi(2.40) - \Phi(-0.80) = .9918 - .2119 = .7799$.

e. $P(|X - 15| \leq 3) = P(-3 \leq X - 15 \leq 3) = P(12 \leq X \leq 18) = P(-2.4 \leq Z \leq 2.4) =$
$\Phi(2.40) - \Phi(-2.40) = .9918 - .0082 = .9836$.

33.

a. $P(X \leq 50) = P\left(Z \leq \frac{50-46.8}{1.75}\right) = P(Z \leq 1.83) = \Phi(1.83) = .9664$.

b. $P(X \geq 48) = P\left(Z \geq \frac{48-46.8}{1.75}\right) = P(Z \geq 0.69) = 1 - \Phi(0.69) = 1 - .7549 = .2451$.

c. The mean and standard deviation aren't important here. The probability a normal random variable is within 1.5 standard deviations of its mean equals $P(-1.5 \leq Z \leq 1.5) =$
$\Phi(1.5) - \Phi(-1.5) = .9332 - .0668 = .8664$.

34. $\mu = .30$, $\sigma = .06$

a. $P(X > .50) = P(Z > 3.33) = 1 - \Phi(3.33) = 1 - .9996 = .0004$.

b. $P(X \leq .20) = \Phi(-0.50) = .3085$.

c. We want the 95th percentile, c, of this normal distribution, so that 5% of the values are higher. The 95th percentile of the standard normal distribution satisfies $\Phi(z) = .95$, which from the normal table yields $z = 1.645$.

So, $c = .30 + (1.645)(.06) = .3987$. The largest 5% of all concentration values are above .3987 mg/cm^3.

35. $\mu = 8.46$ min, $\sigma = 0.913$ min

a. $P(X \geq 10) = P(Z \geq 1.69) = 1 - \Phi(1.69) = 1 - .9545 = .0455$.
Since X is continuous, $P(X > 10) = P(X \geq 10) = .0455$.

b. $P(X > 15) = P(Z > 7.16) \approx 0$.

c. $P(8 \leq X \leq 10) = P(-0.50 \leq Z \leq 1.69) = \Phi(1.69) - \Phi(-0.50) = .9545 - .3085 = .6460$.

d. $P(8.46 - c \leq X \leq 8.46 + c) = .98$, so $8.46 - c$ and $8.46 + c$ are at the 1st and the 99th percentile of the given distribution, respectively. The 99th percentile of the standard normal distribution satisfies $\Phi(z) = .99$, which corresponds to $z = 2.33$.
So, $8.46 + c = \mu + 2.33\sigma = 8.46 + 2.33(0.913) \Rightarrow c = 2.33(0.913) = 2.13$.

e. From **a**, $P(X > 10) = .0455$ and $P(X \leq 10) = .9545$. For four independent selections,
P(at least one haul time exceeds 10) = 1 – P(none of the four exceeds 10) =
1 – P(first doesn't $\cap$ … fourth doesn't) = 1 – (.9545)(.9545)(.9545)(.9545) by independence =
$1 - (.9545)^4 = .1700$.

36.

a. $P(X < 1500) = P(Z < 3) = \Phi(3) = .9987$; $P(X \geq 1000) = P(Z \geq -.33) = 1 - \Phi(-.33) = 1 - .3707 = .6293$.

b. $P(1000 < X < 1500) = P(-.33 < Z < 3) = \Phi(3) - \Phi(-.33) = .9987 - .3707 = .6280$

c. From the table, $\Phi(z) = .02 \Rightarrow z = -2.05 \Rightarrow x = 1050 - 2.05(150) = 742.5$ μm. The smallest 2% of droplets are those smaller than 742.5 μm in size.

d. Let Y = the number of droplets, out of 5, that exceed 1500 μm. Then Y is binomial, with $n = 5$ and $p = .0013$ from **a**. So, $P(Y = 2) = \binom{5}{2}(.0013)^2(.9987)^3 \approx 1.68 \times 10^{-5}$.

37.

a. $P(X = 105) = 0$, since the normal distribution is continuous;
$P(X < 105) = P(Z < 0.2) = P(Z \leq 0.2) = \Phi(0.2) = .5793$;
$P(X \leq 105) = .5793$ as well, since X is continuous.

b. No, the answer does not depend on μ or σ. For any normal rv, $P(|X - \mu| > \sigma) = P(|Z| > 1) =$
$P(Z < -1 \text{ or } Z > 1) = 2P(Z < -1)$ by symmetry $= 2\Phi(-1) = 2(.1587) = .3174$.

c. From the table, $\Phi(z) = .1\% = .001 \Rightarrow z = -3.09 \Rightarrow x = 104 - 3.09(5) = 88.55$ mmol/L. The smallest .1% of chloride concentration values are those less than 88.55 mmol/L

38. Let X denote the diameter of a randomly selected cork made by the first machine, and let Y be defined analogously for the second machine.
$P(2.9 \leq X \leq 3.1) = P(-1.00 \leq Z \leq 1.00) = .6826$, while
$P(2.9 \leq Y \leq 3.1) = P(-7.00 \leq Z \leq 3.00) = .9987$. So, the second machine wins handily.

39. μ = 30 mm, σ = 7.8 mm

a. $P(X \le 20) = P(Z \le -1.28) = .1003$. Since X is continuous, $P(X < 20) = .1003$ as well.

b. Set $\Phi(z) = .75$ to find $z \approx 0.67$. That is, 0.67 is roughly the 75th percentile of a <u>standard</u> normal distribution. Thus, the 75th percentile of X's distribution is $\mu + 0.67\sigma = 30 + 0.67(7.8) = 35.226$ mm.

c. Similarly, $\Phi(z) = .15 \Rightarrow z \approx -1.04 \Rightarrow \eta(.15) = 30 - 1.04(7.8) = 21.888$ mm.

d. The values in question are the 10th and 90th percentiles of the distribution (in order to have 80% in the middle). Mimicking **b** and **c**, $\Phi(z) = .1 \Rightarrow z \approx -1.28$ & $\Phi(z) = .9 \Rightarrow z \approx +1.28$, so the 10th and 90th percentiles are $30 \pm 1.28(7.8) = 20.016$ mm and 39.984 mm.

40.

a. $P(X < 40) = P\left(Z \le \dfrac{40-43}{4.5}\right) = P(Z < -0.667) = .2514$.

$P(X > 60) = P\left(Z > \dfrac{60-43}{4.5}\right) = P(Z > 3.778) \approx 0$.

b. We desire the 25th percentile. Since the 25th percentile of a standard normal distribution is roughly $z = -0.67$, the answer is $43 + (-0.67)(4.5) = 39.985$ ksi.

41. For a single drop, $P(\text{damage}) = P(X < 100) = P\left(Z < \dfrac{100-200}{30}\right) = P(Z < -3.33) = .0004$. So, the probability of <u>no</u> damage on any single drop is $1 - .0004 = .9996$, and
$P(\text{at least one among five is damaged}) = 1 - P(\text{none damaged}) = 1 - (.9996)^5 = 1 - .998 = .002$.

42. The probability X is within .1 of its mean is given by $P(\mu - .1 \le X \le \mu + .1) =$
$P\left(\dfrac{(\mu-.1)-\mu}{\sigma} < Z < \dfrac{(\mu+.1)-\mu}{\sigma}\right) = \Phi\left(\dfrac{.1}{\sigma}\right) - \Phi\left(-\dfrac{.1}{\sigma}\right) = 2\Phi\left(\dfrac{.1}{\sigma}\right) - 1$. If we require this to equal 95%, we find $2\Phi\left(\dfrac{.1}{\sigma}\right) - 1 = .95 \Rightarrow \Phi\left(\dfrac{.1}{\sigma}\right) = .975 \Rightarrow \dfrac{.1}{\sigma} = 1.96$ from the standard normal table. Thus, $\sigma = \dfrac{.1}{1.96} = .0510$.
Alternatively, use the empirical rule: 95% of all values lie within 2 standard deviations of the mean, so we want $2\sigma = .1$, or $\sigma = .05$. (This is not quite as precise as the first answer.)

43.

a. Let μ and σ denote the unknown mean and standard deviation. The given information provides
$.05 = P(X < 39.12) = \Phi\left(\dfrac{39.12-\mu}{\sigma}\right) \Rightarrow \dfrac{39.12-\mu}{\sigma} \approx -1.645 \Rightarrow 39.12 - \mu = -1.645\sigma$ and
$.10 = P(X > 73.24) = 1 - \Phi\left(\dfrac{73.24-\mu}{\sigma}\right) \Rightarrow \dfrac{73.24-\mu}{\sigma} = \Phi^{-1}(.9) \approx 1.28 \Rightarrow 73.24 - \mu = 1.28\sigma$.
Subtract the top equation from the bottom one to get $34.12 = 2.925\sigma$, or $\sigma \approx 11.665$ mph. Then, substitute back into either equation to get $\mu \approx 58.309$ mph.

b. $P(50 \le X \le 65) = \Phi(.57) - \Phi(-.72) = .7157 - .2358 = .4799$.

c. $P(X > 70) = 1 - \Phi(1.00) = 1 - .8413 = .1587$.

44.

a. $P(\mu - 1.5\sigma \le X \le \mu + 1.5\sigma) = P(-1.5 \le Z \le 1.5) = \Phi(1.50) - \Phi(-1.50) = .8664$.

b. $P(X < \mu - 2.5\sigma \text{ or } X > \mu + 2.5\sigma) = 1 - P(\mu - 2.5\sigma \le X \le \mu + 2.5\sigma)$
$= 1 - P(-2.5 \le Z \le 2.5) = 1 - .9876 = .0124$.

c. $P(\mu - 2\sigma \le X \le \mu - \sigma \text{ or } \mu + \sigma \le X \le \mu + 2\sigma) = P(\text{within 2 sd's}) - P(\text{within 1 sd}) =$
$P(\mu - 2\sigma \le X \le \mu + 2\sigma) - P(\mu - \sigma \le X \le \mu + \sigma) = .9544 - .6826 = .2718$.

45. With $\mu = .500$ inches, the acceptable range for the diameter is between .496 and .504 inches, so unacceptable bearings will have diameters smaller than .496 or larger than .504.
The new distribution has $\mu = .499$ and $\sigma = .002$.

$$P(X < .496 \text{ or } X > .504) = P\left(Z < \frac{.496 - .499}{.002}\right) + P\left(Z > \frac{.504 - .499}{.002}\right) = P(Z < -1.5) + P(Z > 2.5) =$$

$\Phi(-1.5) + [1 - \Phi(2.5)] = .073$. 7.3% of the bearings will be unacceptable.

46.

a. $P(67 < X < 75) = P\left(\frac{67-70}{3} < \frac{X-70}{3} < \frac{75-70}{3}\right) = P(-1 < Z < 1.67) = \Phi(1.67) - \Phi(-1) = .9525 -$
$.1587 = .7938$.

b. By the Empirical Rule, c should equal 2 standard deviations. Since $\sigma = 3$, $c = 2(3) = 6$.
We can be a little more precise, as in Exercise 42, and use $c = 1.96(3) = 5.88$.

c. Let Y = the number of acceptable specimens out of 10, so $Y \sim \text{Bin}(10, p)$, where $p = .7938$ from part **a**.
Then $E(Y) = np = 10(.7938) = 7.938$ specimens.

d. Now let Y = the number of specimens out of 10 that have a hardness of less than 73.84, so
$Y \sim \text{Bin}(10, p)$, where

$$p = P(X < 73.84) = P\left(Z < \frac{73.84 - 70}{3}\right) = P(Z < 1.28) = \Phi(1.28) = .8997. \text{ Then}$$

$$P(Y \le 8) = \sum_{y=0}^{8} \binom{10}{y} (.8997)^y (.1003)^{10-y} = .2651.$$

You can also compute $1 - P(Y = 9, 10)$ and use the binomial formula, or round slightly to $p = .9$ and use the binomial table: $P(Y \le 8) = B(8; 10, .9) = .265$.

47. The stated condition implies that 99% of the area under the normal curve with $\mu = 12$ and $\sigma = 3.5$ is to the left of $c - 1$, so $c - 1$ is the 99th percentile of the distribution. Since the 99th percentile of the standard normal distribution is $z = 2.33$, $c - 1 = \mu + 2.33\sigma = 20.155$, and $c = 21.155$.

48.

a. By symmetry, $P(-1.72 \le Z \le -.55) = P(.55 \le Z \le 1.72) = \Phi(1.72) - \Phi(.55)$.

b. $P(-1.72 \le Z \le .55) = \Phi(.55) - \Phi(-1.72) = \Phi(.55) - [1 - \Phi(1.72)]$.

No, thanks to the symmetry of the z curve about 0.

49.

a. $P(X > 4000) = P\left(Z > \frac{4000-3432}{482}\right) = P(Z > 1.18) = 1-\Phi(1.18) = 1-.8810 = .1190$;

$P(3000 < X < 4000) = P\left(\frac{3000-3432}{482} < Z < \frac{4000-3432}{482}\right) = \Phi(1.18) – \Phi(–.90) = .8810 – .1841 = .6969.$

b. $P(X < 2000 \text{ or } X > 5000) = P\left(Z < \frac{2000-3432}{482}\right) + P\left(Z > \frac{5000-3432}{482}\right)$
$= \Phi(-2.97) + [1-\Phi(3.25)] = .0015 + .0006 = .0021$.

c. We will use the conversion 1 lb = 454 g, then 7 lbs = 3178 grams, and we wish to find
$P(X > 3178) = P\left(Z > \frac{3178-3432}{482}\right) = 1-\Phi(-.53) = .7019$.

d. We need the top .0005 and the bottom .0005 of the distribution. Using the *z* table, both .9995 and .0005 have multiple *z* values, so we will use a middle value, ±3.295. Then 3432 ± 3.295(482) = 1844 and 5020. The most extreme .1% of all birth weights are less than 1844 g and more than 5020 g.

e. Converting to pounds yields a mean of 7.5595 lbs and a standard deviation of 1.0608 lbs.
Then $P(X > 7) = P\left(Z > \frac{7-7.5595}{1.0608}\right) = 1-\Phi(-.53) = .7019$. This yields the same answer as in part **c**.

50. We use a normal approximation to the binomial distribution: Let *X* denote the number of people in the sample of 1000 who <u>can</u> taste the difference, so $X \sim$ Bin(1000, .03). Because $\mu = np = 1000(.03) = 30$ and $\sigma = \sqrt{np(1-p)} = 5.394$, *X* is approximately *N*(30, 5.394).

a. Using a continuity correction, $P(X \geq 40) = 1-P(X \leq 39) = 1-P\left(Z \leq \frac{39.5-30}{5.394}\right) =$
$1 – P(Z \leq 1.76) = 1 – \Phi(1.76) = 1 – .9608 = .0392.$

b. 5% of 1000 is 50, and $P(X \leq 50) = P\left(Z \leq \frac{50.5-30}{5.394}\right) = \Phi(3.80) \approx 1$.

51. $P(|X – \mu| \geq \sigma) = 1 – P(|X – \mu| < \sigma) = 1 – P(\mu – \sigma < X < \mu + \sigma) = 1 – P(–1 \leq Z \leq 1) = .3174.$
Similarly, $P(|X – \mu| \geq 2\sigma) = 1 – P(–2 \leq Z \leq 2) = .0456$ and $P(|X – \mu| \geq 3\sigma) = .0026$.
These are considerably less than the bounds 1, .25, and .11 given by Chebyshev.

52.

a. $P(20 \leq X \leq 30) = P(20 – .5 \leq X \leq 30 + .5) = P(19.5 \leq X \leq 30.5) = P(–1.1 \leq Z \leq 1.1) = .7286.$

b. $P(X \leq 30) = P(X \leq 30.5) = P(Z \leq 1.1) = .8643$, while
$P(X < 30) = P(X \leq 29.5) = P(Z < .9) = .8159.$

53. $p = .5 \Rightarrow \mu = 12.5$ & $\sigma^2 = 6.25$; $p = .6 \Rightarrow \mu = 15$ & $\sigma^2 = 6$; $p = .8 \Rightarrow \mu = 20$ and $\sigma^2 = 4$. These mean and standard deviation values are used for the normal calculations below.

a. For the binomial calculation, $P(15 \le X \le 20) = B(20; 25, p) - B(14; 25, p)$.

p	$P(15 \le X \le 20)$	$P(14.5 \le \text{Normal} \le 20.5)$
.5	= .212	$= P(.80 \le Z \le 3.20) = .2112$
.6	= .577	$= P(-.20 \le Z \le 2.24) = .5668$
.8	= .573	$= P(-2.75 \le Z \le .25) = .5957$

b. For the binomial calculation, $P(X \le 15) = B(15; 25, p)$.

p	$P(X \le 15)$	$P(\text{Normal} \le 15.5)$
.5	= .885	$= P(Z \le 1.20) = .8849$
.6	= .575	$= P(Z \le .20) = .5793$
.8	= .017	$= P(Z \le -2.25) = .0122$

c. For the binomial calculation, $P(X \ge 20) = 1 - B(19; 25, p)$.

p	$P(X \ge 20)$	$P(\text{Normal} \ge 19.5)$
.5	= .002	$= P(Z \ge 2.80) = .0026$
.6	= .029	$= P(Z \ge 1.84) = .0329$
.8	= .617	$= P(Z \ge -0.25) = .5987$

54. Use the normal approximation to the binomial, with a continuity correction. With $p = .10$ and $n = 200$, $\mu = np = 20$, and $\sigma^2 = npq = 18$. So, Bin(200, .10) ≈ $N(20, \sqrt{18})$.

a. $P(X \le 30) = \Phi\left(\frac{(30+.5)-20}{\sqrt{18}}\right) = \Phi(2.47) = .9932$.

b. $P(X < 30) = P(X \le 29) = \Phi\left(\frac{(29+.5)-20}{\sqrt{18}}\right) = \Phi(2.24) = .9875$.

c. $P(15 \le X \le 25) = P(X \le 25) - P(X \le 14) = \Phi\left(\frac{(25+.5)-20}{\sqrt{18}}\right) - \Phi\left(\frac{(14+.5)-20}{\sqrt{18}}\right)$

$= \Phi(1.30) - \Phi(-1.30) = .9032 - .0968 = .8064$.

55. Use the normal approximation to the binomial, with a continuity correction. With $p = .75$ and $n = 500$, $\mu = np = 375$, and $\sigma = 9.68$. So, Bin(500, .75) ≈ $N(375, 9.68)$.

a. $P(360 \le X \le 400) = P(359.5 \le X \le 400.5) = P(-1.60 \le Z \le 2.58) = \Phi(2.58) - \Phi(-1.60) = .9409$.

b. $P(X < 400) = P(X \le 399.5) = P(Z \le 2.53) = \Phi(2.53) = .9943$.

56. Let z_{1-p} denote the (100p)th percentile of a standard normal distribution. The claim is the (100p)th percentile of a $N(\mu, \sigma)$ distribution is $\mu + z_{1-p}\sigma$. To verify this,

$P(X \le \mu + z_{1-p}\sigma) = P\left(\frac{X-\mu}{\sigma} \le z_{1-p}\right) = P\left(Z \le z_{1-p}\right) = p$ by definition of z_{1-p}. That establishes $\mu + z_{1-p}\sigma$ as the (100p)th percentile.

57.

a. For any $a > 0$, $F_Y(y) = P(Y \le y) = P(aX + b \le y) = P\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$. This, in turn, implies

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\left(\frac{y-b}{a}\right) = \frac{1}{a}f_X\left(\frac{y-b}{a}\right).$$

Now let X have a normal distribution. Applying this rule,

$$f_Y(y) = \frac{1}{a}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{((y-b)/a-\mu)^2}{2\sigma^2}\right) = \frac{1}{\sqrt{2\pi}a\sigma}\exp\left(-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}\right)$$

. This is the pdf of a normal distribution. In particular, from the exponent we can read that the mean of Y is $E(Y) = a\mu + b$ and the variance of Y is $V(Y) = a^2\sigma^2$. These match the usual rescaling formulas for mean and variance. (The same result holds when $a < 0$.)

b. Temperature in °F would also be normal, with a mean of 1.8(115) + 32 = 239°F and a variance of $1.8^2 2^2 = 12.96$ (i.e., a standard deviation of 3.6°F).

58.

a. $P(Z \ge 1) \approx .5 \cdot \exp\left(\frac{83+351+562}{703+165}\right) = .1587$, which matches $1 - \Phi(1)$.

b. $P(Z < -3) = P(Z > 3) \approx .5 \cdot \exp\left(\frac{-2362}{399.3333}\right) = .0013$, which matches $\Phi(-3)$.

c. $P(Z > 4) \approx .5 \cdot \exp\left(\frac{-3294}{340.75}\right) = .0000317$, so $P(-4 < Z < 4) = 1 - 2P(Z \ge 4) \approx$ $1 - 2(.0000317) = .999937$.

d. $P(Z > 5) \approx .5 \cdot \exp\left(\frac{-4392}{305.6}\right) = .00000029$.

Section 4.4

59.

a. $E(X) = \frac{1}{\lambda} = 1$.

b. $\sigma = \frac{1}{\lambda} = 1$.

c. $P(X \le 4) = 1 - e^{-(1)(4)} = 1 - e^{-4} = .982$.

d. $P(2 \le X \le 5) = (1 - e^{-(1)(5)}) - (1 - e^{-(1)(2)}) = e^{-2} - e^{-5} = .129$.

60.

a. $P(X \le 100) = 1 - e^{-(100)(.01386)} = 1 - e^{-1.386} = .7499$.
$P(X \le 200) = 1 - e^{-(200)(.01386)} = 1 - e^{-2.772} = .9375$.
$P(100 \le X \le 200) = P(X \le 200) - P(X \le 100) = .9375 - .7499 = .1876$.

b. First, since X is exponential, $\mu = \frac{1}{\lambda} = \frac{1}{.01386} = 72.15$, $\sigma = 72.15$. Then
$P(X > \mu + 2\sigma) = P(X > 72.15 + 2(72.15)) = P(X > 216.45) = 1 - (1 - e^{-.01386(216.45)}) = e^{-3} = .0498$.

c. Remember the median is the solution to $F(x) = .5$. Use the formula for the exponential cdf and solve for x: $F(x) = 1 - e^{-.01386x} = .5 \Rightarrow e^{-.01386x} = .5 \Rightarrow -.01386x = \ln(.5) \Rightarrow x = -\frac{\ln(.5)}{.01386} = 50.01$ m.

61. Note that a mean value of 2.725 for the exponential distribution implies $\lambda = \frac{1}{2.725}$. Let X denote the duration of a rainfall event.

a. $P(X \ge 2) = 1 - P(X < 2) = 1 - P(X \le 2) = 1 - F(2; \lambda) = 1 - [1 - e^{-(1/2.725)(2)}] = e^{-2/2.725} = .4800$;
$P(X \le 3) = F(3; \lambda) = 1 - e^{-(1/2.725)(3)} = .6674$; $P(2 \le X \le 3) = .6674 - .4800 = .1874$.

b. For this exponential distribution, $\sigma = \mu = 2.725$, so $P(X > \mu + 2\sigma) =$
$P(X > 2.725 + 2(2.725)) = P(X > 8.175) = 1 - F(8.175; \lambda) = e^{-(1/2.725)(8.175)} = e^{-3} = .0498$.
On the other hand, $P(X < \mu - \sigma) = P(X < 2.725 - 2.725) = P(X < 0) = 0$, since an exponential random variable is non-negative.

62.

a. Clearly $E(X) = 0$ by symmetry, so $V(X) = E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda|x|} dx \ = \ \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx \ = \ \lambda \cdot \frac{\Gamma(3)}{\lambda^3} = \frac{2}{\lambda^2}$.

Solving $\frac{2}{\lambda^2} = V(X) = (40.9)^2$ yields $\lambda = 0.034577$.

b. $P(|X - 0| \le 40.9) = \int_{-40.9}^{40.9} \frac{\lambda}{2} e^{-\lambda|x|} dx = \int_0^{40.9} \lambda e^{-\lambda x} dx \ = 1 - e^{-40.9\,\lambda} = .75688$.

63.

a. If a customer's calls are typically short, the first calling plan makes more sense. If a customer's calls are somewhat longer, then the second plan makes more sense, viz. 99¢ is less than 20min(10¢/min) = \$2 for the first 20 minutes under the first (flat-rate) plan.

b. $h_1(X) = 10X$, while $h_2(X) = 99$ for $X \le 20$ and $99 + 10(X - 20)$ for $X > 20$. With $\mu = 1/\lambda$ for the exponential distribution, it's obvious that $E[h_1(X)] = 10E[X] = 10\mu$. On the other hand,

$E[h_2(X)] = 99 + 10\int_{20}^{\infty} (x - 20)\lambda e^{-\lambda x} dx = 99 + \frac{10}{\lambda} e^{-20\lambda} \ = 99 + 10\mu e^{-20/\mu}$.

When $\mu = 10$, $E[h_1(X)] = 100¢ = \$1.00$ while $E[h_2(X)] = 99 + 100e^{-2} \approx \$1.13$.

When $\mu = 15$, $E[h_1(X)] = 150¢ = \$1.50$ while $E[h_2(X)] = 99 + 150e^{-4/3} \approx \$1.39$.

As predicted, the first plan is better when expected call length is lower, and the second plan is better when expected call length is somewhat higher.

64.

a. $\Gamma(6) = 5! = 120$.

b. $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \left(\frac{3}{4}\right)\sqrt{\pi} \approx 1.329$.

c. $F(4; 5) = .371$ from row 4, column 5 of Table A.4.
$F(5; 4) = .735$ from row 5, column 4 of Table A.4.

d. $P(X \le 5) = F(5; 7) = .238$.

e. $P(3 < X < 8) = P(X < 8) - P(X \le 3) = F(8; 7) - F(3; 7) = .687 - .034 = .653$.

65.

a. From the mean and sd equations for the gamma distribution, $\alpha\beta = 37.5$ and $\alpha\beta^2 = (21.6)2 = 466.56$. Take the quotient to get $\beta = 466.56/37.5 = 12.4416$. Then, $\alpha = 37.5/\beta = 37.5/12.4416 = 3.01408\ldots$.

b. $P(X > 50) = 1 - P(X \le 50) = 1 - F(50/12.4416; 3.014) = 1 - F(4.0187; 3.014)$. If we approximate this by $1 - F(4; 3)$, Table A.4 gives $1 - .762 = .238$. Software gives the more precise answer of .237.

c. $P(50 \le X \le 75) = F(75/12.4416; 3.014) - F(50/12.4416; 3.014) = F(6.026; 3.014) - F(4.0187; 3.014) \approx F(6; 3) - F(4; 3) = .938 - .762 = .176$.

66.

a. If X has a gamma distribution with parameters α, β, γ, then $Y = X - \gamma$ has a gamma distribution with parameters α and β (i.e., threshold 0). So, write $X = Y + \gamma$, from which $E(X) = E(Y) + \gamma = \alpha\beta + \gamma$ and SD(X) = SD(Y) = $\sqrt{\alpha\beta^2}$. For the given values, $E(X) = 12(7) + 40 = 124$ (10^8 m^3) and SD(X) = $\sqrt{12(7)^2}$ = 24.25 (10^8 m^3).

b. Use the same threshold-shift idea as in part **a**: $P(100 \le X \le 150) = P(60 \le X - 40 \le 110) = P(60 \le Y \le 110) = F\left(\frac{110}{7};12\right) - F\left(\frac{60}{7};12\right)$. To evaluate these functions or the equivalent integrals requires software; the answer is .8582 – .1575 = .7007.

c. $P(X > \mu + \sigma) = P(X > 148.25) = P(X - 40 > 108.25) = P(Y > 108.25) = 1 - F\left(\frac{108.25}{7};12\right)$. From software, the answer is .1559.

d. Set $.95 = P(X \le x) = P(Y \le x - 40) = F\left(\frac{x-40}{7};12\right)$. From software, the 95th percentile of the standard gamma distribution with $\alpha = 12$ is 18.21, so $\frac{x-40}{7}$ = 18.21, or x = 167.47 (10^8 m^3).

67. Notice that $\mu = 24$ and $\sigma^2 = 144 \Rightarrow \alpha\beta = 24$ and $\alpha\beta^2 = 144 \Rightarrow \beta = \frac{144}{24} = 6$ and $\alpha = \frac{24}{\beta} = 4$.

a. $P(12 \le X \le 24) = F(4; 4) - F(2; 4) = .424$.

b. $P(X \le 24) = F(4; 4) = .567$, so while the mean is 24, the median is <u>less</u> than 24, since $P(X \le \tilde{\mu}) = .5$. This is a result of the positive skew of the gamma distribution.

c. We want a value x for which $F\left(\frac{x}{\beta}, \alpha\right) = F\left(\frac{x}{6}, 4\right) = .99$. In Table A.4, we see $F(10; 4) = .990$. So $x/6 = 10$, and the 99th percentile is 6(10) = 60.

d. We want a value t for which $P(X > t) = .005$, i.e. $P(X \le t) = .005$. The left-hand side is the cdf of X, so we really want $F\left(\frac{t}{6}, 4\right) = .995$. In Table A.4, $F(11; 4) = .995$, so $t/6 = 11$ and $t = 6(11) = 66$. At 66 weeks, only .5% of all transistors would still be operating.

68.

a. $E(X) = \alpha\beta = n\frac{1}{\lambda} = \frac{n}{\lambda}$; for $\lambda = .5$ and $n = 10$, $E(X) = 20$.

b. $P(X \le 30) = F(30/2; 10) = F(15; 10) = .930$.

c. $F(x; \lambda, n) = P(X \le t) = P$(the nth event occurs in $[0, t]$) = P(<u>at least</u> n events occur in $[0, t]$) $= P(Y \ge n)$, where Y = the number of events in $[0, t] \sim$ Poisson with parameter λt.

Thus $F(x; \lambda, n) = 1 - P(Y < n) = 1 - P(Y \le n - 1) = 1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$.

69.

a. $\{X \geq t\}$ = {the lifetime of the system is at least t}. Since the components are connected in series, this equals {all 5 lifetimes are at least t} = $A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$.

b. Since the events A_i are assumed to be independent, $P(X \geq t) = P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4) \cdot P(A_5)$. Using the exponential cdf, for any i we have
$P(A_i) = P(\text{component lifetime is} \geq t) = 1 - F(t) = 1 - [1 - e^{-.01t}] = e^{-.01t}$.
Therefore, $P(X \geq t) = (e^{-.01t}) \cdots (e^{-.01t}) = e^{-.05t}$, and $F_X(t) = P(X \leq t) = 1 - e^{-.05t}$.
Taking the derivative, the pdf of X is $f_X(t) = .05e^{-.05t}$ for $t \geq 0$. Thus X also has an exponential distribution, but with parameter $\lambda = .05$.

c. By the same reasoning, $P(X \leq t) = 1 - e^{-n\lambda t}$, so X has an exponential distribution with parameter $n\lambda$.

70. To find the (100p)th percentile, set $F(x) = p$ and solve for x: $p = F(x) = 1 - e^{-\lambda x} \Rightarrow e^{-\lambda x} = 1 - p \Rightarrow$
$-\lambda x = \ln(1-p) \Rightarrow x = -\dfrac{\ln(1-p)}{\lambda}$.

To find the median, set $p = .5$ to get $\tilde{\mu} = -\dfrac{\ln(1-.5)}{\lambda} = \dfrac{.693}{\lambda}$.

71.

a. $\{X^2 \leq y\} = \{-\sqrt{y} \leq X \leq \sqrt{y}\}$.

b. $F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$. To find the pdf of Y, use the identity (Leibniz's rule):

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \cdot \frac{d\sqrt{y}}{dy} - \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2/2} \cdot \frac{d(-\sqrt{y})}{dy}$$
$$= \frac{1}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{1}{2\sqrt{y}} - \frac{1}{\sqrt{2\pi}} e^{-y/2} \cdot \frac{-1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}$$

This is valid for $y > 0$. We recognize this as the chi-squared pdf with $\nu = 1$.

Section 4.5

72.

a. $E(X) = 3\Gamma\left(1+\frac{1}{2}\right) = 3\Gamma\left(\frac{3}{2}\right) = 3\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = 3\cdot\frac{1}{2}\sqrt{\pi} = 2.66$,

$V(X) = 9\left[\Gamma\left(1+\frac{2}{2}\right) - \Gamma^2\left(1+\frac{1}{2}\right)\right] = 9\left[\Gamma(2) - \Gamma^2\left(\frac{3}{2}\right)\right] = 1.926$.

b. $P(X \le\ 6) = 1-e^{-(6/\beta)^{\alpha}} = 1-e^{-(6/3)^2} = 1-e^{-4} = .982$.

c. $P(1.5 \le X \le 6) = (1-e^{-(6/3)^2}) - (1-e^{-(1.5/3)^2}) = e^{-.25} - e^{-4} = .760$.

73.

a. $P(X \le 250) = F(250;\, 2.5,\, 200) = 1-e^{-(250/200)^{2.5}} = 1-e^{-1.75} = .8257$.
$P(X < 250) = P(X \le 250) = .8257$.
$P(X > 300) = 1 - F(300;\, 2.5,\, 200) = e^{-(1.5)^{2.5}} = .0636$.

b. $P(100 \le X \le 250) = F(250;\, 2.5,\, 200) - F(100;\, 2.5,\, 200) = .8257 - .162 = .6637$.

c. The question is asking for the median, $\tilde{\mu}$. Solve $F(\tilde{\mu}) = .5$: $.5 = 1 - e^{-(\tilde{\mu}/200)^{2.5}} \Rightarrow$
$e^{-(\tilde{\mu}/200)^{2.5}} = .5 \Rightarrow (\tilde{\mu}/200)^{2.5} = -\ln(.5) \Rightarrow \tilde{\mu} = 200(-\ln(.5))^{1/2.5} = 172.727$ hours.

74. Let $Y = X - \gamma = X - .5$, so Y has a Weibull distribution with $\alpha = 2.2$ and $\beta = 1.1$.

a. $P(1 < X < 2) = P(.5 < Y < 1.5) = [1 - \exp(-(1.5/1.1)^{2.2})] - [1 - \exp(-(.5/1.1)^{2.2})] = .8617 - .1618 = .7$.

b. $P(X > 1.5) = P(Y > 1) = 1 - P(Y \le 1) = 1 - [1 - \exp(-(1/1.1)^{2.2})] = \exp(-(1/1.1)^{2.2}) = .4445$.

c. First, the 90th percentile of Y's distribution is determined by $.9 = F(y) = 1 - \exp(-(y/1.1)^{2.2}) \Rightarrow$ $\exp(-(y/1.1)^{2.2}) = .1 \Rightarrow y \approx 1.607$. Then, since $Y = X - .5$ (aka $X = Y + .5$), the 90th percentile of X's distribution is $1.607 + .5 = 2.107$ days.

d. The mean and variance of Y are $\mu = 1.1\Gamma\left(1+\frac{1}{2.2}\right) \approx 0.974$ and $\sigma^2 = 1.1^2\left\{\Gamma\left(1+\frac{2}{2.2}\right) - \left[\Gamma\left(1+\frac{1}{2.2}\right)\right]^2\right\}$ ≈ 0.2185. Since $X = Y + .5$, $E(X) = E(Y) + .5 = 0.974 + .5 = 1.474$ days, and $\sigma_X = \sigma_Y = \sqrt{0.2185} \approx$ 0.467 days.

75. Using the substitution $y = \left(\frac{x}{\beta}\right)^{\alpha} = \frac{x^{\alpha}}{\beta^{\alpha}}$. Then $dy = \frac{\alpha x^{\alpha-1}}{\beta^{\alpha}}dx$, and $\mu = \int_0^{\infty} x\cdot\frac{\alpha}{\beta^{\alpha}}x^{\alpha-1}e^{-\left(x/\beta\right)^{\alpha}}dx =$ $\int_0^{\infty}(\beta^{\alpha}y)^{1/\alpha}\cdot e^{-y}dy = \beta\int_0^{\infty}y^{1/\alpha}e^{-y}dy = \beta\cdot\Gamma\left(1+\frac{1}{\alpha}\right)$ by definition of the gamma function.

76.

a. $E(X) = e^{\mu+\sigma^2/2} = e^{1.9+.9^2/2} = e^{2.305} = 10.024$.

$V(X) = \left(e^{2(1.9)+.9^2}\right)\cdot\left(e^{.9^2}-1\right)$ = 125.394 ⇒ σ_X = 11.198.

b. $P(X \leq 10) = \Phi\left(\frac{\ln(10)-1.9}{.9}\right) = \Phi(0.447) \approx .6736$.

$P(5 \leq X \leq 10) = \Phi\left(\frac{\ln(10)-1.9}{.9}\right) - \Phi\left(\frac{\ln(5)-1.9}{.9}\right)$ = .6736 – Φ(–0.32) = .6736 – .3745 = .2991.

77.

a. $E(X) = e^{\mu+\sigma^2/2} = e^{4.82} = 123.97$.

$V(X) = \left(e^{2(4.5)+.8^2}\right)\cdot\left(e^{.8^2}-1\right) = 13{,}776.53 \Rightarrow \sigma = 117.373$.

b. $P(X \leq 100) = \Phi\left(\frac{\ln(100)-4.5}{.8}\right) = \Phi\left(0.13\right) = .5517$.

c. $P(X \geq 200) = 1 - P(X < 200) = 1 - \Phi\left(\frac{\ln(200)-4.5}{.8}\right) = 1 - \Phi\left(1.00\right) = 1 - .8413 = .1587.$ Since X is continuous, $P(X > 200)$ = .1587 as well.

78.

a. $P(X \leq 0.5) = F(0.5) = 1 - \exp[-(0.5/\beta)^{\alpha}] = .3099$.

b. Using a computer, $\Gamma\left(1+\frac{1}{1.817}\right) = \Gamma(1.55) = 0.889$ and $\Gamma\left(1+\frac{2}{1.817}\right) = \Gamma(2.10) = 1.047$. From these we find $\mu = (.863)(0.889) = 0.785$ and $\sigma^2 = (.863)^2[1.047 - 0.889^2] = .1911$, or $\sigma = .437$.
Hence, $P(X > \mu + \sigma) = P(X > 1.222) = 1 - F(1.222) = \exp[-(1.222/\beta)^{\alpha}] = .1524$.

c. Set $F(x) = .5$ and solve for x: $p = F(x) = 1 - e^{-(x/\beta)^{\alpha}} \Rightarrow x = \beta[-\ln(1-p)]^{1/\alpha} =$
$.863(-\ln(1-.5))^{1/1.817} = .7054$.

d. Using the same math as part **c**, $\eta(p) = \beta(-\ln(1-p))^{1/\alpha} = .863(-\ln(1-p))^{1/1.817}$.

79. Notice that μ_X and σ_X are the mean and standard deviation of the lognormal variable X in this example; they are <u>not</u> the parameters μ and σ which usually refer to the mean and standard deviation of ln(X). We're given $\mu_X = 10{,}281$ and $\sigma_X/\mu_X = .40$, from which $\sigma_X = .40\mu_X = 4112.4$.

a. To find the mean and standard deviation of ln(X), set the lognormal mean and variance equal to the appropriate quantities: 10,281 = $E(X) = e^{\mu+\sigma^2/2}$ and $(4112.4)^2 = V(X) = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$. Square the first equation: $(10{,}281)^2 = e^{2\mu+\sigma^2}$. Now divide the variance by this amount:

$$\frac{(4112.4)^2}{(10,281)^2} = \frac{e^{2\mu+\sigma^2}(e^{\sigma^2}-1)}{e^{2\mu+\sigma^2}} \Rightarrow e^{\sigma^2}-1=(.40)^2=.16 \Rightarrow \sigma=\sqrt{\ln(1.16)}=.38525$$

That's the standard deviation of ln(X). Use this in the formula for $E(X)$ to solve for μ:

$10,281 = e^{\mu+(.38525)^2/2} = e^{\mu+.07421} \Rightarrow \mu = 9.164$. That's $E(\ln(X))$.

b. $P(X \le 15{,}000) = P\left(Z \le \frac{\ln(15,000)-9.164}{.38525}\right) = P(Z \le 1.17) = \Phi(1.17) = .8790$.

c. $P(X \ge \mu_X) = P(X \ge 10{,}281) = P\left(Z \ge \frac{\ln(10,281)-9.164}{.38525}\right) = P(Z \ge .19) = 1 - \Phi(0.19) = .4247$. Even though the normal distribution is symmetric, the lognormal distribution is <u>not</u> a symmetric distribution. (See the lognormal graphs in the textbook.) So, the mean and the median of X aren't the same and, in particular, the probability X exceeds its own mean doesn't equal .5.

d. One way to check is to determine whether $P(X < 17{,}000) = .95$; this would mean 17,000 is indeed the 95th percentile. However, we find that $P(X < 17{,}000) = \Phi\left(\frac{\ln(17,000)-9.164}{.38525}\right) = \Phi(1.50) = .9332$, so 17,000 is <u>not</u> the 95th percentile of this distribution (it's the 93.32%ile).

80.

a. $.5 = F(\tilde{\mu}) = \Phi\left(\frac{\ln(\tilde{\mu})-\mu}{\sigma}\right)$, where $\tilde{\mu}$ refers to the median of the lognormal distribution and μ and σ to the mean and sd of the normal distribution. Since $\Phi(0) = .5$, $\frac{\ln(\tilde{\mu})-\mu}{\sigma} = 0$, so $\tilde{\mu} = e^{\mu}$. For the power distribution, $\tilde{\mu} = e^{3.5} = 33.12$.

b. Use the same math: $1 - \alpha = \Phi(z_\alpha) = P(Z \le z_\alpha) = \left(\frac{\ln(X)-\mu}{\sigma} \le z_\alpha\right) = P(\ln(X) \le \mu + \sigma z_\alpha)$

$= P(X \le e^{\mu+\sigma z_\alpha})$, so the $100(1-\alpha)$th percentile is $e^{\mu+\sigma z_\alpha}$. For the power distribution, the 95th percentile is $e^{3.5+(1.645)(1.2)} = e^{5.474} = 238.41$.

81.

a. $V(X) = e^{2(2.05)+.06}(e^{.06}-1) = 3.96 \Rightarrow \text{SD}(X) = 1.99$ months.

b. $P(X > 12) = 1 - P(X \le 12) = 1 - P\left(Z \le \frac{\ln(12)-2.05}{\sqrt{.06}}\right) = 1 - \Phi(1.78) = .0375$.

c. The mean of X is $E(X) = e^{2.05+.06/2} = 8.00$ months, so $P(\mu_X - \sigma_X < X < \mu_X + \sigma_X) = P(6.01 < X < 9.99) = \Phi\left(\frac{\ln(9.99)-2.05}{\sqrt{.06}}\right) - \Phi\left(\frac{\ln(6.01)-2.05}{\sqrt{.06}}\right) = \Phi(1.03) - \Phi(-1.05) = .8485 - .1469 = .7016$.

d. $.5 = F(x) = \Phi\left(\frac{\ln(x)-2.05}{\sqrt{.06}}\right) \Rightarrow \frac{\ln(x)-2.05}{\sqrt{.06}} = \Phi^{-1}(.5) = 0 \Rightarrow \ln(x) - 2.05 = 0 \Rightarrow$ the median is given by $x = e^{2.05} = 7.77$ months.

e. Similarly, $\frac{\ln(\eta_{.99})-2.05}{\sqrt{.06}} = \Phi^{-1}(.99) = 2.33 \Rightarrow \eta_{.99} = e^{2.62} = 13.75$ months.

f. The probability of exceeding 8 months is $P(X > 8) = 1 - \Phi\left(\frac{\ln(8)-2.05}{\sqrt{.06}}\right) = 1 - \Phi(.12) = .4522$, so the expected number that will exceed 8 months out of $n = 10$ is just $10(.4522) = 4.522$.

82.

a. Let $Y = X - \gamma = X - 1.0$, so Y has a (non-shifted) lognormal distribution. The mean and variance of Y are $\mu_Y = \mu_X - 1.0 = 2.16 - 1.0 = 1.16$ and $\sigma_Y = \sigma_X = 1.03$. Using the same algebra as in Exercise 79, $e^{\sigma^2} - 1 = \frac{\sigma_Y^2}{\mu_Y^2} = \frac{1.03^2}{1.16^2} \Rightarrow \sigma = 0.76245$, and $e^{\mu+\sigma^2/2} = \mu_Y = 1.16 \Rightarrow \mu = -0.14225$.

b. Using the information in **a**, $P(X > 2) = P(Y > 1) = 1 - P(Y \le 1) = 1 - \Phi\left(\frac{\ln(1)-\mu}{\sigma}\right) = \Phi\left(\frac{0.14225}{0.76245}\right) = \Phi(.19) = .5753$.

c. From the previous exercise, the median of the distribution of Y is $\eta_Y = e^{\mu} = e^{-0.14225} = .8674$. Since $X = Y + 1$, $\eta_X = \eta_Y + 1 = 1.8674$.

d. First, find the 99th percentile of Y: $.99 = \Phi\left(\frac{\ln(y)-\mu}{\sigma}\right) \Rightarrow \frac{\ln(y)-\mu}{\sigma} = 2.33 \Rightarrow$
$y = e^{\mu+2.33\sigma} = e^{-.14225+2.33(.76245)} = 5.126$. Since $X = Y + 1$, the 99th percentile of X is 6.126.

83. Since the standard beta distribution lies on (0, 1), the point of symmetry must be ½, so we require that $f\left(\tfrac{1}{2}-\mu\right) = f\left(\tfrac{1}{2}+\mu\right)$. Cancelling out the constants, this implies $\left(\tfrac{1}{2}-\mu\right)^{\alpha-1}\left(\tfrac{1}{2}+\mu\right)^{\beta-1} = \left(\tfrac{1}{2}+\mu\right)^{\alpha-1}\left(\tfrac{1}{2}-\mu\right)^{\beta-1}$, which (by matching exponents on both sides) in turn implies that $\alpha = \beta$.

Alternatively, symmetry about ½ requires μ = ½, so $\frac{\alpha}{\alpha+\beta} = .5$. Solving for α gives $\alpha = \beta$.

84.

a. $E(X)$ = $\frac{5}{5+2}=\frac{5}{7}=.714$, $V(X)$ = $\frac{10}{(49)(8)}=.0255$.

b. $f(x)=\frac{\Gamma(7)}{\Gamma(5)\Gamma(2)}\cdot x^4\cdot(1-x)=30\left(x^4-x^5\right)$ for $0 \le x \le 1$, so $P(X \le .2)$ = $\int_0^{.2}30(x^4-x^5)\,dx=.0016$.

c. Similarly, $P(.2 \le X \le .4)$ = $\int_{.2}^{.4}30(x^4-x^5)\,dx=.03936$.

d. If X is the proportion covered by the plant, then $1 - X$ is the proportion not covered by the plant. $E(1 - X) = 1 - E(X) = 1 - \frac{5}{7}=\frac{2}{7}=.286$.

85.

a. Notice from the definition of the standard beta pdf that, since a pdf must integrate to 1,

$$1=\int_0^1\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\,dx \Rightarrow \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Using this, $E(X)$ = $\int_0^1 x\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\,dx=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha}(1-x)^{\beta-1}\,dx=$

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)} = \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\frac{\Gamma(\alpha+\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)}=\frac{\alpha}{\alpha+\beta}.$$

b. Similarly, $E[(1 - X)^m]$ = $\int_0^1(1-x)^m\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\,dx=$

$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\alpha-1}(1-x)^{m+\beta-1}\,dx=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha)\Gamma(m+\beta)}{\Gamma(\alpha+m+\beta)}=\frac{\Gamma(\alpha+\beta)\cdot\Gamma(m+\beta)}{\Gamma(\alpha+m+\beta)\Gamma(\beta)}.$$

If X represents the proportion of a substance consisting of an ingredient, then $1 - X$ represents the proportion not consisting of this ingredient. For $m = 1$ above,

$E(1 - X)$ = $\frac{\Gamma(\alpha+\beta)\cdot\Gamma(1+\beta)}{\Gamma(\alpha+1+\beta)\Gamma(\beta)}=\frac{\Gamma(\alpha+\beta)\cdot\beta\Gamma(\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)\Gamma(\beta)}=\frac{\beta}{\alpha+\beta}$.

86.

a. $E(Y) = 10 \Rightarrow E\left(\frac{Y}{20}\right)=\frac{1}{2}=\frac{\alpha}{\alpha+\beta}$ and $V(Y)$ = $\frac{100}{7}\Rightarrow V\left(\frac{Y}{20}\right)=\frac{100}{7(20)^2}=\frac{1}{28}$ = $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$. After some algebra, the solutions are $\alpha = \beta = 3$.

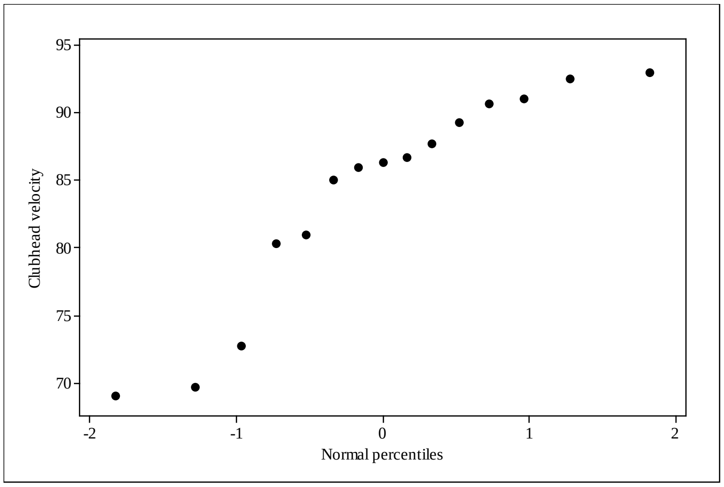
b. $f(y) = 30y^2(1 - y)^2$, so $P(8 \le Y \le 12)$ = $\int_{.4}^{.6}30y^2(1-y)^2\,dy=.365$.

c. We expect it to snap at 10, so $P(Y < 8$ or $Y > 12) = 1 - P(8 \le Y \le 12)$
$= 1 - .365 = .665$.

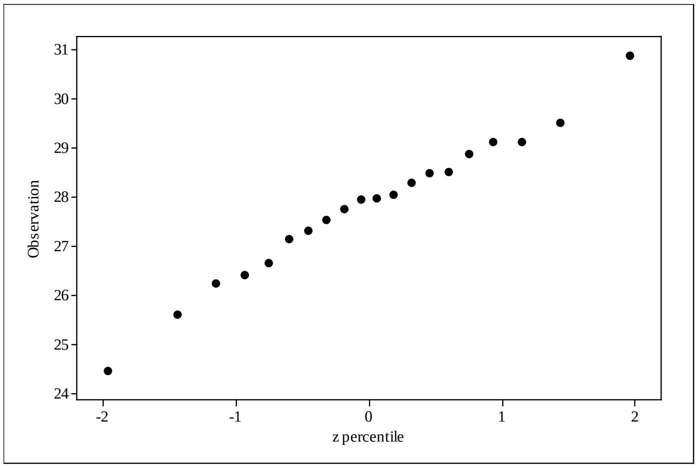
Section 4.6

87. The given probability plot is quite linear, and thus it is quite plausible that the tension distribution is normal.

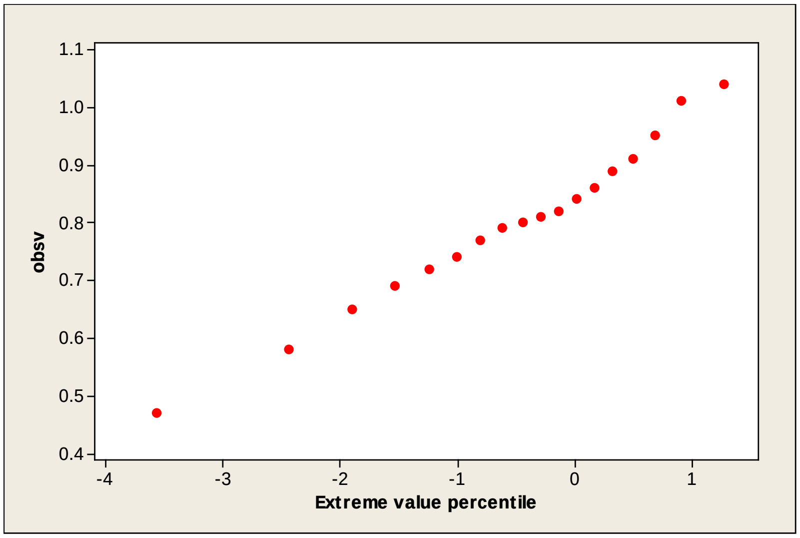
88. The data values and *z* percentiles provided result in the probability plot below. The plot shows some non-trivial departures from linearity, especially in the lower tail of the distribution. This indicates a normal distribution might not be a good fit to the population distribution of clubhead velocities for female golfers.



89. The plot below shows the (observation, *z* percentile) pairs provided. Yes, we would feel comfortable using a normal probability model for this variable, because the normal probability plot exhibits a strong, linear pattern.

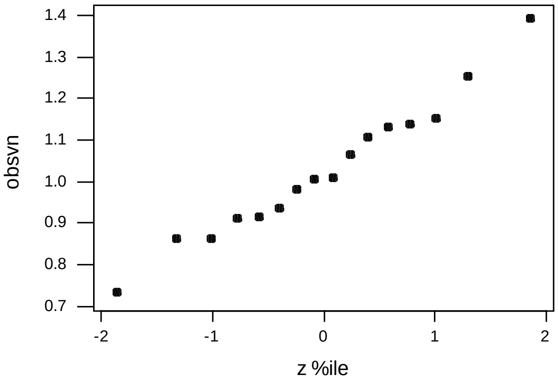


90. The Weibull plot uses ln(observations) and the extreme value percentiles of the p_i values given; i.e., $\eta(p) = \ln[-\ln(1-p)]$. The accompanying probability plot appears sufficiently straight to lead us to agree with the argument that the distribution of fracture toughness in concrete specimens could well be modeled by a Weibull distribution.



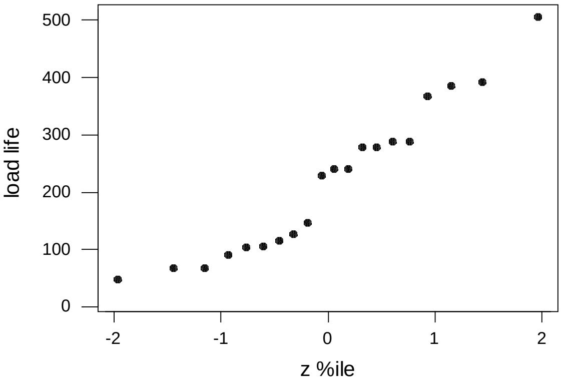
91. The (*z* percentile, observation) pairs are (–1.66, .736), (–1.32, .863), (–1.01, .865), (–.78, .913), (–.58, .915), (–.40, .937), (–.24, .983), (–.08, 1.007), (.08, 1.011), (.24, 1.064), (.40, 1.109), (.58, 1.132), (.78, 1.140), (1.01, 1.153), (1.32, 1.253), (1.86, 1.394).

The accompanying probability plot is straight, suggesting that an assumption of population normality is plausible.

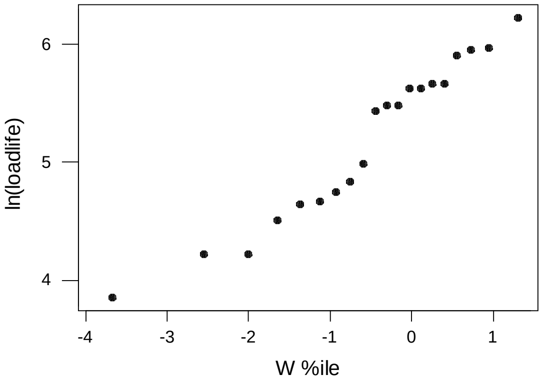


92.

a. The 10 largest *z* percentiles are 1.96, 1.44, 1.15, .93, .76, .60, .45, .32, .19 and .06; the remaining 10 are the negatives of these values. The accompanying normal probability plot is reasonably straight. An assumption of population distribution normality is plausible.

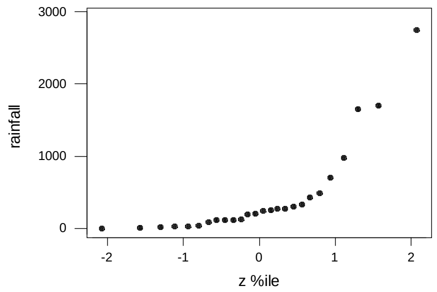


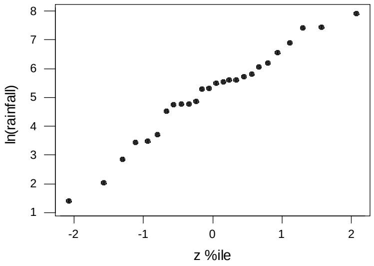
b. For a Weibull probability plot, the natural logs of the observations are plotted against extreme value percentiles; these percentiles are –3.68, –2.55, –2.01, –1.65, –1.37, –1.13, –.93, –.76, –.59, –.44, –.30, –.16, –.02, .12, .26, .40, .56, .73, .95, and 1.31. The accompanying probability plot is roughly as straight as the one for checking normality.



A plot of ln(*x*) versus the *z* percentiles, appropriate for checking the plausibility of a lognormal distribution, is also reasonably straight. Any of 3 different families of population distributions seems plausible.

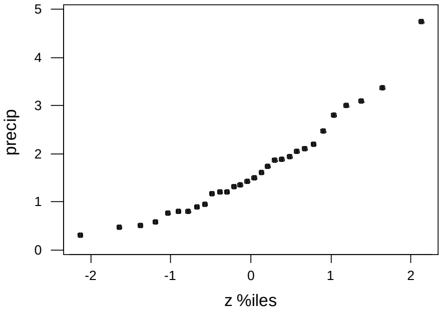
93. To check for plausibility of a lognormal population distribution for the rainfall data of Exercise 81 in Chapter 1, take the natural logs and construct a normal probability plot. This plot and a normal probability plot for the original data appear below. Clearly the log transformation gives quite a straight plot, so lognormality is plausible. The curvature in the plot for the original data implies a positively skewed population distribution — like the lognormal distribution.



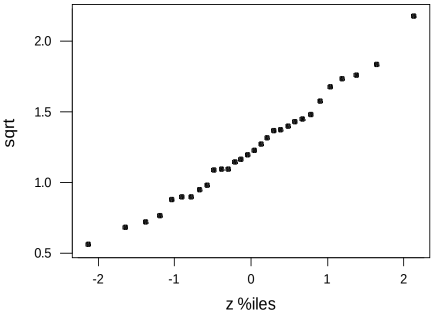


94.

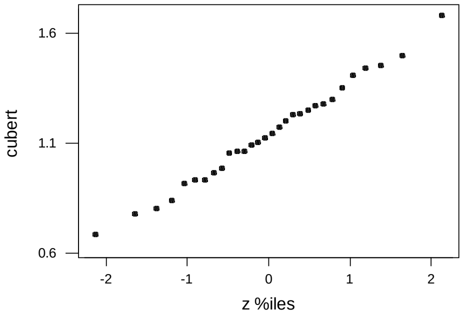
a. The plot of the original (untransformed) data appears somewhat curved.



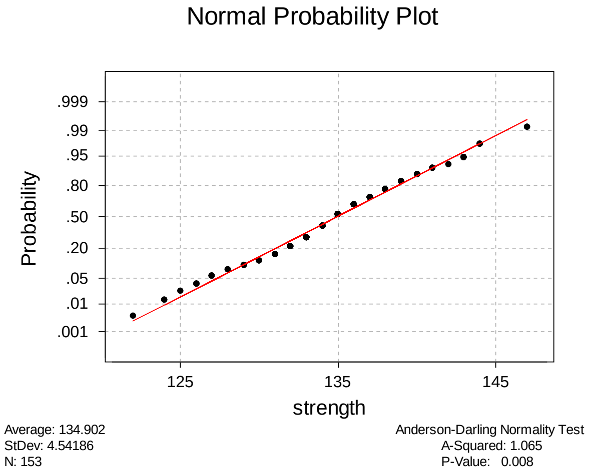
b. The square root transformation results in a very straight plot. It is reasonable that this distribution is normally distributed.



c. The cube root transformation also results in a very straight plot. It is very reasonable that the distribution is normally distributed.

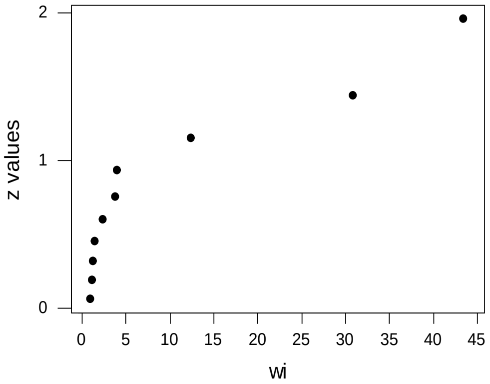


95. The pattern in the plot (below, generated by Minitab) is reasonably linear. By visual inspection alone, it is plausible that strength is normally distributed.



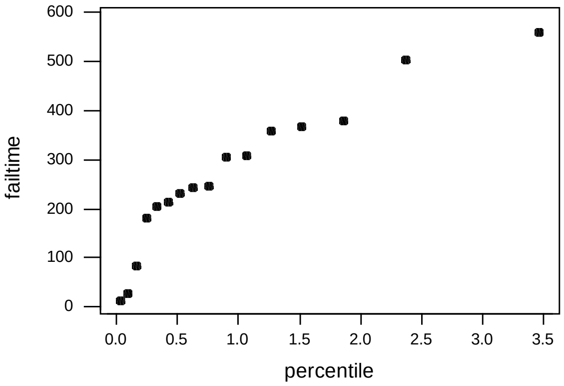
96. We use the data (table below) to create the desired plot. This half–normal plot reveals some extreme values, without which the distribution may appear to be normal.

ordered absolute values (*w*'s)	**probabilities**	***z* values**
0.89	0.525	0.063
1.15	0.575	0.19
1.27	0.625	0.32
1.44	0.675	0.454
2.34	0.725	0.6
3.78	0.775	0.755
3.96	0.825	0.935
12.38	0.875	1.15
30.84	0.925	1.44
43.4	0.975	1.96



97. The $(100p)^{\text{th}}$ percentile $\eta(p)$ for the exponential distribution with $\lambda = 1$ is given by the formula $\eta(p) = -\ln(1 - p)$. With $n = 16$, we need $\eta(p)$ for $p = \frac{0.5}{16}, \frac{1.5}{16}, \ldots, \frac{15.5}{16}$. These are .032, .398, .170, .247, .330, .421, .521, .633, .758, .901, 1.068, 1.269, 1.520, 1.856, 2.367, 3.466.

The accompanying plot of (percentile, failure time value) pairs exhibits substantial curvature, casting doubt on the assumption of an exponential population distribution.



Because λ is a scale parameter (as is σ for the normal family), $\lambda = 1$ can be used to assess the plausibility of the entire exponential family. If we used a different value of λ to find the percentiles, the slope of the graph would change, but not its linearity (or lack thereof).

Supplementary Exercises

98. The pdf of X is $f(x) = \frac{1}{25}$ for $0 \le x \le 25$ and is = 0 otherwise.

a. $P(10 \le X \le 20) = \frac{10}{25} = .4$.

b. $P(X \ge 10) = P(10 \le X \le 25) = \frac{15}{25} = .6$.

c. For $0 \le x \le 25$, $F(x) = \int_0^x \frac{1}{25} dy = \frac{x}{25}$. $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 25$.

d. $E(X) = \frac{A+B}{2} = \frac{0+25}{2} = 12.5$; $V(X) = \frac{(B-A)^2}{12} = \frac{625}{12} = 52.083$, so $\sigma_X = 7.22$.

99.

a. For $0 \le y \le 25$, $F(y) = \frac{1}{24}\int_0^y \left(u - \frac{u^2}{12}\right) du = \frac{1}{24}\left(\frac{u^2}{2} - \frac{u^3}{36}\right)\Bigg]_0^y = \frac{y^2}{48} - \frac{y^3}{864}$. Thus

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{y^2}{48} - \frac{y^3}{864} & 0 \le y \le 12 \\ 1 & y > 12 \end{cases}$$

b. $P(Y \le 4) = F(4) = .259$. $P(Y > 6) = 1 - F(6) = .5$.
$P(4 \le X \le 6) = F(6) - F(4) = .5 - .259 = .241$.

c. $E(Y) = \int_0^{12} y \cdot \frac{1}{24} y\left(1 - \frac{y}{12}\right) dy = \frac{1}{24}\int_0^{12}\left(y^2 - \frac{y^3}{12}\right) dy = \frac{1}{24}\left[\frac{y^3}{3} - \frac{y^4}{48}\right]_0^{12} = 6$ inches.

$E(Y^2) = \frac{1}{24}\int_0^{12}\left(y^3 - \frac{y^4}{12}\right) dy = 43.2$, so $V(Y) = 43.2 - 36 = 7.2$.

d. $P(Y < 4 \text{ or } Y > 8) = 1 - P(4 \le Y \le 8) = 1 - [F(8) - F(4)] = .518$.

e. The shorter segment has length equal to $\min(Y, 12 - Y)$, and

$E[\min(Y, 12 - Y)] = \int_0^{12} \min(y, 12-y) \cdot f(y) dy = \int_0^{6} \min(y, 12-y) \cdot f(y) dy$

$+ \int_6^{12} \min(y, 12-y) \cdot f(y) dy = \int_0^6 y \cdot f(y) dy + \int_6^{12} (12-y) \cdot f(y) dy = \frac{90}{24} = 3.75$ inches.

100.

a. Clearly $f(x) \geq 0$. Now check that the function integrates to 1:

$$\int_0^\infty \frac{32}{(x+4)^3}dx = \int_0^\infty 32(x+4)^{-3}\,dx = -\frac{16}{(x+4)^2}\Bigg|_0^\infty = 0 - -\frac{16}{(0+4)^2} = 1.$$

b. For $x \leq 0$, $F(x) = 0$. For $x > 0$,

$$F(x) = \int_{-\infty}^x f(y)dy = \int_0^x \frac{32}{(y+4)^3}dy = -\frac{1}{2}\cdot\frac{32}{(y+4)^2}\Bigg]_0^x = 1-\frac{16}{(x+4)^2}.$$

c. $P(2 \leq X \leq 5) = F(5) - F(2) = 1-\frac{16}{81}-\left(1-\frac{16}{36}\right)=.247$.

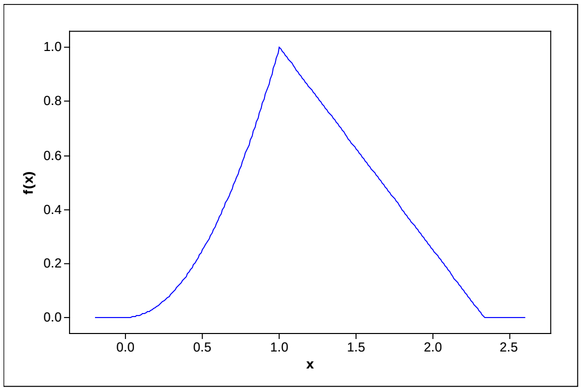
d. $$E(X) = \int_{-\infty}^\infty x\cdot f(x)dx = \int_{-\infty}^\infty x\cdot\frac{32}{(x+4)^3}dx = \int_0^\infty (x+4-4)\cdot\frac{32}{(x+4)^3}dx$$

$$= \int_0^\infty \frac{32}{(x+4)^2}dx - 4\int_0^\infty \frac{32}{(x+4)^3}dx = 8-4=4 \text{ years.}$$

e. $$E\left(\frac{100}{X+4}\right) = \int_0^\infty \frac{100}{x+4}\cdot\frac{32}{(x+4)^3}dx = 3200\int_0^\infty \frac{1}{(x+4)^4}dx = \frac{3200}{(3)(64)} = 16.67\ .$$

101.

a. By differentiation, $f(x) = \begin{cases} x^2 & 0 \leq x < 1 \\ \frac{7}{4}-\frac{3}{4}x & 1 \leq x < \frac{7}{3} \\ 0 & \text{otherwise} \end{cases}$



b. $P(.5 \leq X \leq 2) = F(2) - F(.5) = 1-\frac{1}{2}\left(\frac{7}{3}-2\right)\left(\frac{7}{4}-\frac{3}{4}\cdot 2\right)-\frac{(.5)^3}{3}=\frac{11}{12}=.917.$

c. Using the pdf from **a**, $E(X) = \int_0^1 x\cdot x^2dx + \int_1^{7/3} x\cdot\left(\frac{7}{4}-\frac{3}{4}x\right)dx = \frac{131}{108} = 1.213.$

102. Since we're using a continuous distribution (Weibull) to approximate a discrete variable (number of individuals), a continuity correction is in order. With $\alpha = 10$ and $\beta = 20$,
$P(15 \le X \le 20) = P(14.5 < X < 20.5)$ continuity correction
$= F(20.5) - F(14.5) = [1 - \exp(-(20.5/20)^{10})] - [1 - \exp(-(14.5/20)^{10})] = .7720 - .0393 = .7327$.

103.

a. $P(X > 135) = 1 - \Phi\left(\frac{135-137.2}{1.6}\right) = 1 - \Phi(-1.38) = 1 - .0838 = .9162$.

b. With Y = the number among ten that contain more than 135 oz, $Y \sim \text{Bin}(10, .9162)$.
So, $P(Y \ge 8) = b(8; 10, .9162) + b(9; 10, .9162) + b(10; 10, .9162) = .9549$.

c. We want $P(X > 135) = .95$, i.e. $1 - \Phi\left(\frac{135-137.2}{\sigma}\right) = .95$ or $\Phi\left(\frac{135-137.2}{\sigma}\right) = .05$. From the standard normal table, $\frac{135-137.2}{\sigma} = -1.65 \Rightarrow \sigma = 1.33$.

104.

a. Let X = the number of defectives in the batch of 250, so $X \sim \text{Bin}(250, .05)$. We can approximate X by a normal distribution, since $np = 12.5 \ge 10$ and $nq = 237.5 \ge 10$. The mean and sd of X are $\mu = np = 12.5$ and $\sigma = 3.446$. Using a continuity correction and realizing 10% of 250 is 25,
$P(X \ge 25) = 1 - P(X < 25) = 1 - P(X \le 24.5) \approx 1 - \Phi\left(\frac{24.5-12.5}{3.446}\right) = 1 - \Phi(3.48) =$
$1 - .9997 = .0003$. (The exact binomial probability, from software, is .00086.)

b. Using the same normal approximation with a continuity correction, $P(X = 10) =$
$P(9.5 \le X \le 10.5) \approx \Phi\left(\frac{10.5-12.5}{3.446}\right) - \Phi\left(\frac{9.5-12.5}{3.446}\right) = \Phi(-.58) - \Phi(-.87) = .2810 - .1922 = .0888$.
(The exact binomial probability is $\binom{250}{10}(.05)^{10}(.95)^{240} = .0963$.)

105. Let A = the cork is acceptable and B = the first machine was used. The goal is to find $P(B \mid A)$, which can be obtained from Bayes' rule:

$$P(B \mid A) = \frac{P(B)P(A \mid B)}{P(B)P(A \mid B) + P(B')P(A \mid B')} = \frac{.6P(A \mid B)}{.6P(A \mid B) + .4P(A \mid B')} .$$

From Exercise 38, $P(A \mid B)$ = P(machine 1 produces an acceptable cork) = .6826 and $P(A \mid B')$ = P(machine 2 produces an acceptable cork) = .9987. Therefore,

$$P(B \mid A) = \frac{.6(.6826)}{.6(.6826) + .4(.9987)} = .5062.$$

106.

a. $F(x) = 0$ for $x < 1$ and $F(x) = 1$ for $x > 3$. For $1 \le x \le 3$, $F(x) = \int_1^x \frac{3}{2} \cdot \frac{1}{y^2}\,dy = 1.5\left(1 - \frac{1}{x}\right)$.

b. $P(X \le 2.5) = F(2.5) = 1.5(1 - .4) = .9$; $P(1.5 \le X \le 2.5) = F(2.5) - F(1.5) = .4$.

c. $E(X) = = \int_1^3 x \cdot \frac{3}{2} \cdot \frac{1}{x^2}\,dx = \frac{3}{2}\int_1^3 \frac{1}{x}\,dx = 1.5\ln(x)\Big]_1^3 = 1.648.$

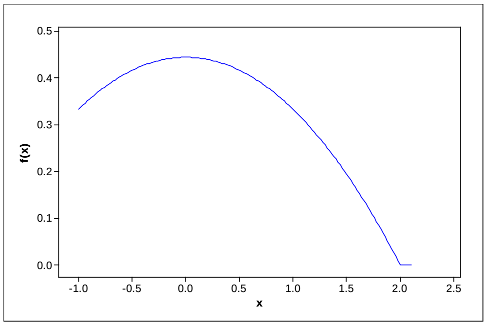
d. $E(X^2) = = \int_1^3 x^2 \cdot \frac{3}{2} \cdot \frac{1}{x^2}\,dx = \frac{3}{2}\int_1^3 dx = 3$, so $V(X) = E(X^2) - [E(X)]^2 = .284$ and $\sigma = .553$.

e. From the description, $h(x) = 0$ if $1 \le x \le 1.5$; $h(x) = x - 1.5$ if $1.5 \le x \le 2.5$ (one second later), and $h(x) = 1$ if $2.5 \le x \le 3$. Using those terms,

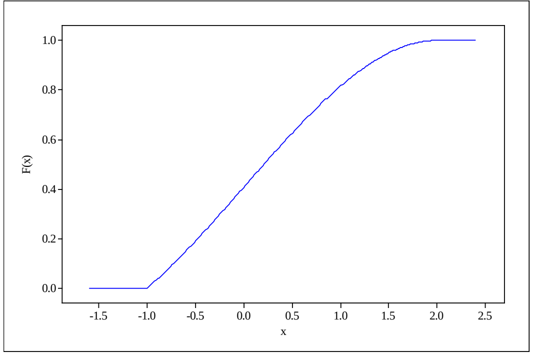
$$E[h(X)] = \int_1^3 h(x)\,dx = \int_{1.5}^{2.5} (x - 1.5) \cdot \frac{3}{2} \cdot \frac{1}{x^2}\,dx + \int_{2.5}^3 1 \cdot \frac{3}{2} \cdot \frac{1}{x^2}\,dx = .267 .$$

107.

a.



b. $F(x) = 0$ for $x < -1$, and $F(x) = 1$ for $x > 2$. For $-1 \le x \le 2$, $F(x)=\int_{-1}^{x}\frac{1}{9}(4-y^2)dy=\frac{11+12x-x^3}{27}$. This is graphed below.



c. The median is 0 iff $F(0) = .5$. Since $F(0) = \frac{11}{27}$, this is not the case. Because $\frac{11}{27} < .5$, the median must be greater than 0. (Looking at the pdf in **a** it's clear that the line $x = 0$ does not evenly divide the distribution, and that such a line must lie to the right of $x = 0$.)

d. Y is a binomial rv, with $n = 10$ and $p = P(X > 1) = 1 - F(1) = \frac{5}{27}$.

108.

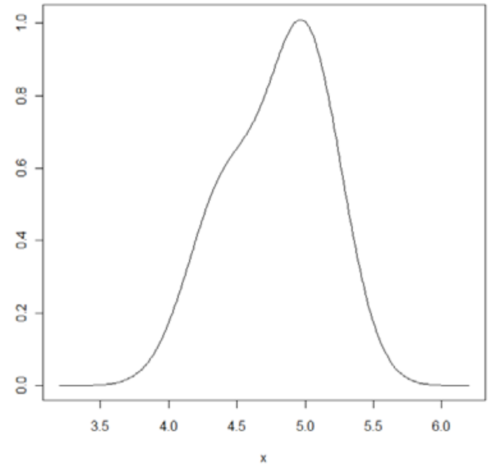
a. The expected value is just a weighted average: $E(X) = p\mu_1 + (1 - p)\mu_2 = .35(4.4) + .65(5.0) = 4.79$ μm. (This can be shown rigorously using a similar "distribution of integrals" technique as we'll see in **b**.)

b. Using the hint,

$$P(4.4 < X < 5.0) = \int_{4.4}^{5.0} f(x)dx = \int_{4.4}^{5.0}[pf_1(x;\mu_1,\sigma)+(1-p)f_2(x;\mu_2,\sigma)]dx$$
$$= p\int_{4.4}^{5.0} f_1(x;\mu_1,\sigma)dx+(1-p)\int_{4.4}^{5.0} f_2(x;\mu_2,\sigma)dx$$
$$= pP(4.4 < X_1 < 5.0)+(1-p)P(4.4 < X_2 < 5.0)$$

where X_1 and X_2 are normal rvs with parameters specified in f_1 and f_2.
$P(4.4 < X_1 < 5.0) = P(0 < Z < 2.22) = .9868 - .5 = .4868$ and
$P(4.4 < X_1 < 5.0) = P(-2.22 < Z < 0) = .5 - .0132 = .4868$; the choice of endpoints 4.4 and 5.0 makes these equal. Putting it all together, $P(4.4 < X < 5.0) = .35(.4868) + .65(.4868) = .4868$.

c. Similarly, $P(X < 4.79) = .35P(X_1 < 4.79) + .65P(X_2 < 4.79) = .35\Phi(1.44) + .65\Phi(–0.78) = .35(.9251) + .65(.2177) = .4653$.
Since $P(X < \mu_X) \neq .5$, the mean is not equal to the median (the mean is only the 46th percentile). So, in particular, the shape of the pdf of X is not symmetric, even though the individual normal distributions from which X was created are symmetric. (The pdf of X appears below.)



109. Below, exp(u) is alternative notation for e^u.

a. $P(X \le 150) = \exp\left[-\exp\left(\frac{-(150-150)}{90}\right)\right] = \exp[-\exp(0)] = \exp(-1) = .368$,

$P(X \le 300) = \exp[-\exp(-1.6667)] = .828$, and $P(150 \le X \le 300) = .828 – .368 = .460$.

b. The desired value c is the 90th percentile, so c satisfies
$.9 = \exp\left[-\exp\left(\frac{-(c-150)}{90}\right)\right]$. Taking the natural log of each side twice in succession yields $\frac{-(c-150)}{90}$ = ln[–ln(.9)] = –2.250367, so $c = 90(2.250367) + 150 = 352.53$.

c. Use the chain rule: $f(x) = F'(x) = \exp\left[-\exp\left(\frac{-(x-\alpha)}{\beta}\right)\right] \cdot -\exp\left(\frac{-(x-\alpha)}{\beta}\right) \cdot -\frac{1}{\beta} =$

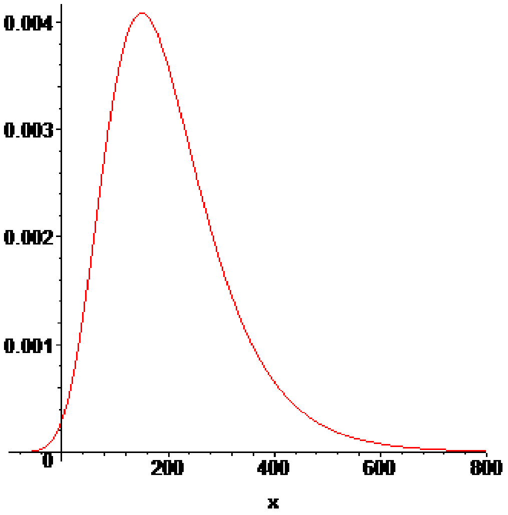
$$\frac{1}{\beta}\exp\left[-\exp\left(\frac{-(x-\alpha)}{\beta}\right) - \frac{(x-\alpha)}{\beta}\right].$$

d. We wish the value of x for which $f(x)$ is a maximum; from calculus, this is the same as the value of x for which ln[$f(x)$] is a maximum, and ln[$f(x)$] = $-\ln\beta - e^{-(x-\alpha)/\beta} - \frac{(x-\alpha)}{\beta}$. The derivative of ln[$f(x)$] is

$\frac{d}{dx}\left[-\ln\beta - e^{-(x-\alpha)/\beta} - \frac{(x-\alpha)}{\beta}\right] = 0 + \frac{1}{\beta}e^{-(x-\alpha)/\beta} - \frac{1}{\beta}$; set this equal to 0 and we get $e^{-(x-\alpha)/\beta} = 1$, so

$\frac{-(x-\alpha)}{\beta} = 0$, which implies that $x = \alpha$. Thus the mode is α.

e. $E(X) = .5772\beta + \alpha = 201.95$, whereas the mode is $\alpha = 150$ and the median is the solution to $F(x) = .5$. From **b**, this equals $-90\ln[-\ln(.5)] + 150 = 182.99$.
Since mode < median < mean, the distribution is positively skewed. A plot of the pdf appears below.



110. We have a random variable $T \sim N(\mu, \sigma)$. Let $f(t)$ denote its pdf.

a. The "expected loss" is the expected value of a piecewise-defined function, so we should first write the function out in pieces (two integrals, as seen below). Call this expected loss $Q(a)$, to emphasize we're interested in its behavior as a function of a. We have:

$$\begin{aligned} Q(a) &= E[L(a,T)] \\ &= \int_{-\infty}^{a} k(a-t)f(t)\,dt + \int_{a}^{\infty} (t-a)f(t)\,dt \\ &= ka\int_{-\infty}^{a} f(t)\,dt - k\int_{-\infty}^{a} tf(t)\,dt + \int_{a}^{\infty} tf(t)\,dt - a\int_{a}^{\infty} f(t)\,dt \\ &= kaF(a) - k\int_{-\infty}^{a} tf(t)\,dt + \int_{a}^{\infty} tf(t)\,dt - a[1-F(a)] \end{aligned}$$

where $F(a)$ denotes the cdf of T. To minimize this expression, take the first derivative with respect to a, using the product rule and the fundamental theorem of calculus where appropriate:

$$\begin{aligned} Q'(a) &= kaF(a) - k\int_{-\infty}^{a} tf(t)\,dt + \int_{a}^{\infty} tf(t)\,dt - a[1-F(a)] \\ &= kF(a) + kaF'(a) - kaf(a) + 0 - af(a) - 1 + F(a) + aF'(a) \\ &= kF(a) + kaf(a) - kaf(a) - af(a) - 1 + F(a) + af(a) \\ &= (k+1)F(a) - 1 \end{aligned}$$

Finally, set this equal to zero, and use the fact that, because T is a normal random variable, $F(a) = \Phi\left(\frac{a-\mu}{\sigma}\right)$:

$$(k+1)F(a) - 1 = 0 \Rightarrow (k+1)\Phi\left(\frac{a-\mu}{\sigma}\right) - 1 = 0 \Rightarrow \Phi\left(\frac{a-\mu}{\sigma}\right) = \frac{1}{k+1} \Rightarrow a = \mu + \sigma\cdot\Phi^{-1}\left(\frac{1}{k+1}\right)$$

This is the critical value, a^*, as desired.

b. With the values provided, $a^* = 100,000 + 10,000\Phi^{-1}\left(\frac{1}{2+1}\right) = 100,000 + 10,000\Phi^{-1}(0.33)$ = 100,000 + 10,000(–0.44) from the standard normal table = 100,000 – 4,400 = \$95,600. The probability of an over-assessment equals $P(95{,}600 > T) = P(T < 96{,}500) = \Phi\left(\frac{95,600-100,000}{10,000}\right) = \Phi(-0.44) = .3300$, or 33%. Notice that, in general, the probability of an over-assessment using the optimal value of a is equal to $\frac{1}{k+1}$.

111.

a. From a graph of the normal pdf or by differentiation of the pdf, $x^* = \mu$.

b. No; the density function has constant height for $A \le x \le B$.

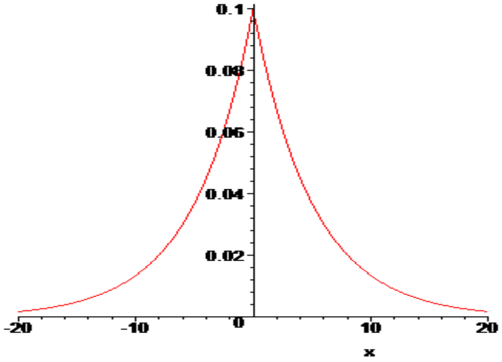
c. $f(x; \lambda)$ is largest for $x = 0$ (the derivative at 0 does not exist since f is not continuous there), so $x^* = 0$.

d. $\ln[f(x;\alpha,\beta)] = -\ln(\beta^\alpha) - \ln(\Gamma(\alpha)) + (\alpha-1)\ln(x) - \frac{x}{\beta}$, and $\frac{d}{dx}\ln[f(x;\alpha,\beta)] = \frac{\alpha-1}{x} - \frac{1}{\beta}$. Setting this equal to 0 gives the mode: $x^* = (\alpha - 1)\beta$.

e. The chi-squared distribution is the gamma distribution with $\alpha = \nu/2$ and $\beta = 2$. From **d**, $x^* = \left(\frac{\nu}{2} - 1\right)(2) = \nu - 2$.

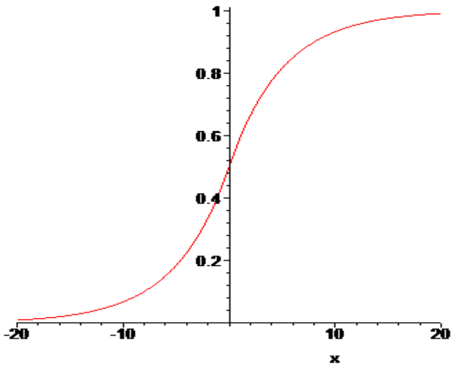
112.

a. $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} .1e^{.2x}dx + \int_{0}^{\infty} .1e^{-.2x}dx = .5 + .5 = 1.$



b. For $x < 0$, $F(x) = \int_{-\infty}^{x} .1e^{.2y}dy = \frac{1}{2}e^{.2x}$.

For $x \ge 0$, $F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{-\infty}^{0} .1e^{-.2y}\,dy + \int_{0}^{x} .1e^{-.2y}dy = \frac{1}{2} + \int_{0}^{x} .1e^{-.2y}dy = 1 - \frac{1}{2}e^{-.2x}$.



c. $P(X < 0) = F(0) = .5$; $P(X < 2) = F(2) = 1 – .5e^{–.4} = .665$; $P(–1 \leq X \leq 2) = F(2) – F(–1) = .256$; and $P(|X| > 2) = 1 – (–2 \leq X \leq 2) = 1 – [F(2) – F(–2)] = .670$.

113.

a. $E(X)$ = $\int_0^\infty x\cdot\left[p\lambda_1 e^{-\lambda_1 x}+(1-p)\lambda_2 e^{-\lambda_2 x}\right]dx$ $= p\int_0^\infty x\lambda_1 e^{-\lambda_1 x}dx+(1-p)\int_0^\infty x\lambda_2 e^{-\lambda_2 x}dx$ $=\frac{p}{\lambda_1}+\frac{(1-p)}{\lambda_2}$. (Each of the two integrals represents the expected value of an exponential random variable, which is the reciprocal of λ.) Similarly, since the mean-square value of an exponential rv is $E(Y^2) = V(Y) + [E(Y)]^2 = 1/\lambda^2 + [1/\lambda]^2 = 2/\lambda^2$, $E(X^2)$ = $\int_0^\infty x^2 f(x)dx$ = … = $\frac{2p}{\lambda_1^2}+\frac{2(1-p)}{\lambda_2^2}$. From this,

$V(X)$ = $\frac{2p}{\lambda_1^2}+\frac{2(1-p)}{\lambda_2^2}-\left[\frac{p}{\lambda_1}+\frac{(1-p)}{\lambda_2}\right]^2$.

b. For $x > 0$, $F(x;\lambda_1,\lambda_2,p)=\int_0^x f(y;\lambda_1,\lambda_2,p)dy$ = $\int_0^x\left[p\lambda_1 e^{-\lambda_1 y}+(1-p)\lambda_2 e^{-\lambda_2 y}\right]dy$

$= p\int_0^x \lambda_1 e^{-\lambda_1 y}dy+(1-p)\int_0^x \lambda_2 e^{-\lambda_2 y}dy = p(1-e^{-\lambda_1 x})+(1-p)(1-e^{-\lambda_2 x})$. For $x \le 0$, $F(x) = 0$.

c. $P(X > .01) = 1 - F(.01) = 1 - [.5(1 - e^{-40(.01)}) + (1 - .5)(1 - e^{-200(.01)})] = .5e^{-0.4} + .5e^{-2} = .403$.

d. Using the expressions in **a**, $\mu = .015$ and $\sigma^2 = .000425 \Rightarrow \sigma = .0206$. Hence,
$P(\mu - \sigma < X < \mu + \sigma) = P(-.0056 < X < .0356) = P(X < .0356)$ because X can't be negative
$= F(.0356) = \ldots = .879$.

e. For an exponential rv, CV = $\frac{\sigma}{\mu}=\frac{1/\lambda}{1/\lambda}=1$. For X hyperexponential,

CV = $\frac{\sigma}{\mu}=\frac{\sqrt{E(X^2)-\mu^2}}{\mu}=\sqrt{\frac{E(X^2)}{\mu^2}-1}=\sqrt{\frac{2p/\lambda_1^2+2(1-p)/\lambda_2^2}{\left[p/\lambda_1+(1-p)/\lambda_2\right]^2}-1}$ =

$\sqrt{\frac{2\left(p\lambda_2^2+(1-p)\lambda_1^2\right)}{\left(p\lambda_2+(1-p)\lambda_1\right)^2}-1}=\sqrt{2r-1}$, where r = $\frac{p\lambda_2^2+(1-p)\lambda_1^2}{\left(p\lambda_2+(1-p)\lambda_1\right)^2}$. But straightforward algebra shows that $r > 1$ when $\lambda_1 \ne \lambda_2$, so that $CV > 1$.

f. For the Erlang distribution, $\mu=\frac{n}{\lambda}$ and $\sigma=\frac{\sqrt{n}}{\lambda}$, so CV = $\frac{1}{\sqrt{n}}<1$ for $n > 1$.

114.

a. Provided $\alpha > 1$, $1=\int_5^\infty \frac{k}{x^\alpha}dx=k\cdot\frac{5^{1-\alpha}}{\alpha-1}\Rightarrow k=(\alpha-1)5^{\alpha-1}$.

b. For $x \ge 5$, $F(x)$ = $\int_5^x \frac{(\alpha-1)5^{\alpha-1}}{y^\alpha}dy=-5^{\alpha-1}\left[x^{1-\alpha}-5^{1-\alpha}\right]=1-\left(\frac{5}{x}\right)^{\alpha-1}$. For $x < 5$, $F(x) = 0$.

c. Provided $\alpha > 2$, $E(X)$ = $\int_5^\infty x\cdot\frac{k}{x^\alpha}dx=\int_5^\infty\frac{(\alpha-1)5^{\alpha-1}}{x^{\alpha-1}}dx=5\frac{\alpha-1}{\alpha-2}$.

d. Let $Y = \ln(X/5)$. Then $F_Y(y)$ = $P\left(\ln\left(\frac{X}{5}\right)\le y\right)=P\left(\frac{X}{5}\le e^y\right)=P\left(X\le 5e^y\right)=F(5e^y)$ = $1-\left(\frac{5}{5e^y}\right)^{\alpha-1}$ =

$1-e^{-(\alpha-1)y}$, the cdf of an exponential rv with parameter $\alpha - 1$.

115.

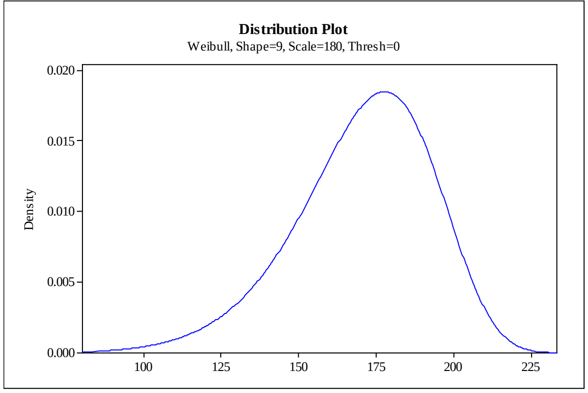
a. Since $\ln\left(\frac{I_o}{I_i}\right)$ has a normal distribution, by definition $\frac{I_o}{I_i}$ has a lognormal distribution.

b. $P(I_o > 2I_i) = P\left(\frac{I_o}{I_i} > 2\right) = P\left(\ln\left(\frac{I_o}{I_i}\right) > \ln 2\right) = 1 - P\left(\ln\left(\frac{I_o}{I_i}\right) \le \ln 2\right) = 1 - P(X \le \ln 2) =$
$1 - \Phi\left(\frac{\ln 2 - 1}{.05}\right) = 1 - \Phi(-6.14) = 1.$

c. $E\left(\frac{I_o}{I_i}\right) = e^{1+.0025/2} = 2.72$ and $V\left(\frac{I_o}{I_i}\right) = e^{2+.0025} \cdot \left(e^{.0025} - 1\right) = .0185$.

116.

a. The accompanying Weibull pdf plot was created in Minitab.



b. $P(X > 175) = 1 - F(175; 9, 180) = e^{-(175/180)^9} = .4602.$
$P(150 \le X \le 175) = F(175; 9, 180) - F(150; 9, 180) = .5398 - .1762 = .3636.$

c. From **b**, the probability a specimen is not between 150 and 175 equals 1 – .3636 = .6364. So, P(at least one is between 150 and 175) = 1 – P(neither is between 150 and 175) = $1 - (.6364)^2 = .5950$.

d. We want the 10th percentile: $.10 = F(x; 9, 180) = 1 - e^{-(x/180)^9}$. A small bit of algebra leads us to $x = 180(-\ln(1-.10))^{1/9} = 140.178$. Thus, 10% of all tensile strengths will be less than 140.178 MPa.

117. $F(y) = P(Y \le y) = P(\sigma Z + \mu \le y) = P\left(Z \le \frac{y-\mu}{\sigma}\right) = \int_{-\infty}^{\frac{y-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \Rightarrow$ by the fundamental theorem of calculus, $f(y) = F'(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \cdot \frac{1}{\sigma} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$, a normal pdf with parameters μ and σ.

118.

a. $F_Y(y) = P(Y \le y) = P(60X \le y) = P\left(X \le \frac{y}{60}\right) = F\left(\frac{y}{60\beta};\alpha\right)$. Thus $f_Y(y) =$

$f\left(\frac{y}{60\beta};\alpha\right)\cdot\frac{1}{60\beta} = \frac{1}{(60\beta)^\alpha\,\Gamma(\alpha)}y^{\alpha-1}e^{\frac{-y}{60\beta}}$, which shows that Y has a gamma distribution with parameters α and 60β.

b. With c replacing 60 in **a**, the same argument shows that cX has a gamma distribution with parameters α and $c\beta$.

119.

a. $Y = –\ln(X) \Rightarrow x = e^{–y} = k(y)$, so $k'(y) = –e^{–y}$. Thus since $f(x) = 1$, $g(y) = 1 \cdot |–e^{–y}| = e^{–y}$ for $0 < y < \infty$. Y has an exponential distribution with parameter $\lambda = 1$.

b. $y = \sigma Z + \mu \Rightarrow z = k(y) = \frac{y-\mu}{\sigma}$ and $k'(y) = \frac{1}{\sigma}$, from which the result follows easily.

c. $y = h(x) = cx \Rightarrow x = k(y) = \frac{y}{c}$ and $k'(y) = \frac{1}{c}$, from which the result follows easily.

120.

a. If we let $\alpha = 2$ and $\beta = \sqrt{2}\sigma$, then we can manipulate $f(v)$ as follows:

$f(v) = \frac{v}{\sigma^2}e^{-v^2/2\sigma^2} = \frac{2}{2\sigma^2}ve^{-v^2/2\sigma^2} = \frac{2}{(\sqrt{2}\sigma)^2}v^{2-1}e^{-\left(v/\sqrt{2}\sigma\right)^2} = \frac{\alpha}{\beta^\alpha}v^{\alpha-1}e^{-(v/\beta)^\alpha}$, which is in the Weibull family of distributions.

b. Use the Weibull cdf: $P(V \le 25) = F(25;2,\sqrt{2}\sigma) = 1 - e^{-\left(\frac{25}{\sqrt{2}\sigma}\right)^2} = 1 - e^{-\frac{625}{800}}$ = 1 – .458 = .542.

121.

a. Assuming the three birthdays are independent and that all 365 days of the calendar year are equally likely, P(all 3 births occur on March 11) = $\left(\frac{1}{365}\right)^3$.

b. P(all 3 births on the same day) = P(all 3 on Jan. 1) + P(all 3 on Jan. 2) + … = $\left(\frac{1}{365}\right)^3 + \left(\frac{1}{365}\right)^3 + ... =$

$365\left(\frac{1}{365}\right)^3 = \left(\frac{1}{365}\right)^2$.

c. Let X = deviation from due date, so $X \sim N(0, 19.88)$. The baby due on March 15 was 4 days early, and $P(X = -4) \approx P(-4.5 < X < -3.5) = \Phi\left(\frac{-3.5}{19.88}\right) - \Phi\left(\frac{-4.5}{19.88}\right) =$
$\Phi(-.18) - \Phi(-.237) = .4286 - .4090 = .0196$.
Similarly, the baby due on April 1 was 21 days early, and $P(X = -21) \approx$
$\Phi\left(\frac{-20.5}{19.88}\right) - \Phi\left(\frac{-21.5}{19.88}\right) = \Phi(-1.03) - \Phi(-1.08) = .1515 - .1401 = .0114$.
Finally, the baby due on April 4 was 24 days early, and $P(X = -24) \approx .0097$.

Again assuming independence, P(all 3 births occurred on March 11) = (.0196)(.0114)(.0097) = .0002145.

d. To calculate the probability of the three births happening on any day, we could make similar calculations as in part **c** for each possible day, and then add the probabilities.

122.

a. $f(x) = \lambda e^{-\lambda x}$ and $F(x) = 1 - e^{-\lambda x}$, so $r(x) = \frac{\lambda e^{-\lambda x}}{1-(1-e^{-\lambda x})} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda$, a constant; this is consistent with the memoryless property of the exponential distribution.

b. For the Weibull distribution, $r(x) = \frac{f(x)}{1-F(x)} = \left(\frac{\alpha}{\beta^{\alpha}}\right)x^{\alpha-1}$. For $\alpha > 1$, $r(x)$ is increasing (since the exponent on x is positive), while for $\alpha < 1$ it is a decreasing function.

c. $\ln(1 - F(x)) = \, = -\int r(x)\,dx - \int \alpha\left(1-\frac{x}{\beta}\right)dx = -\alpha\left(x - \frac{x^2}{2\beta}\right) \Rightarrow F(x) = 1 - e^{-\alpha\left(x-\frac{x^2}{2\beta}\right)}$.

$f(x) = F'(x) = \alpha\left(1-\frac{x}{\beta}\right)e^{-\alpha\left(x-\frac{x^2}{2\beta}\right)}$ for $0 \le x \le \beta$.

123.

a. $F(x) = P(X \le x) = P\left(-\frac{1}{\lambda}\ln(1-U) \le x\right) = P\left(\ln(1-U) \ge -\lambda x\right) = P\left(1-U \ge e^{-\lambda x}\right)$
$= P\left(U \le 1-e^{-\lambda x}\right) = 1-e^{-\lambda x}$ since the cdf of a uniform rv on [0, 1] is simply $F(u) = u$. Thus X has an exponential distribution with parameter λ.

b. By taking successive random numbers $u_1, u_2, u_3, \ldots$ and computing $x_i = -\frac{1}{10}\ln(1-u_i)$ for each one, we obtain a sequence of values generated from an exponential distribution with parameter $\lambda = 10$.

124.

a. $E(g(X)) \approx E[g(\mu) + g'(\mu)(X - \mu)] = E(g(\mu)) + g'(\mu) \cdot [E(X) - \mu]$, but $E(X) - \mu = 0$ and $E(g(\mu)) = g(\mu)$ (since $g(\mu)$ is constant), giving $E(g(X)) \approx g(\mu)$.

$V(g(X)) \approx V[g(\mu) + g'(\mu)(X - \mu)] = V[g'(\mu)(X - \mu)] = (g'(\mu))^2 \cdot V(X - \mu) = (g'(\mu))^2 \cdot V(X)$.

b. $g(I) = \frac{v}{I}, g'(I) = \frac{-v}{I^2}$, so $\mu_R = E\left(g(I)\right) \approx g(\mu_I) = \frac{v}{\mu_I} = \frac{v}{20}$ and

$$\sigma_R^2 = V\left(g(I)\right) \approx (g'(\mu_I)) \cdot V(I) = \left(\frac{-v}{\mu_I^2}\right)^2 \cdot \sigma_I^2 \Rightarrow \sigma_R \approx \frac{v}{\mu_I^2} \cdot \sigma_I = .025v\ .$$

125. If $g(x)$ is convex in a neighborhood of μ, then $g(\mu) + g'(\mu)(x - \mu) \leq g(x)$. Replace x by X:
$E[g(\mu) + g'(\mu)(X - \mu)] \leq E[g(X)] \Rightarrow E[g(X)] \geq g(\mu) + g'(\mu)E[(X - \mu)] = g(\mu) + g'(\mu) \cdot 0 = g(\mu)$.
That is, if $g(x)$ is convex, $g(E(X)) \leq E[g(X)]$.

126. For $y > 0$, $F(y) = P(Y \leq y) = P\left(\frac{2X^2}{\beta^2} \leq y\right) = P\left(X^2 \leq \frac{\beta^2 y}{2}\right) = P\left(X \leq \frac{\beta\sqrt{y}}{\sqrt{2}}\right)$. Now take the cdf of X (Weibull), replace x by $\frac{\beta\sqrt{y}}{\sqrt{2}}$, and then differentiate with respect to y to obtain the desired result $f(y)$.

127.

a. $E(X) = 150 + (850 - 150)\frac{8}{8+2} = 710$ and $V(X) = \frac{(850-150)^2(8)(2)}{(8+2)^2(8+2+1)} = 7127.27 \Rightarrow SD(X) \approx 84.423$.
Using software, $P(|X - 710| \leq 84.423) = P(625.577 \leq X \leq 794.423) =$
$\int_{625.577}^{794.423} \frac{1}{700} \frac{\Gamma(10)}{\Gamma(8)\Gamma(2)} \left(\frac{x-150}{700}\right)^7 \left(\frac{850-x}{700}\right)^1 dx = .684$.

b. $P(X > 750) = \int_{750}^{850} \frac{1}{700} \frac{\Gamma(10)}{\Gamma(8)\Gamma(2)} \left(\frac{x-150}{700}\right)^7 \left(\frac{850-x}{700}\right)^1 dx = .376$. Again, the computation of the requested integral requires a calculator or computer.

128.

a. For $w < 0$, $F(w) = 0$.
$F(0) = P(V \le v_d) = 1 - \exp(-\lambda v_d)$, since V is exponential.
For $w > 0$, $F(w) = P(W \le w) = P(k(V - v_d) \le w) = P(V \le v_d + w/k) = 1 - \exp(-\lambda[v_d + w/k])$. This can be written more compactly as

$$F(w) = \begin{cases} 0 & w < 0 \\ 1 - e^{-\lambda[v_d + w/k]} & w \ge 0 \end{cases}$$

b. For $w > 0$, $f(w) = F'(w) = \dfrac{\lambda}{k}\exp(-\lambda[v_d + w/k])$. Since the only other possible value of W is zero, which would not contribute to the expected value, we can compute $E(W)$ as the appropriate integral:

$$E(W) = \int_0^\infty w\frac{\lambda}{k}\exp(-\lambda[v_d + w/k])dw = \exp(-\lambda v_d)\int_0^\infty w\frac{\lambda}{k}\exp(-\lambda w/k)dw \ = \frac{k}{\lambda}e^{-\lambda v_d}.$$

CHAPTER 5

Section 5.1

1.

a. $P(X = 1, Y = 1) = p(1,1) = .20$.

b. $P(X \leq 1 \text{ and } Y \leq 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = .42$.

c. At least one hose is in use at both islands. $P(X \neq 0 \text{ and } Y \neq 0) = p(1,1) + p(1,2) + p(2,1) + p(2,2) = .70$.

d. By summing row probabilities, $p_X(x) = .16, .34, .50$ for $x = 0, 1, 2$, By summing column probabilities, $p_Y(y) = .24, .38, .38$ for $y = 0, 1, 2$. $P(X \leq 1) = p_X(0) + p_X(1) = .50$.

e. $p(0,0) = .10$, but $p_X(0) \cdot p_Y(0) = (.16)(.24) = .0384 \neq .10$, so X and Y are not independent.

2.

a. For each coordinate, independence means $p(x, y) = p_X(x) \cdot p_Y(y)$.

				x			
	$p(x, y)$	0	1	2	3	4	
	0	.01	.02	.03	.03	.02	.1
y	1	.03	.06	.09	.09	.06	.3
	2	.04	.08	.12	.12	.08	.4
	3	.02	.04	.06	.06	.04	.2
		.1	.2	.3	.3	.2	

b. From the table, $P(X \leq 1 \text{ and } Y \leq 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = .01 + .03 + .02 + .06 = .12$. Using the marginal distributions, $P(X \leq 1) = .1 + .2 = .3$, while $P(Y \leq 1) = .1 + .3 = .4$. Sure enough, $(.3)(.4) = .12$.

c. $P(X + Y \leq 1) = p(0,0) + p(1,0) + p(0,1) = .01 + .02 + .03 = .06$.

d. $P(X = 0 \cup Y = 0) = P(X = 0) + P(Y = 0) - P(X = 0 \cap Y = 0) =.1 + .1 - .01 = .19$.

3.

a. $p(1,1) = .15$, the entry in the 1st row and 1st column of the joint probability table.

b. $P(X_1 = X_2) = p(0,0) + p(1,1) + p(2,2) + p(3,3) = .08 + .15 + .10 + .07 = .40$.

c. $A = \{X_1 \geq 2 + X_2 \cup X_2 \geq 2 + X_1\}$, so $P(A) = p(2,0) + p(3,0) + p(4,0) + p(3,1) + p(4,1) + p(4,2) + p(0,2) + p(0,3) + p(1,3) =.22$.

d. $P(X_1 + X_2 = 4) = p(1,3) + p(2,2) + p(3,1) + p(4,0) = .17$.
$P(X_1 + X_2 \geq 4) = P(X_1 + X_2 = 4) + p(4,1) + p(4,2) + p(4,3) + p(3,2) + p(3,3) + p(2,3)=.46$.

4.

a. $p_1(0) = P(X_1 = 0) = p(0,0) + p(0,1) + p(0,2) + p(0,3) = .19$
$p_1(1) = P(X_1 = 1) = p(1,0) + p(1,1) + p(1,2) + p(1,3) = .30$, etc.

x_1	0	1	2	3	4
$p_1(x_1)$	.19	.30	.25	.14	.12

b. $p_2(0) = P(X_2 = 0) = p(0,0) + p(1,0) + p(2,0) + p(3,0) + p(4,0) = .19$, etc.

x_2	0	1	2	3
$p_2(x_2)$	.19	.30	.28	.23

c. $p(4,0) = 0$, yet $p_1(4) = .12 > 0$ and $p_2(0) = .19 > 0$, so $p(x_1, x_2) \neq p_1(x_1) \cdot p_2(x_2)$ for every (x_1, x_2), and the two variables are <u>not</u> independent.

5.

a. $p(3, 3) = P(X = 3, Y = 3) = P$(3 customers, each with 1 package)
$= P$(each has 1 package | 3 customers) · P(3 customers) $= (.6)^3 \cdot (.25) = .054$.

b. $p(4, 11) = P(X = 4, Y = 11) = P$(total of 11 packages | 4 customers) · P(4 customers).
Given that there are 4 customers, there are four different ways to have a total of 11 packages: 3, 3, 3, 2 or 3, 3, 2, 3 or 3, 2, 3, 3 or 2, 3, 3, 3. Each way has probability $(.1)^3(.3)$, so $p(4, 11) = 4(.1)^3(.3)(.15) = .00018$.

6.

a. $p(4,2) = P(Y = 2 \mid X = 4) \cdot P(X = 4) = \left[\binom{4}{2}(.6)^2(.4)^2\right] \cdot (.15) = .0518$.

b. $P(X = Y) = p(0,0) + p(1,1) + p(2,2) + p(3,3) + p(4,4) = .1 + (.2)(.6) + (.3)(.6)^2 + (.25)(.6)^3 + (.15)(.6)^4 = .4014$.

c. $p(x,y) = 0$ unless $y = 0, 1, \ldots, x$ and $x = 0, 1, 2, 3, 4$. For any such pair,

$p(x,y) = P(Y = y \mid X = x) \cdot P(X = x) = \binom{x}{y}(.6)^y(.4)^{x-y} \cdot p_X(x)$. As for the marginal pmf of Y,

$p_Y(4) = P(Y = 4) = P(X = 4, Y = 4) = p(4,4) = (.6)^4 \cdot (.15) = .0194$;

$p_Y(3) = p(3,3) + p(4,3) = (.6)^3(.25) + \binom{4}{3}(.6)^3(.4)(.15) = .1058$; similarly,

$p_Y(2) = p(2,2) + p(3,2) + p(4,2) = (.6)^2(.3) + \binom{3}{2}(.6)^2(.4)(.25) + \binom{4}{2}(.6)^2(.4)^2(.15) = .2678$, $p_Y(1) = p(1,1) + p(2,1) + p(3,1) + p(4,1) = .3590$, and
$p_Y(0) = 1 - [.3590+.2678+.1058+.0194] = .2480$.

7.

a. $p(1,1) = .030$.

b. $P(X \le 1 \text{ and } Y \le 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = .120$.

c. $P(X = 1) = p(1,0) + p(1,1) + p(1,2) = .100$; $P(Y = 1) = p(0,1) + \ldots + p(5,1) = .300$.

d. $P(\text{overflow}) = P(X + 3\text{Y} > 5) = 1 - P(X + 3\text{Y} \le 5) = 1 - P((X,Y)=(0,0)$ or …or $(5,0)$ or $(0,1)$ or $(1,1)$ or $(2,1)) = 1 - .620 = .380$.

e. The marginal probabilities for X (row sums from the joint probability table) are $p_X(0) = .05$, $p_X(1) = .10$, $p_X(2) = .25$, $p_X(3) = .30$, $p_X(4) = .20$, $p_X(5) = .10$; those for Y (column sums) are $p_Y(0) = .5$, $p_Y(1) = .3$, $p_Y(2) = .2$. It is now easily verified that for every (x,y), $p(x,y) = p_X(x) \cdot p_Y(y)$, so X and Y are independent.

8.

a. $p(3, 2) = P(X = 3 \text{ and } Y = 2) = P(\text{3 supplier 1 comps \& 2 supplier 2 comps selected}) =$

$$P(\text{3 from supplier 1, 2 from supplier 2, 1 from supplier 3}) = \frac{\binom{8}{3}\binom{10}{2}\binom{12}{1}}{\binom{30}{6}} = .0509.$$

b. Replace 3 with x, 2 with y, and 1 with the remainder $6 - x - y$. The resulting formula is

$$p(x,y) = \frac{\binom{8}{x}\binom{10}{y}\binom{12}{6-x-y}}{\binom{30}{6}}$$

This formula is valid for non-negative integers x and y such that $x + y \le 6$.

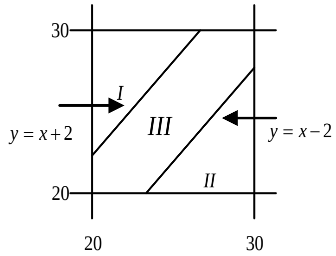
9.

a. $$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_{20}^{30}\int_{20}^{30} K(x^2+y^2)dxdy = K\int_{20}^{30}\int_{20}^{30} x^2dydx + K\int_{20}^{30}\int_{20}^{30} y^2dxdy$$
$$= 10K\int_{20}^{30} x^2dx + 10K\int_{20}^{30} y^2dy = 20K \cdot \left(\frac{19{,}000}{3}\right) \Rightarrow K = \frac{3}{380{,}000}.$$

b. $$P(X < 26 \text{ and } Y < 26) = \int_{20}^{26}\int_{20}^{26} K(x^2+y^2)dxdy = K\int_{20}^{26}\left[x^2y + \frac{y^3}{3}\right]_{20}^{26} dx = K\int_{20}^{26}(6x^2+3192)\,dx =$$

$K(38{,}304) = .3024$.

c. The region of integration is labeled *III* below.



$$P(|X-Y|\le 2) = \iint_{III} f(x,y)dxdy = 1-\iint_{I} f(x,y)dxdy - \iint_{II} f(x,y)dxdy =$$

$$1-\int_{20}^{28}\int_{x+2}^{30} f(x,y)dydx - \int_{22}^{30}\int_{20}^{x-2} f(x,y)dydx = .3593\text{(after much algebra).}$$

d. $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{20}^{30} K(x^2+y^2)dy = 10Kx^2 + K\left.\frac{y^3}{3}\right|_{20}^{30}$ = $10Kx^2$ + .05, for $20 \le x \le 30$.

e. $f_Y(y)$ can be obtained by substituting y for x in (d); clearly $f(x,y) \ne f_X(x)\cdot f_Y(y)$, so X and Y are not independent.

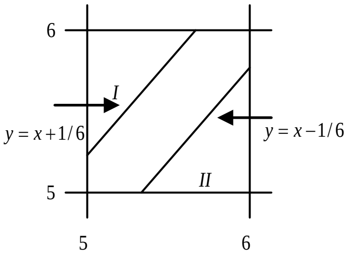
10.

a. Since $f_X(x) = \frac{1}{6-5} = 1$ for $5 \le x \le 6$, similarly $f_Y(y) = 1$ for $5 \le y \le 6$, and X and Y are independent,

$$f(x,y) = f_X(x)\cdot f_Y(y) = \begin{cases} 1 & 5\le x\le 6, 5\le y\le 6 \\ 0 & \text{otherwise}\end{cases}$$

b. $P(5.25 \le X \le 5.75, 5.25 \le Y \le 5.75) = P(5.25 \le X \le 5.75)\cdot P(5.25 \le Y \le 5.75)$ by independence = (.5)(.5) = .25.

c. The region A is the diagonal stripe below.



$P((X,Y) \in A) = \iint_A 1\, dxdy$ = the area of A = 1 – (area of I + area of II)

$$= 1 - \left(\frac{1}{2}\cdot\frac{5}{6}\cdot\frac{5}{6}+\frac{1}{2}\cdot\frac{5}{6}\cdot\frac{5}{6}\right) = 1-\frac{25}{36} = \frac{11}{36} = .306.$$

11.

a. Since X and Y are independent, $p(x,y)$ = $p_X(x)\cdot p_Y(y)=\dfrac{e^{-\mu_1}\mu_1^x}{x!}\cdot\dfrac{e^{-\mu_2}\mu_2^y}{y!}=\dfrac{e^{-\mu_1-\mu_2}\mu_1^x\mu_2^y}{x!y!}$ for x = 0, 1, 2, …; y = 0, 1, 2, ….

b. $P(X + Y \le 1) = p(0,0) + p(0,1) + p(1,0) = \ldots = e^{-\mu_1-\mu_2}\left[1+\mu_1+\mu_2\right]$.

c. $P(X + Y= m)$ = $\sum_{k=0}^{m}P(X=k,Y=m-k)=e^{-\mu_1-\mu_2}\sum_{k=0}^{m}\dfrac{\mu_1^k}{k!}\dfrac{\mu_2^{m-k}}{(m-k)!}$ = $\dfrac{e^{-\mu_1-\mu_2}}{m!}\sum_{k=0}^{m}\dfrac{m!}{k!(m-k)!}\mu_1^k\mu_2^{m-k}$ = $\dfrac{e^{-\mu_1-\mu_2}}{m!}\sum_{k=0}^{m}\binom{m}{k}\mu_1^k\mu_2^{m-k}$ = $\dfrac{e^{-\mu_1-\mu_2}}{m!}(\mu_1+\mu_2)^m$ by the binomial theorem. We recognize this as the pmf of a Poisson random variable with parameter $\mu_1+\mu_2$. Therefore, the total number of errors, $X + Y$, also has a Poisson distribution, with parameter $\mu_1+\mu_2$.

12.

a. $P(X> 3)$ = $\int_3^\infty\int_0^\infty xe^{-x(1+y)}dydx=\int_3^\infty e^{-x}dx=.050.$

b. The marginal pdf of X is $f_X(x)=\int_0^\infty xe^{-x(1+y)}dy=e^{-x}$ for $x \ge 0$. The marginal pdf of Y is $f_Y(y)=\int_3^\infty xe^{-x(1+y)}dx=\dfrac{1}{(1+y)^2}$ for $y \ge 0$. It is now clear that $f(x,y)$ is not the product of the marginal pdfs, so the two rvs are not independent.

c. P(at least one exceeds 3) = $P(X > 3$ or $Y > 3) = 1 – P(X \le 3$ and $Y \le 3)$

$=1-\int_0^3\int_0^3 xe^{-x(1+y)}dydx=1-\int_0^3\int_0^3 xe^{-x}e^{-xy}dy$

$=1-\int_0^3 e^{-x}(1-e^{-3x})dx=e^{-3}+.25-.25e^{-12}=.300.$

13.

a. $f(x,y) = f_X(x) \cdot f_Y(y)$ = $\begin{cases} e^{-x-y} & x\ge 0, y\ge 0 \\ 0 & \text{otherwise}\end{cases}$

b. By independence, $P(X \le 1$ and $Y \le 1) = P(X \le 1) \cdot P(Y \le 1) = (1 – e^{–1}) (1 – e^{–1}) = .400.$

c. $P(X + Y \le 2)$ = $\int_0^2\int_0^{2-x}e^{-x-y}dydx=\int_0^2 e^{-x}\left[1-e^{-(2-x)}\right]dx$ $=\int_0^2(e^{-x}-e^{-2})dx=1-e^{-2}-2e^{-2}=.594.$

d. $P(X + Y \le 1)$ = $\int_0^1 e^{-x}\left[1-e^{-(1-x)}\right]dx=1-2e^{-1}=.264$,

so $P(1 \le X + Y \le 2) = P(X + Y \le 2) – P(X + Y \le 1) = .594 – .264 = .330.$

14.

a. $P(X_1 < t, X_2 < t, \ldots, X_{10} < t) = P(X_1 < t) \cdots P(X_{10} < t) = (1-e^{-\lambda t})^{10}$.

b. If "success" = {fail before t}, then $p = P(\text{success}) = 1-e^{-\lambda t}$,
and $P(k$ successes among 10 trials$) = \binom{10}{k}(1-e^{-\lambda t})^k(e^{-\lambda t})^{10-k}$.

c. P(exactly 5 fail) = P(5 with par. λ fail, other 4 don't, 1 with par. θ doesn't fail) +
P(4 with par. λ fail, other 5 don't, 1 with par. θ fails) =

$$\binom{9}{5}\left(1-e^{-\lambda t}\right)^5(e^{-\lambda t})^4\left(e^{-\theta t}\right)+\binom{9}{4}\left(1-e^{-\lambda t}\right)^4(e^{-\lambda t})^5\left(1-e^{-\theta t}\right).$$

15.

a. Each X_i has cdf $F(x) = P(X_i \le x) = 1-e^{-\lambda x}$. Using this, the cdf of Y is
$F(y) = P(Y \le y) = P(X_1 \le y \cup [X_2 \le y \cap X_3 \le y])$
$= P(X_1 \le y) + P(X_2 \le y \cap X_3 \le y) - P(X_1 \le y \cap [X_2 \le y \cap X_3 \le y])$
$= (1-e^{-\lambda y})+(1-e^{-\lambda y})^2-(1-e^{-\lambda y})^3$ for $y > 0$.

The pdf of Y is $f(y) = F'(y) = \lambda e^{-\lambda y}+2(1-e^{-\lambda y})\left(\lambda e^{-\lambda y}\right)-3(1-e^{-\lambda y})^2\left(\lambda e^{-\lambda y}\right) = 4\lambda e^{-2\lambda y}-3\lambda e^{-3\lambda y}$
for $y > 0$.

b. $E(Y) = \int_0^\infty y\cdot\left(4\lambda e^{-2\lambda y}-3\lambda e^{-3\lambda y}\right)dy = 2\left(\frac{1}{2\lambda}\right)-\frac{1}{3\lambda}=\frac{2}{3\lambda}$.

16.

a. $f(x_1, x_3) = \int_{-\infty}^{\infty} f(x_1,x_2,x_3)dx_2 = \int_0^{1-x_1-x_3}144x_1x_2\left(1-x_3\right)dx_2 = 72x_1\left(1-x_3\right)\left(1-x_1-x_3\right)^2$ for $0 \le x_1$, $0 \le x_3$, $x_1 + x_3 \le 1$.

b. $P(X_1 + X_3 \le .5) = \int_0^{.5}\int_0^{.5-x_1}72x_1(1-x_3)(1-x_1-x_3)^2\,dx_3dx_1$ = (after much algebra) .53125.

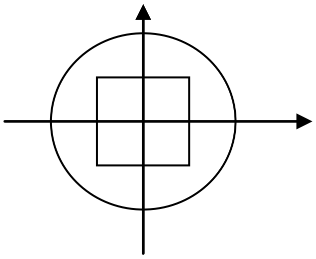
c. $f_{X_1}(x_1) = \int_{-\infty}^{\infty} f(x_1,x_3)dx_3 = \int_0^{1-x_1}72x_1\left(1-x_3\right)\left(1-x_1-x_3\right)^2dx_3 = 18x_1-48x_1^2+36x_1^3-6x_1^5$ for $0 \le x_1 \le 1$.

17.

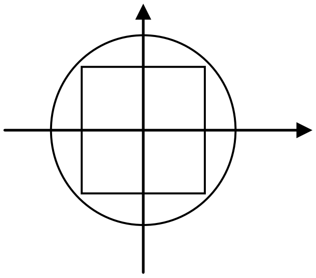
a. Let A denote the disk of radius R/2. Then $P((X,Y)$ lies in $A)$ = $\iint_A f(x,y)dxdy$

$= \iint_A \frac{1}{\pi R^2}dxdy = \frac{1}{\pi R^2}\iint_A dxdy = \frac{\text{area of } A}{\pi R^2} = \frac{\pi(R/2)^2}{\pi R^2} = \frac{1}{4} = .25$. Notice that, since the joint pdf of X and Y is a constant (i.e., (X,Y) is <u>uniform</u> over the disk), it will be the case for any subset A that $P((X,Y)$ lies in $A)$ = $\frac{\text{area of } A}{\pi R^2}$.

b. By the same ratio-of-areas idea, $P\left(-\frac{R}{2} \le X \le \frac{R}{2}, -\frac{R}{2} \le Y \le \frac{R}{2}\right) = \frac{R^2}{\pi R^2} = \frac{1}{\pi}$. This region is the square depicted in the graph below.



c. Similarly, $P\left(-\frac{R}{\sqrt{2}} \le X \le \frac{R}{\sqrt{2}}, -\frac{R}{\sqrt{2}} \le Y \le \frac{R}{\sqrt{2}}\right) = \frac{2R^2}{\pi R^2} = \frac{2}{\pi}$. This region is the slightly larger square depicted in the graph below, whose corners actually touch the circle.



d. $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}dy = \frac{2\sqrt{R^2-x^2}}{\pi R^2}$ for $-R \le x \le R$.

Similarly, $f_Y(y) = \frac{2\sqrt{R^2-y^2}}{\pi R^2}$ for $-R \le y \le R$. X and Y are <u>not</u> independent, since the joint pdf is not the product of the marginal pdfs: $\frac{1}{\pi R^2} \ne \frac{2\sqrt{R^2-x^2}}{\pi R^2} \cdot \frac{2\sqrt{R^2-y^2}}{\pi R^2}$.

18.

a. $p_{Y|X}(y \mid 1)$ results from dividing each entry in $x = 1$ row of the joint probability table by $p_X(1) = .34$:

$$p_{Y|X}(0|1) = \frac{.08}{.34} = .2353 \qquad p_{Y|X}(1|1) = \frac{.20}{.34} = .5882 \qquad p_{Y|X}(2|1) = \frac{.06}{.34} = .1765$$

b. $p_{Y|X}(y \mid 2)$ is requested; to obtain this divide each entry in the $x = 2$ row by $p_X(2) = .50$:

y	0	1	2
$p_{Y\|X}(y \mid 2)$	.12	.28	.60

c. $P(Y \le 1 \mid X = 2) = p_{Y|X}(0 \mid 2) + p_{Y|X}(1 \mid 2) = .12 + .28 = .40$.

d. $p_{X|Y}(x \mid 2)$ results from dividing each entry in the $y = 2$ column by $p_y(2) = .38$:

x	0	1	2
$p_{X\|Y}(x \mid 2)$	.0526	.1579	.7895

19. Throughout these solutions, $K = \frac{3}{380{,}000}$, as calculated in Exercise 9.

a. $$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{K(x^2+y^2)}{10Kx^2+.05} \text{ for } 20 \le y \le 30.$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{K(x^2+y^2)}{10Ky^2+.05} \text{ for } 20 \le x \le 30.$$

b. $P(Y \ge 25 \mid X = 22) = \int_{25}^{30} f_{Y|X}(y|22)dy = \int_{25}^{30} \frac{K((22)^2+y^2)}{10K(22)^2+.05}dy = .5559.$

$P(Y \ge 25) = \int_{25}^{30} f_Y(y)dy = \int_{25}^{30}(10Ky^2+.05)dy = .75$. So, given that the right tire pressure is 22 psi, it's much less likely that the left tire pressure is at least 25 psi.

c. $E(Y \mid X = 22) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|22)dy = \int_{20}^{30} y \cdot \frac{K((22)^2+y^2)}{10K(22)^2+.05}dy = 25.373$ psi.

$E(Y^2 \mid X = 22) = \int_{20}^{30} y^2 \cdot \frac{k((22)^2+y^2)}{10k(22)^2+.05}dy = 652.03 \Rightarrow$

$V(Y \mid X = 22) = E(Y^2 \mid X = 22) - [E(Y \mid X = 22)]^2 = 652.03 - (25.373)^2 = 8.24 \Rightarrow$

$SD(Y \mid X = 22) = 2.87$ psi.

20.

a. $P(X_1 = 2, \ldots, X_6 = 2) = \dfrac{12!}{2!2!2!2!2!2!}(.24)^2(.13)^2(.16)^2(.20)^2(.13)^2(.14)^2 = .00247.$

b. The marginal pmf of X_4, the number of orange candies, is Bin(n = 20, $p = p_4$ = .2). Therefore, $P(X_4 \le 5) = B(5; 20, .2) = .8042$.

c. Let $Y = X_1 + X_3 + X_4$ = the number of blue, green, or orange candies. Then Y is also binomial, but with parameter $p = p_1 + p_3 + p_4 = .24 + .16 + .20 = .60$. Therefore, $P(Y \ge 10) = 1 - P(Y \le 9) = 1 - B(9; 20, .60) = .8725$.

21.

a. $f_{X_3|X_1,X_2}(x_3 \mid x_1, x_2) = \dfrac{f(x_1,x_2,x_3)}{f_{X_1,X_2}(x_1,x_2)}$, where $f_{X_1,X_2}(x_1,x_2) =$ the marginal joint pdf of X_1 and X_2,

i.e. $f_{X_1,X_2}(x_1,x_2) = \int_{-\infty}^{\infty} f(x_1,x_2,x_3)\, dx_3$.

b. $f_{X_2,X_3|X_1}(x_2, x_3 \mid x_1) = \dfrac{f(x_1,x_2,x_3)}{f_{X_1}(x_1)}$, where $f_{X_1}(x_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2,x_3)dx_2dx_3$, the marginal pdf of X_1.

Section 5.2

22.

a. $E(X+Y) = \sum\sum (x+y)p(x,y)$ = (0 + 0)(.02) + (5 + 0)(.04) + … + (10 + 15)(.01) = 14.10.

Note: It can be shown that $E(X + Y)$ always equals $E(X) + E(Y)$, so in this case we could also work out the means of X and Y from their marginal distributions: $E(X) = 5.55$, $E(Y) = 8.55$, so $E(X + Y) = 5.55 + 8.55 = 14.10$.

b. For each coordinate, we need the maximum; e.g., max(0,0) = 0, while max(5,0) = 5 and max(5,10) = 10. Then calculate the sum: $E(\max(X,Y)) = \sum\sum \max(x,y)\cdot p(x,y) =$ max(0,0)(.02) + max(5,0)(.04) + … + max(10,15)(.01) = 0(.02) + 5(.04) + … + 15(.01) = 9.60.

23. $E(X_1 - X_2) = \sum_{x_1=0}^{4}\sum_{x_2=0}^{3}(x_1 - x_2)\cdot p(x_1,x_2)$ = (0 – 0)(.08) + (0 – 1)(.07) + … + (4 – 3)(.06) = .15.

Note: It can be shown that $E(X_1 - X_2)$ always equals $E(X_1) - E(X_2)$, so in this case we could also work out the means of X_1 and X_2 from their marginal distributions: $E(X_1) = 1.70$ and $E(X_2) = 1.55$, so $E(X_1 - X_2) = E(X_1) - E(X_2) = 1.70 - 1.55 = .15$.

24. Let $h(X, Y)$ = the number of individuals who handle the message. A table of the possible values of (X, Y) and of $h(X, Y)$ are displayed in the accompanying table.

		y					
	h(x, y)	1	2	3	4	5	6
	1	-	2	3	4	3	2
	2	2	-	2	3	4	3
x	3	3	2	-	2	3	4
	4	4	3	2	-	2	3
	5	3	4	3	2	-	2
	6	2	3	4	3	2	-

Since $p(x,y) = \frac{1}{30}$ for each possible (x, y),

$E[h(X,Y)] = \sum_x\sum_y h(x,y)\cdot p(x,y) = \sum_x\sum_y h(x,y)\cdot\frac{1}{30} = \cdots = \frac{84}{30} = 2.80$.

25. The expected value of X, being uniform on $[L - A, L + A]$, is simply the midpoint of the interval, L. Since Y has the same distribution, $E(Y) = L$ as well. Finally, since X and Y are independent,
$E(\text{area}) = E(XY) = E(X)\cdot E(Y) = L\cdot L = L^2$.

26. Revenue = $3X + 10Y$, so $E(\text{revenue}) = E(3X + 10Y)$

$= \sum_{x=0}^{5}\sum_{y=0}^{2}(3x+10y)\cdot p(x,y) = 0\cdot p(0,0)+\ldots+35\cdot p(5,2) = 15.4$ = \$15.40.

27. The amount of time Annie waits for Alvie, if Annie arrives first, is $Y - X$; similarly, the time Alvie waits for Annie is $X - Y$. Either way, the amount of time the first person waits for the second person is $h(X, Y) = |X - Y|$. Since X and Y are independent, their joint pdf is given by $f_X(x)\cdot f_Y(y) = (3x^2)(2y) = 6x^2y$. From these, the expected waiting time is

$E[h(X,Y)] = \int_0^1\int_0^1 |x-y|\cdot f(x,y)dxdy = \int_0^1\int_0^1 |x-y|\cdot 6x^2ydxdy$

$= \int_0^1\int_0^x (x-y)\cdot 6x^2ydydx + \int_0^1\int_x^1 (x-y)\cdot 6x^2ydydx = \frac{1}{6}+\frac{1}{12} = \frac{1}{4}$ hour, or 15 minutes.

28. $E(XY) = \sum_x\sum_y xy\cdot p(x,y) = \sum_x\sum_y xy\cdot p_x(x)\cdot p_y(y) = \sum_x xp_x(x)\cdot\sum_y yp_y(y) = E(X)\cdot E(Y)$. For the continuous case, replace summation by integration and pmfs by pdfs.

29. $\text{Cov}(X,Y) = -\frac{2}{75}$ and $\mu_X = \mu_Y = \frac{2}{5}$.

$E(X^2) = \int_0^1 x^2\cdot f_X(x)dx = 12\int_0^1 x^3(1-x^2dx) = \frac{12}{60} = \frac{1}{5}$, so $V(X) = \frac{1}{5} - \left(\frac{2}{5}\right)^2 = \frac{1}{25}$.

Similarly, $V(Y) = \frac{1}{25}$, so $\rho_{X,Y} = \frac{-\frac{2}{75}}{\sqrt{\frac{1}{25}}\cdot\sqrt{\frac{1}{25}}} = -\frac{50}{75} = -\frac{2}{3}$.

30.

a. $E(X) = 5.55$, $E(Y) = 8.55$, $E(XY) = (0)(.02) + (0)(.06) + \ldots + (150)(.01) = 44.25$, so $\text{Cov}(X,Y) = 44.25 - (5.55)(8.55) = -3.20$.

b. By direct computation, $\sigma_X^2 = 12.45$ and $\sigma_Y^2 = 19.15$, so $\rho_{X,Y} = \dfrac{-3.20}{\sqrt{(12.45)(19.15)}} = -.207$.

31.

a. $E(X) = \int_{20}^{30} x f_X(x)dx = \int_{20}^{30} x\left[10Kx^2 + .05\right]dx = \dfrac{1925}{76} = 25.329 = E(Y)$,

$E(XY) = \int_{20}^{30}\int_{20}^{30} xy \cdot K(x^2 + y^2)dxdy = \dfrac{24375}{38} = 641.447 \Rightarrow$

$\text{Cov}(X, Y) = 641.447 - (25.329)^2 = -.1082$.

b. $E(X^2) = \int_{20}^{30} x^2\left[10Kx^2 + .05\right]dx = \dfrac{37040}{57} = 649.8246 = E(Y^2) \Rightarrow$

$V(X) = V(Y) = 649.8246 - (25.329)^2 = 8.2664 \Rightarrow \rho = \dfrac{-.1082}{\sqrt{(8.2664)(8.2664)}} = -.0131$.

32. $E(XY) = \int_0^\infty \int_0^\infty xy \cdot xe^{-x(1+y)}dydx = \cdots = 1$. Yet, since the marginal pdf of Y is $f_Y(y) = \dfrac{1}{(1-y)^2}$ for $y \ge 0$, $E(Y) = \int_0^\infty \dfrac{y}{(1+y)^2}dy = \infty$. Therefore, Cov($X$, Y) and Corr(X, Y) do not exist, since they require this integral (among others) to be convergent.

33. Since $E(XY) = E(X) \cdot E(Y)$, $\text{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = E(X) \cdot E(Y) - E(X) \cdot E(Y) = 0$, and since $\text{Corr}(X, Y) = \dfrac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$, then $\text{Corr}(X, Y) = 0$.

34.

a. In the discrete case, $V[h(X,Y)] = E\{[h(X,Y) - E(h(X,Y))]^2\} =$ $\sum_x \sum_y [h(x,y) - E(h(X,Y))]^2 p(x,y)$ or $\sum_x \sum_y [h(x,y)^2 p(x,y)] - [E(h(X,Y))]^2$ For the continuous case, replace the double sums with double integrals and the pmf with the pdf.

b. $E[h(X, Y)] = E[\max(X, Y)] = 9.60$, and $E[h^2(X, Y)] = E[(\max(X, Y))^2] =$
$(0)^2(.02) + (5)^2(.06) + \ldots + (15)^2(.01) = 105.5$, so $V[\max(X, Y)] = 105.5 - (9.60)^2 = 13.34$.

35.

a. $\text{Cov}(aX + b, cY + d) = E[(aX + b)(cY + d)] - E(aX + b) \cdot E(cY + d)$
$= E[acXY + adX + bcY + bd] - (aE(X) + b)(cE(Y) + d)$
$= acE(XY) + adE(X) + bcE(Y) + bd - [acE(X)E(Y) + adE(X) + bcE(Y) + bd]$
$= acE(XY) - acE(X)E(Y) = ac[E(XY) - E(X)E(Y)] = ac\text{Cov}(X, Y)$.

b. Corr($aX + b, cY + d$) = $\dfrac{\text{Cov}(aX+b, cY+d)}{SD(aX+b)SD(cY+d)} = \dfrac{ac\text{Cov}(X,Y)}{|a|\cdot|c|\,SD(X)SD(Y)} = \dfrac{ac}{|ac|}\text{Corr}(X,Y)$. When a and c have the same signs, $ac = |ac|$, and we have
Corr($aX + b, cY + d$) = Corr(X, Y)

c. When a and c differ in sign, $|ac| = –ac$, and we have Corr($aX + b, cY + d$) = –Corr(X, Y).

36. Use the previous exercise: Cov(X, Y) = Cov($X, aX + b$) = aCov(X, X) = $aV(X)$ ⇒
so Corr(X,Y) = $\dfrac{aV(X)}{\sigma_X \cdot \sigma_Y} = \dfrac{aV(X)}{\sigma_X \cdot |a|\,\sigma_X} = \dfrac{a}{|a|}$ = 1 if $a > 0$, and –1 if $a < 0$.

Section 5.3

37. The joint pmf of X_1 and X_2 is presented below. Each joint probability is calculated using the independence of X_1 and X_2; e.g., $p(25, 25) = P(X_1 = 25) \cdot P(X_2 = 25) = (.2)(.2) = .04$.

		x_1			
	$p(x_1, x_2)$	25	40	65	
	25	.04	.10	.06	.2
x_2	40	.10	.25	.15	.5
	65	.06	.15	.09	.3
		.2	.5	.3	

a. For each coordinate in the table above, calculate $\bar{x}$. The six possible resulting $\bar{x}$ values and their corresponding probabilities appear in the accompanying pmf table.

$\bar{x}$	25	32.5	40	45	52.5	65
$p(\bar{x})$	.04	.20	.25	.12	.30	.09

From the table, $E(\bar{X}) = (25)(.04) + 32.5(.20) + ... + 65(.09) = 44.5$. From the original pmf, μ = 25(.2) + 40(.5) + 65(.3) = 44.5. So, $E(\bar{X}) = \mu$.

b. For each coordinate in the joint pmf table above, calculate $s^2 = \dfrac{1}{2-1}\sum_{i=1}^{2}(x_i - \bar{x})^2$. The four possible resulting s^2 values and their corresponding probabilities appear in the accompanying pmf table.

s^2	0	112.5	312.5	800
$p(s^2)$	.38	.20	.30	.12

From the table, $E(S^2) = 0(.38) + … + 800(.12) = 212.25$. From the original pmf,
$\sigma^2 = (25 – 44.5)^2(.2) + (40 – 44.5)^2(.5) + (65 – 44.5)^2(.3) = 212.25$. So, $E(S^2) = \sigma^2$.

38.

a. Since each X is 0 or 1 or 2, the possible values of T_o are 0, 1, 2, 3, 4.
$P(T_o = 0) = P(X_1 = 0 \text{ and } X_2 = 0) = (.2)(.2) = .04$ since X_1 and X_2 are independent.
$P(T_o = 1) = P(X_1 = 1 \text{ and } X_2 = 0, \text{ or } X_1 = 0 \text{ and } X_2 = 1) = (.5)(.2) + (.2)(.5) = .20$.
Similarly, $P(T_o = 2) = .37$, $P(T_o = 3) = .30$, and $P(T_o = 4) = .09$. These values are displayed in the pmf table below.

t_o	0	1	2	3	4
$p(t_o)$	.04	.20	.37	.30	.09

b. $E(T_o) = 0(.04) + 1(.20) + 2(.37) + 3(.30) + 4(.09) = 2.2$. This is exactly twice the population mean: $E(T_o) = 2\mu$.

c. First, $E(T_o^2) = 0^2(.04) + 1^2(.20) + 2^2(.37) + 3^2(.30) + 4^2(.09) = 5.82$. Then $V(T_o) = 5.82 – (2.2)^2 = .98$. This is exactly twice the population variance: $V(T_o) = 2\sigma^2$.

d. Assuming the pattern persists (and it does), when $T_o = X_1 + X_2 + X_3 + X_4$ we have $E(T_o) = 4\mu = 4(1.1) = 4.4$ and $V(T_o) = 4\sigma^2 = 4(.49) = 1.96$.

e. The event $\{T_o = 8\}$ occurs iff we encounter 2 lights on all four trips; i.e., $X_i = 2$ for each X_i. So, assuming the X_i are independent,
$P(T_o = 8) = P(X_1 = 2 \cap X_2 = 2 \cap X_3 = 2 \cap X_4 = 2) = P(X_1 = 2)\cdots P(X_4 = 2) = (.3)^4 = 0081$.
Similarly, $T_o = 7$ iff exactly three of the X_i are 2 and the remaining X_i is 1. The probability of that event is $P(T_o = 7) = (.3)(.3)(.3)(.5) + (.3)(.3)(.5)(.3) + \ldots = 4(.3)^3(.5) = .054$. Therefore, $P(T_o \geq 7) = P(T_o = 7) + P(T_o = 8) = .054 + .0081 = .0621$.

39. X is a binomial random variable with $n = 15$ and $p = .8$. The values of X, then $X/n = X/15$ along with the corresponding probabilities $b(x; 15, .8)$ are displayed in the accompanying pmf table.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$x/15$	0	.067	.133	.2	.267	.333	.4	.467	.533	.6	.667	.733	.8	.867	.933	1
$p(x/15)$	.000	.000	.000	.000	.000	.000	.001	.003	.014	.043	.103	.188	.250	.231	.132	.035

40.

a. There are only three possible values of M: 0, 5, and 10. Let's find the probabilities associated with 0 and 10, since they're the easiest.
$P(M = 0) = P(\text{all three draws are } 0) = P(X_1 = 0) \cdot P(X_2 = 0) \cdot P(X_3 = 0) = (5/10)(5/10)(5/10) = .125$.
$P(M = 10) = P(\text{at least one draw is a } 10) = 1 – P(\underline{\text{none}} \text{ of the three draws is a } 10) =$
$1 – P(X_1 \neq 10) \cdot P(X_2 \neq 10) \cdot P(X_3 \neq 10) = 1 – (8/10)(8/10)(8/10) = .488$.
Calculating all the options for $M = 5$ would be complicated; however, the three probabilities must sum to 1, so $P(M = 5) = 1 – [.125 + .488] = .387$. The probability distribution of M is displayed in the pmf table below.

m	0	5	10
$p(m)$	.125	.387	.488

An alternative solution would be to list all 27 possible combinations using a tree diagram and computing probabilities directly from the tree.

b. The statistic of interest is M, the maximum of X_1, X_2, or X_3. The population distribution for the X_i is as follows:

x	0	5	10
$p(x)$	5/10	3/10	2/10

Write a computer program to generate the digits 0-9 from a uniform distribution. Assign a value of $x = 0$ to the digits 0-4, a value of $x = 5$ to digits 5-7, and a value of $x = 10$ to digits 8 and 9. Generate samples of increasing sizes, keeping the number of replications constant, and compute $M = \max(X_1, \dots, X_n)$ from each sample. As n, the sample size, increases, $P(M = 0)$ goes to zero and $P(M = 10)$ goes to one. Furthermore, $P(M = 5)$ goes to zero, but at a slower rate than $P(M = 0)$.

41. The tables below delineate all 16 possible (x_1, x_2) pairs, their probabilities, the value of $\bar{x}$ for that pair, and the value of r for that pair. Probabilities are calculated using the independence of X_1 and X_2.

(x_1, x_2)	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4
probability	.16	.12	.08	.04	.12	.09	.06	.03
$\bar{x}$	1	1.5	2	2.5	1.5	2	2.5	3
r	0	1	2	3	1	0	1	2

(x_1, x_2)	3,1	3,2	3,3	3,4	4,1	4,2	4,3	4,4
probability	.08	.06	.04	.02	.04	.03	.02	.01
$\bar{x}$	2	2.5	3	3.5	2.5	3	3.5	4
r	2	1	0	1	3	2	1	2

a. Collecting the $\bar{x}$ values from the table above yields the pmf table below.

$\bar{x}$	1	1.5	2	2.5	3	3.5	4
$p(\bar{x})$	.16	.24	.25	.20	.10	.04	.01

b. $P(\bar{X} \le 2.5) = .16 + .24 + .25 + .20 = .85$.

c. Collecting the r values from the table above yields the pmf table below.

r	0	1	2	3
$p(r)$	.30	.40	.22	.08

d. With $n = 4$, there are numerous ways to get a sample average of at most 1.5, since $\bar{X} \le 1.5$ iff the sum of the X_i is at most 6. Listing out all options, $P(\bar{X} \le 1.5) = P(1,1,1,1) + P(2,1,1,1) + \dots + P(1,1,1,2) + P(1,1,2,2) + \dots + P(2,2,1,1) + P(3,1,1,1) + \dots + P(1,1,1,3)$
$= (.4)^4 + 4(.4)^3(.3) + 6(.4)^2(.3)^2 + 4(.4)^2(.2)^2 = .2400$.

42.

a. For each of the $\binom{6}{2} = 15$ pairs of employees, we calculate the sample mean. The collected values of $\overline{x}$ are displayed in the table below.

$\overline{x}$	27.75	28.0	29.7	29.95	31.65	31.9	33.6
$p(\overline{x})$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{2}{15}$	$\frac{1}{15}$

b. There are now only 3 possible $\overline{x}$ values, corresponding to the 3 offices.

$\overline{x}$	27.75	31.65	31.9
$p(\overline{x})$	1/3	1/3	1/3

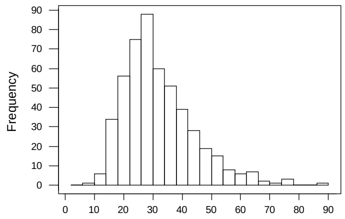
c. All three mean values are the same: 30.4333.

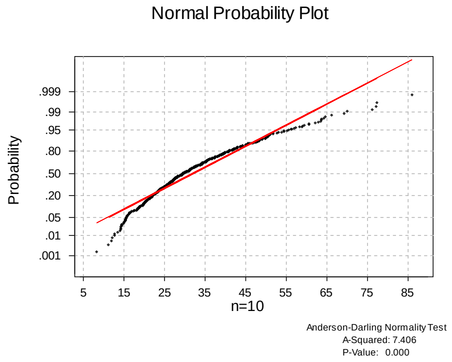
43. The statistic of interest is the fourth spread, or the difference between the medians of the upper and lower halves of the data. The population distribution is uniform with $A = 8$ and $B = 10$. Use a computer to generate samples of sizes $n = 5$, 10, 20, and 30 from a uniform distribution with $A = 8$ and $B = 10$. Keep the number of replications the same (say 500, for example). For each replication, compute the upper and lower fourth, then compute the difference. Plot the sampling distributions on separate histograms for $n = 5$, 10, 20, and 30.

44. Use a computer to generate samples of sizes $n = 5$, 10, 20, and 30 from a Weibull distribution with parameters as given, keeping the number of replications the same. For each replication, calculate the mean. The sampling distribution of $\overline{x}$ for $n = 5$ appears to be normal. Since larger sample sizes will produce distributions that are closer to normal, the others will also appear normal.

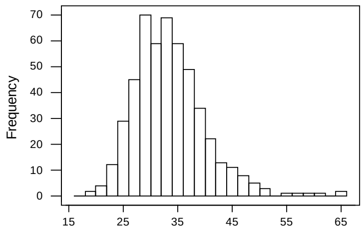
45. Using Minitab to generate the necessary sampling distribution, we can see that as n increases, the distribution slowly moves toward normality. However, even the sampling distribution for $n = 50$ is not yet approximately normal.

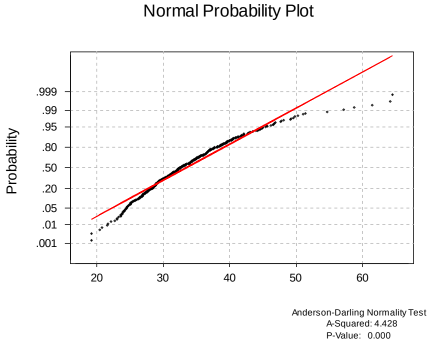
$n = 10$





$n = 50$





Section 5.4

46. $\mu = 70$ GPa, $\sigma = 1.6$ GPa

a. The sampling distribution of $\bar X$ is centered at $E(\bar X) = \mu = 70$ GPa, and the standard deviation of the $\bar X$ distribution is $\sigma_{\bar X} = \frac{\sigma_{\bar X}}{\sqrt{n}} = \frac{1.6}{\sqrt{16}} = 0.4$ GPa.

b. With $n = 64$, the sampling distribution of $\bar X$ is still centered at $E(\bar X) = \mu = 70$ GPa, but the standard deviation of the $\bar X$ distribution is $\sigma_{\bar X} = \frac{\sigma_{\bar X}}{\sqrt{n}} = \frac{1.6}{\sqrt{64}} = 0.2$ GPa.

c. $\bar X$ is more likely to be within 1 GPa of the mean (70 GPa) with the second, larger, sample. This is due to the decreased variability of $\bar X$ that comes with a larger sample size.

47.

a. In the previous exercise, we found $E(\bar X) = 70$ and $SD(\bar X) = 0.4$ when $n = 16$. If the diameter distribution is normal, then $\bar X$ is also normal, so
$P(69 \le \bar X \le 71) = P\left(\frac{69-70}{0.4} \le Z \le \frac{71-70}{0.4}\right) = P(-2.5 \le Z \le 2.5) = \Phi(2.5) - \Phi(-2.5) = .9938 - .0062 =$ .9876.

b. With $n = 25$, $E(\bar X) = 70$ but $SD(\bar X) = \frac{1.6}{\sqrt{25}} = 0.32$ GPa. So, $P(\bar X > 71) = P\left(Z > \frac{71-70}{0.32}\right) =$
$1 - \Phi(3.125) = 1 - .9991 = .0009$.

48.

a. No, it doesn't seem plausible that waist size distribution is approximately normal. The normal distribution is symmetric; however, for this data the mean is 86.3 cm and the median is 81.3 cm (these should be nearly equal). Likewise, for a symmetric distribution the lower and upper quartiles should be equidistant from the mean (or median); that isn't the case here.

If anything, since the upper percentiles stretch much farther than the lower percentiles do from the median, we might suspect a right-skewed distribution, such as the exponential distribution (or gamma or Weibull or …) is appropriate.

b. Irrespective of the population distribution's shape, the Central Limit Theorem tells us that $\overline{X}$ is (approximately) normal, with a mean equal to μ = 85 cm and a standard deviation equal to $\sigma/\sqrt{n} = 15/\sqrt{277} = .9$ cm. Thus,

$$P(\overline{X} \geq 86.3) = P\left(Z \geq \frac{86.3-85}{.9}\right) = 1 - \Phi(1.44) = .0749$$

c. Replace 85 with 82 in (b):

$$P(\overline{X} \geq 86.3) = P\left(Z \geq \frac{86.3-82}{.9}\right) = 1 - \Phi(4.77) \approx 1 - 1 = 0$$

That is, if the population mean waist size is 82 cm, there would be almost no chance of observing a sample mean waist size of 86.3 cm (or higher) in a random sample if 277 men. Since a sample mean of 86.3 was actually observed, it seems incredibly implausible that μ would equal 82 cm.

49.

a. 11 P.M. – 6:50 P.M. = 250 minutes. With $T_o = X_1 + \ldots + X_{40}$ = total grading time, $\mu_{T_o} = n\mu = (40)(6) = 240$ and $\sigma_{T_o} = \sigma \cdot \sqrt{n} = 37.95$, so $P(T_o \leq 250) \approx$ $P\left(Z \leq \frac{250-240}{37.95}\right) = P\left(Z \leq .26\right) = .6026.$

b. The sports report begins 260 minutes after he begins grading papers.

$P\left(T_0 > 260\right) = P\left(Z > \frac{260-240}{37.95}\right) = P\left(Z > .53\right) = .2981.$

50.

a. No, courtship time cannot plausibly be normally distributed. Since X must be non-negative, realistically the interval $\mu \pm 3\sigma$ should be entirely non-negative; however, with $\mu = 120$ and $\sigma = 110$, the left boundary of the distribution is barely more than 1 standard deviation below the mean.

b. By the Central Limit Theorem, $\overline{X}$ is approximately normal, with mean μ = 120 min and standard deviation $\sigma/\sqrt{n} = 110/\sqrt{50}$ min. Hence,

$P(100 \leq \overline{X} \leq 1250) \approx P\left(\frac{100-120}{110/\sqrt{50}} \leq Z \leq \frac{125-120}{110/\sqrt{50}}\right) = P(–0.64 \leq Z \leq 0.32) = \Phi(0.32) – \Phi(–0.64) =$ .6255 – .2611 = .3644.

c. Similarly, $P(\overline{X} > 150) \approx P\left(Z > \frac{150-125}{110/\sqrt{50}}\right)$ = 1 – Φ(1.61) = 1 – .9463 = .0537.

d. No. According to the guideline given in Section 5.4, n should be greater than 30 in order to apply the CLT. Thus, using the same procedure for n = 15 as was used for n = 50 would not be appropriate.

51. Individual times are given by $X \sim N(10, 2)$. For day 1, $n = 5$, and so

$$P(\overline{X} \le 11) = P\left(Z \le \frac{11-10}{2/\sqrt{5}}\right) = P(Z \le 1.12) = .8686 .$$

For day 2, $n = 6$, and so

$$P(\overline{X} \le 11) = P(\overline{X} \le 11) = P\left(Z \le \frac{11-10}{2/\sqrt{6}}\right) = P(Z \le 1.22) = .8888 .$$

Finally, assuming the results of the two days are independent (which seems reasonable), the probability the sample average is at most 11 min on both days is (.8686)(.8888) = .7720.

52. We have $X \sim N(10,1)$, $n = 4$, $\mu_{T_o} = n\mu = (4)(10) = 40$ and $\sigma_{T_o} = \sigma\sqrt{n} = 2$. Hence, $T_o \sim N(40, 2)$. We desire the 95th percentile of T_o: 40 + (1.645)(2) = 43.29 hours.

53.

a. With the values provided,

$$P(\overline{X} \ge 51) = P\left(Z \ge \frac{51-50}{1.2/\sqrt{9}}\right) = P(Z \ge 2.5) = 1 - .9938 = .0062 .$$

b. Replace $n = 9$ by $n = 40$, and

$$P(\overline{X} \ge 51) = P\left(Z \ge \frac{51-50}{1.2/\sqrt{40}}\right) = P(Z \ge 5.27) \approx 0 .$$

54.

a. With $n = 5$, $\mu_{\overline{X}} = \mu = 2.65$, $\sigma_{\overline{X}} = \frac{\sigma_X}{\sqrt{n}} = \frac{.85}{\sqrt{25}} = .17$.

$$P(\overline{X} \le 3.00) = P\left(Z \le \frac{3.00 - 2.65}{.17}\right) = P(Z \le 2.06) = .9803.$$

$$P(2.65 \le \overline{X} \le 3.00) = P(\overline{X} \le 3.00) - P(\overline{X} \le 2.65) = .4803.$$

b. $P(\overline{X} \le 3.00) = P\left(Z \le \frac{3.00-2.65}{.85/\sqrt{n}}\right) = .99$ implies that $\frac{3.00-2.65}{85/\sqrt{n}} = 2.33,$ from which $n = 32.02$.
Thus, $n = 33$ will suffice.

55.

a. With Y = # of tickets, Y has approximately a normal distribution with $\mu = 50$ and $\sigma = \sqrt{\mu} = 7.071$. So, using a continuity correction from [35, 70] to [34.5, 70.5],

$$P(35 \le Y \le 70) \approx P\left(\frac{34.5-50}{7.071} \le Z \le \frac{70.5-50}{7.071}\right) = P(-2.19 \le Z \le 2.90) = .9838.$$

b. Now $\mu = 5(50) = 250$, so $\sigma = \sqrt{250} = 15.811$.
Using a continuity correction from [225, 275] to [224.5, 275.5], $P(225 \le Y \le 275) \approx$

$$P\left(\frac{224.5-250}{15.811} \le Z \le \frac{275.5-250}{15.811}\right) = P(-1.61 \le Z \le 1.61) = .8926.$$

c. Using software, part (a) = $\sum_{y=35}^{70} \frac{e^{-50}50^y}{y!}$ = .9862 and part (b) = $\sum_{y=225}^{275} \frac{e^{-250}250^y}{y!}$ = .8934. Both of the approximations in (a) and (b) are correct to 2 decimal places.

56.

a. Let X = the number of erroneous bits out of 1000, so $X \sim \text{Bin}(1000, .10)$. If we approximate X by a normal rv with $\mu = np = 100$ and $\sigma^2 = npq = 90$, then with a continuity correction $P(X \leq 125) = P(X \leq 125.5) \approx P\left(Z \leq \frac{125.5 - 100}{\sqrt{90}}\right) = P(Z \leq 2.69) = \Phi(2.69) = .9964$.

b. Let Y = the number of errors in the second transmission, so $Y \sim \text{Bin}(1000, .10)$ and is independent of X. To find $P(|X - Y| \leq 50)$, use the facts that $E[X - Y] = 100 - 100 = 0$ and $V(X - Y) = V(X) + V(Y) = 90 + 90 = 180$. So, using a normal approximation to both binomial rvs, $P(|X - Y| \leq 50) \approx P\left(|Z| \leq \frac{50}{\sqrt{180}}\right) = P(|Z| \leq 3.73) \approx 1$.

57. With the parameters provided, $E(X) = \alpha\beta = 100$ and $V(X) = \alpha\beta^2 = 200$. Using a normal approximation, $P(X \leq 125) \approx P\left(Z \leq \frac{125 - 100}{\sqrt{200}}\right) = P(Z \leq 1.77) = .9616$.

Section 5.5

58.

a. $E(27X_1 + 125X_2 + 512X_3) = 27E(X_1) + 125E(X_2) + 512E(X_3)$
$= 27(200) + 125(250) + 512(100) = 87{,}850$.
$V(27X_1 + 125X_2 + 512X_3) = 27^2\, V(X_1) + 125^2\, V(X_2) + 512^2\, V(X_3)$
$= 27^2\,(10)^2 + 125^2\,(12)^2 + 512^2\,(8)^2 = 19{,}100{,}116$.

b. The expected value is still correct, but the variance is not because the covariances now also contribute to the variance.

59.

a. $E(X_1 + X_2 + X_3) = 180$, $V(X_1 + X_2 + X_3) = 45$, $SD(X_1 + X_2 + X_3) = \sqrt{45} = 6.708$.
$P(X_1 + X_2 + X_3 \leq 200) = P\left(Z \leq \frac{200 - 180}{6.708}\right) = P(Z \leq 2.98) = .9986$.
$P(150 \leq X_1 + X_2 + X_3 \leq 200) = P(-4.47 \leq Z \leq 2.98) \approx .9986$.

b. $\mu_{\bar{X}} = \mu = 60$ and $\sigma_{\bar{X}} = \frac{\sigma_X}{\sqrt{n}} = \frac{\sqrt{15}}{\sqrt{3}} = 2.236$, so
$P(\bar{X} \geq 55) = P\left(Z \geq \frac{55 - 60}{2.236}\right) = P(Z \geq -2.236) = .9875$ and
$P(58 \leq \bar{X} \leq 62) = P(-.89 \leq Z \leq .89) = .6266$.

c. $E(X_1 - .5X_2 - .5X_3) = \mu - .5\,\mu - .5\,\mu = 0$, while
$V(X_1 - .5X_2 - .5X_3) = \sigma_1^2 + .25\sigma_2^2 + .25\sigma_3^2 = 22.5 \Rightarrow SD(X_1 - .5X_2 - .5X_3) = 4.7434$. Thus,
$P(-10 \le X_1 - .5X_2 - .5X_3 \le 5) = P\left(\frac{-10-0}{4.7434} \le Z \le \frac{5-0}{4.7434}\right) = P\left(-2.11 \le Z \le 1.05\right) = .8531 - .0174 =$ .8357.

d. $E(X_1 + X_2 + X_3) = 150$, $V(X_1 + X_2 + X_3) = 36 \Rightarrow SD(X_1 + X_2 + X_3) = 6$, so
$P(X_1 + X_2 + X_3 \le 200) = P\left(Z \le \frac{160-150}{6}\right) = P(Z \le 1.67) = .9525.$
Next, we want $P(X_1 + X_2 \ge 2X_3)$, or, written another way, $P(X_1 + X_2 - 2X_3 \ge 0)$.
$E(X_1 + X_2 - 2X_3) = 40 + 50 - 2(60) = -30$ and $V(X_1 + X_2 - 2X_3) = \sigma_1^2 + \sigma_2^2 + 4\sigma_3^2 = 78 \Rightarrow$
$SD(X_1 + X_2 - 2X_3) = 8.832$, so
$P(X_1 + X_2 - 2X_3 \ge 0) = P\left(Z \ge \frac{0-(-30)}{8.832}\right) = P(Z \ge 3.40) = .0003.$

60. The fuel efficiency of the first two cars have parameters $\mu = 22$ and $\sigma = 1.2$, while the last three cars have parameters $\mu = 26$ and $\sigma = 1.5$. Since the X_is are normally distributed, Y is also normally distributed, with
$\mu_Y = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 - \frac{1}{3}\mu_3 - \frac{1}{3}\mu_4 - \frac{1}{3}\mu_5 = \frac{1}{2}(22) + \frac{1}{2}(22) - \frac{1}{3}(26) - \frac{1}{3}(26) - \frac{1}{3}(26) = 22 - 26 = -4$ mpg, and
$\sigma_Y^2 = (\frac{1}{2})^2\sigma_1^2 + (\frac{1}{2})^2\sigma_2^2 + (-\frac{1}{3})^2\sigma_3^2 + (-\frac{1}{3})^2\sigma_4^2 + (-\frac{1}{3})^2\sigma_5^2 = \frac{1}{4}(1.2)^2 + \frac{1}{4}(1.2)^2 + \frac{1}{9}(1.5)^2 + \frac{1}{9}(1.5)^2 + \frac{1}{9}(1.5)^2 = 1.47$
$\Rightarrow \sigma_Y = 1.212$ mpg.
Thus, $P(Y \ge 0) = 1 - \Phi\left(\frac{0-(-4)}{1.212}\right) = 1 - \Phi(3.30) = 1 - .9995 = .0005$. Similarly,
$P(Y > -2) = 1 - \Phi\left(\frac{-2-(-4)}{1.212}\right) = 1 - \Phi(1.65) = 1 - .9505 = .0495$.

61.

a. The marginal pmfs of X and Y are given in the solution to Exercise 7, from which $E(X) = 2.8$, $E(Y) = .7$, $V(X) = 1.66$, and $V(Y) = .61$. Thus, $E(X + Y) = E(X) + E(Y) = 3.5$, $V(X + Y) = V(X) + V(Y) = 2.27$, and the standard deviation of $X + Y$ is 1.51.

b. $E(3X + 10Y) = 3E(X) + 10E(Y) = 15.4$, $V(3X + 10Y) = 9V(X) + 100V(Y) = 75.94$, and the standard deviation of revenue is 8.71.

62. $E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3) = 15 + 30 + 20 = 65$ min, and
$V(X_1 + X_2 + X_3) = 1^2 + 2^2 + 1.5^2 = 7.25 \Rightarrow SD(X_1 + X_2 + X_3) = 2.6926$ min.
Thus, $P(X_1 + X_2 + X_3 \le 60) = P\left(Z \le \frac{60-65}{2.6926}\right) = P(Z \le -1.86) = .0314.$

63.

a. $E(X_1) = 1.70$, $E(X_2) = 1.55$, $E(X_1X_2) = \sum_{x_1}\sum_{x_2} x_1 x_2\, p(x_1, x_2) = \cdots = 3.33$, so
$\text{Cov}(X_1, X_2) = E(X_1X_2) - E(X_1)\,E(X_2) = 3.33 - 2.635 = .695$.

b. $V(X_1 + X_2) = V(X_1) + V(X_2) + 2\text{Cov}(X_1, X_2) = 1.59 + 1.0875 + 2(.695) = 4.0675$. This is much larger than $V(X_1) + V(X_2)$, since the two variables are positively correlated.

64. Let $X_1, \ldots, X_5$ denote morning times and $X_6, \ldots, X_{10}$ denote evening times.

a. $E(X_1 + \ldots + X_{10}) = E(X_1) + \ldots + E(X_{10}) = 5E(X_1) + 5E(X_6) = 5(4) + 5(5) = 45$ min.

b. $V(X_1 + \ldots + X_{10}) = V(X_1) + \ldots + V(X_{10}) = 5V(X_1) + 5V(X_6) = 5\left[\frac{64}{12} + \frac{100}{12}\right] = \frac{820}{12} = 68.33$.

c. $E(X_1 - X_6) = E(X_1) - E(X_6) = 4 - 5 = -1$ min, while
$V(X_1 - X_6) = V(X_1) + V(X_6) = \frac{64}{12} + \frac{100}{12} = \frac{164}{12} = 13.67$.

d. $E[(X_1 + \ldots + X_5) - (X_6 + \ldots + X_{10})] = 5(4) - 5(5) = -5$ min, while
$V[(X_1 + \ldots + X_5) - (X_6 + \ldots + X_{10})] = V(X_1 + \ldots + X_5) + V(X_6 + \ldots + X_{10}) = 68.33$, the same variance as for the sum in (b).

65.

a. $E(\overline{X} - \overline{Y}) = 0;\ V(\overline{X} - \overline{Y}) = \frac{\sigma^2}{25} + \frac{\sigma^2}{25} = .0032 \Rightarrow \sigma_{\overline{X}-\overline{Y}} = \sqrt{.0032} = .0566$
$\Rightarrow P\left(-.1 \le \overline{X} - \overline{Y} \le .1\right) = P\left(-1.77 \le Z \le 1.77\right) = .9232$.

b. $V(\overline{X} - \overline{Y}) = \frac{\sigma^2}{36} + \frac{\sigma^2}{36} = .0022222 \Rightarrow \sigma_{\overline{X}-\overline{Y}} = .0471$
$\Rightarrow P\left(-.1 \le \overline{X} - \overline{Y} \le .1\right) \approx P\left(-2.12 \le Z \le 2.12\right) = .9660$. The normal curve calculations are still justified here, even though the populations are not normal, by the Central Limit Theorem (36 is a sufficiently "large" sample size).

66.

a. With $M = 5X_1 + 10X_2$, $E(M) = 5(2) + 10(4) = 50$,
$V(M) = 5^2\,(.5)^2 + 10^2\,(1)^2 = 106.25$ and $\sigma_M = 10.308$.

b. $P(75 < M) = P\left(\frac{75-50}{10.308} < Z\right) = P(2.43 < Z) = .0075$.

c. $M = A_1X_1 + A_2X_2$ with the A_i and X_i all independent, so
$E(M) = E(A_1X_1) + E(A_2X_2) = E(A_1)E(X_1) + E(A_2)E(X_2) = 50$.

d. $V(M) = E(M^2) - [E(M)]^2$. Recall that for any rv Y, $E(Y^2) = V(Y) + [E(Y)]^2$.
Thus, $E(M^2) = E\left(A_1^2X_1^2 + 2A_1X_1A_2X_2 + A_2^2X_2^2\right)$
$= E\left(A_1^2\right)E\left(X_1^2\right) + 2E\left(A_1\right)E\left(X_1\right)E\left(A_2\right)E\left(X_2\right) + E\left(A_2^2\right)E\left(X_2^2\right)$ (by independence)
$= (.25 + 25)(.25 + 4) + 2(5)(2)(10)(4) + (.25 + 100)(1 + 16) = 2611.5625$, so
$V(M) = 2611.5625 - (50)^2 = 111.5625$.

e. $E(M) = 50$ still, but now $\text{Cov}(X_1, X_2) = (.5)(.5)(1.0) = .25$, so
$V(M) = a_1^2V(X_1) + 2a_1a_2\text{Cov}(X_1, X_2) + a_2^2V(X_2) = 6.25 + 2(5)(10)(.25) + 100 = 131.25$.

67. Letting X_1, X_2, and X_3 denote the lengths of the three pieces, the total length is $X_1 + X_2 – X_3$. This has a normal distribution with mean value 20 + 15 – 1 = 34 and variance .25 + .16 + .01 = .42 from which the standard deviation is .6481. Standardizing gives $P(34.5 \le X_1 + X_2 - X_3 \le 35) = P(.77 \le Z \le 1.54) = .1588$.

68. Let X_1 and X_2 denote the (constant) speeds of the two planes.

a. After two hours, the planes have traveled $2X_1$ km and $2X_2$ km, respectively, so the second will not have caught the first if $2X_1 + 10 > 2X_2$, i.e. if $X_2 – X_1 < 5$.
$X_2 – X_1$ has a mean 500 – 520 = –20, variance 100 + 100 = 200, and standard deviation 14.14. Thus,

$$P(X_2 - X_1 < 5) = P\left(Z < \frac{5-(-20)}{14.14}\right) = P(Z < 1.77) = .9616.$$

b. After two hours, #1 will be $10 + 2X_1$ km from where #2 started, whereas #2 will be $2X_2$ from where it started. Thus, the separation distance will be at most 10 if
$|2X_2 – 10 – 2X_1| \le 10$, i.e. $–10 \le 2X_2 – 10 – 2X_1 \le 10$ or $0 \le X_2 – X_1 \le 10$. The corresponding probability is $P(0 \le X_2 – X_1 \le 10) = P(1.41 \le Z \le 2.12) = .9830 – .9207 = .0623$.

69.

a. $E(X_1 + X_2 + X_3) = 800 + 1000 + 600 = 2400$.

b. Assuming independence of X_1, X_2 , X_3, $V(X_1 + X_2 + X_3) = (16)^2 + (25)^2 + (18)^2 = 1205$.

c. $E(X_1 + X_2 + X_3) =$ 2400 as before, but now $V(X_1 + X_2 + X_3)$
$= V(X_1) + V(X_2) + V(X_3) + 2\text{Cov}(X_1, X_2) + 2\text{Cov}(X_1, X_3) + 2\text{Cov}(X_2, X_3) = 1745$, from which the standard deviation is 41.77.

70.

a. $E(Y_i) = \frac{1}{2}$, so $E(W) = \sum_{i=1}^{n} i \cdot E(Y_i) = \frac{1}{2}\sum_{i=1}^{n} i = \frac{n(n+1)}{4}$.

b. $V(Y_i) = \frac{1}{4}$, so $V(W) = \sum_{i=1}^{n} i^2 \cdot V(Y_i) = \frac{1}{4}\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{24}$.

71.

a. $M = a_1X_1 + a_2X_2 + W\int_0^{12} x\,dx = a_1X_1 + a_2X_2 + 72W$, so
$E(M) = (5)(2) + (10)(4) + (72)(1.5) = 158$ and
$\sigma_M^2 = (5)^2(.5)^2 + (10)^2(1)^2 + (72)^2(.25)^2 = 430.25 \Rightarrow \sigma_M = 20.74$.

b. $P(M \le 200) = P\left(Z \le \frac{200-158}{20.74}\right) = P(Z \le 2.03) = .9788$.

72. The total elapsed time between leaving and returning is $T_o = X_1 + X_2 + X_3 + X_4$, with $E(T_o) = 40$, $\sigma^2_{T_o} = 30$, $\sigma_{T_o} = 5.477$. T_o is normally distributed, and the desired value t is the 99th percentile of the lapsed time distribution added to 10 A.M.:
10:00 + [40 + 2.33(5.477)] = 10:52.76 A.M.

73.

a. Both are approximately normal by the Central Limit Theorem.

b. The difference of two rvs is just an example of a linear combination, and a linear combination of normal rvs has a normal distribution, so $\overline{X} - \overline{Y}$ has approximately a normal distribution with $\mu_{\overline{X}-\overline{Y}} = 5$ and $\sigma_{\overline{X}-\overline{Y}} = \sqrt{\frac{8^2}{40} + \frac{6^2}{35}} = 1.621$.

c. $P\left(-1 \le \overline{X} - \overline{Y} \le 1\right) \approx P\left(\frac{-1-5}{1.6213} \le Z \le \frac{1-5}{1.6213}\right) = P(-3.70 \le Z \le -2.47) \approx .0068.$

d. $P\left(\overline{X} - \overline{Y} \ge 10\right) \approx P\left(Z \ge \frac{10-5}{1.6213}\right) = P(Z \ge 3.08) = .0010.$ This probability is quite small, so such an occurrence is unlikely if $\mu_1 - \mu_2 = 5$, and we would thus doubt this claim.

74. X is approximately normal with $\mu_1 = (50)(.7) = 35$ and $\sigma_1^2 = (50)(.7)(.3) = 10.5$, as is Y with $\mu_2 = 30$ and $\sigma_2^2 = 12$. Thus $\mu_{X-Y} = 5$ and $\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y = 22.5$, so
$P\left(-5 \le X - Y \le 5\right) \approx P\left(\frac{-10}{4.74} \le Z \le \frac{0}{4.74}\right) = P(-2.11 \le Z \le 0) = .4826$.

Supplementary Exercises

75.

a. $p_X(x)$ is obtained by adding joint probabilities across the row labeled x, resulting in $p_X(x)$ = .2, .5, .3 for x = 12, 15, 20 respectively. Similarly, from column sums $p_y(y)$ = .1, .35, .55 for y = 12, 15, 20 respectively.

b. $P(X \le 15 \text{ and } Y \le 15) = p(12,12) + p(12,15) + p(15,12) + p(15,15) = .25$.

c. $p_X(12) \cdot p_Y(12) = (.2)(.1) \ne .05 = p(12,12)$, so X and Y are not independent. (Almost any other (x, y) pair yields the same conclusion).

d. $E(X+Y) = \sum\sum (x+y)p(x,y) = 33.35$ (or = $E(X) + E(Y) = 33.35$).

e. $E(|X-Y|) = \sum\sum |x-y|\, p(x,y) = \cdots = 3.85$.

76. The roll-up procedure is not valid for the 75th percentile unless $\sigma_1 = 0$ and/or $\sigma_2 = 0$, as described below.

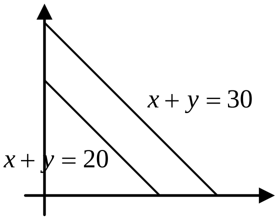
Sum of percentiles: $(\mu_1 + z\sigma_1) + (\mu_2 + z\sigma_2) = \mu_1 + \mu_2 + z(\sigma_1 + \sigma_2)$

Percentile of sums: $(\mu_1 + \mu_2) + z\sqrt{\sigma_1^2 + \sigma_2^2}$

These are equal when $z = 0$ (i.e. for the median) or in the unusual case when $\sigma_1 + \sigma_2 = \sqrt{\sigma_1^2 + \sigma_2^2}$, which happens when $\sigma_1 = 0$ and/or $\sigma_2 = 0$.

77.

a. $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_0^{20}\int_{20-x}^{30-x} kxydydx + \int_{20}^{30}\int_0^{30-x} kxydydx = \frac{81,250}{3}\cdot k \Rightarrow k = \frac{3}{81,250}$.



b. $$f_X(x) = \begin{cases} \int_{20-x}^{30-x} kxydy = k(250x - 10x^2) & 0 \le x \le 20 \\ \int_0^{30-x} kxydy = k(450x - 30x^2 + \tfrac{1}{2}x^3) & 20 \le x \le 30 \end{cases}$$

By symmetry, $f_Y(y)$ is obtained by substituting y for x in $f_X(x)$.
Since $f_X(25) > 0$ and $f_Y(25) > 0$, but $f(25, 25) = 0$, $f_X(x) \cdot f_Y(y) \ne f(x,y)$ for all (x, y), so X and Y are not independent.

c. $P(X + Y \le 25) = \int_0^{20}\int_{20-x}^{25-x} kxydydx + \int_{20}^{25}\int_0^{25-x} kxydydx = \frac{3}{81,250}\cdot\frac{230,625}{24} = .355.$

d. $E(X+Y) = E(X) + E(Y) = 2E(X) = 2\left\{\int_0^{20} x\cdot k\left(250x - 10x^2\right)dx\right.$
$\left.+\int_{20}^{30} x\cdot k\left(450x - 30x^2 + \tfrac{1}{2}x^3\right)dx\right\} = 2k(351,666.67) = 25.969.$

e. $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\cdot f(x,y)dxdy = \int_0^{20}\int_{20-x}^{30-x} kx^2y^2dydx$
$+\int_{20}^{30}\int_0^{30-x} kx^2y^2dydx = \frac{k}{3}\cdot\frac{33,250,000}{3} = 136.4103$, so
$\text{Cov}(X, Y) = 136.4103 - (12.9845)^2 = -32.19.$
$E(X^2) = E(Y^2) = 204.6154$, so $\sigma_X^2 = \sigma_Y^2 = 204.6154 - (12.9845)^2 = 36.0182$ and $\rho = \frac{-32.19}{36.0182} = -.894.$

f. $V(X + Y) = V(X) + V(Y) + 2\text{Cov}(X, Y) = 7.66.$

78. As suggested in the hint, for non-negative integers x and y write
$P(X(t) = x \text{ and } Y(t) = y) = P(X(t) = x \text{ and } W = x + y) = P(W = x + y) \cdot P(X(t) = x \mid W = x + y)$.

The first probability is, by assumption, Poisson: $P(W = x + y) = \dfrac{e^{-\mu}\mu^{x+y}}{(x+y)!}$. As for the second probability, conditional on $W = w$, each of these w loose particles has been released by time t with probability $G(t)$ and not yet released with probability $1 – G(t)$, independent of the other particles. Thus $X(t)$, the number of loose particles not yet released by time t, has a <u>binomial</u> distribution with parameters $n = w$ and $p = 1 – G(t)$ conditional on $W = w$.

Calculate this binomial probability and multiply by the Poisson answer above:

$$P(W = x + y) \cdot P(X(t) = x \mid W = x + y) = \frac{e^{-\mu}\mu^{x+y}}{(x+y)!}\cdot\binom{x+y}{x}(1-G(t))^{x}G(t)^{y} = \frac{e^{-\mu}\mu^{x+y}}{x!y!}(1-G(t))^{x}G(t)^{y}.$$

Then, to find the (marginal) distribution of $X(t)$, eliminate the variable y:

$$p_X(x) = \sum_{y=0}^{\infty}\frac{e^{-\mu}\mu^{x+y}}{x!y!}(1-G(t))^{x}G(t)^{y} = \frac{e^{-\mu}\mu^{x}}{x!}(1-G(t))^{x}\sum_{y=0}^{\infty}\frac{\mu^{y}}{y!}G(t)^{y}$$

$$= \frac{e^{-\mu}(\mu[1-G(t)])^{x}}{x!}\sum_{y=0}^{\infty}\frac{(\mu G(t))^{y}}{y!} = \frac{e^{-\mu}(\mu[1-G(t)])^{x}}{x!}\cdot e^{\mu G(t)} = \frac{e^{-\mu[1-G(t)]}(\mu[1-G(t)])^{x}}{x!}$$

This is precisely the Poisson pmf with parameter $\mu[1 – G(t)]$, as claimed.

79. $E(\bar{X}+\bar{Y}+\bar{Z}) = 500+900+2000 = 3400.$

$V(\bar{X}+\bar{Y}+\bar{Z}) = \dfrac{50^2}{365}+\dfrac{100^2}{365}+\dfrac{180^2}{365} = 123.014 \Rightarrow SD(\bar{X}+\bar{Y}+\bar{Z}) = 11.09.$

$P(\bar{X}+\bar{Y}+\bar{Z} \le 3500) = P(Z \le 9.0) \approx 1.$

80.

a. Let $X_1, \ldots, X_{12}$ denote the weights for the business-class passengers and $Y_1, \ldots, Y_{50}$ denote the tourist-class weights. Then T = total weight = $X_1 + \ldots + X_{12} + Y_1 + \ldots + Y_{50} = X + Y$.
$E(X) = 12E(X_1) = 12(30) = 360$; $V(X) = 12V(X_1) = 12(36) = 432$.
$E(Y) = 50E(Y_1) = 50(40) = 2000$; $V(Y) = 50V(Y_1) = 50(100) = 5000$.
Thus $E(T) = E(X) + E(Y) = 360 + 2000 = 2360$,
and $V(T) = V(X) + V(Y) = 432 + 5000 = 5432 \Rightarrow SD(T) = 73.7021$.

b. $P(T \le 2500) = P\left(Z \le \dfrac{2500-2360}{73.7021}\right) = P\left(Z \le 1.90\right) = .9713.$

81.

a. $E(N) \cdot \mu = (10)(40) = 400$ minutes.

b. We expect 20 components to come in for repair during a 4 hour period,
so $E(N) \cdot \mu = (20)(3.5) = 70$.

82. $X \sim \text{Bin}(200, .45)$ and $Y \sim \text{Bin}(300, .6)$. Because both ns are large, both X and Y are approximately normal, so $X + Y$ is approximately normal with mean $(200)(.45) + (300)(.6) = 270$, variance $200(.45)(.55) + 300(.6)(.4) = 121.40$, and standard deviation 11.02. Thus,

$P(X + Y \ge 250) = P\left(Z \ge \dfrac{249.5-270}{11.02}\right) = P\left(Z \ge -1.86\right) = .9686.$

83. $0.95 = P(\mu - .02 \le \overline{X} \le \mu + .02) = P\left(\frac{-.02}{.1/\sqrt{n}} \le Z \le \frac{.02}{.1/\sqrt{n}}\right) = P\left(-.2\sqrt{n} \le Z \le .2\sqrt{n}\right)$; since $P\left(-1.96 \le Z \le 1.96\right) = .95$, $.2\sqrt{n} = 1.96 \Rightarrow n = 97$. The Central Limit Theorem justifies our use of the normal distribution here.

84. I have 192 oz. The amount which I would consume if there were no limit is $T_o = X_1 + \ldots + X_{14}$ where each X_i is normally distributed with $\mu = 13$ and $\sigma = 2$. Thus T_o is normal with $\mu_{T_o} = 182$ and $\sigma_{T_o} = 7.483$, so $P(T_o < 192) = P(Z < 1.34) = .9099$.

85. The expected value and standard deviation of volume are 87,850 and 4370.37, respectively, so
$$P(\text{volume} \le 100{,}000) = P\left(Z \le \frac{100{,}000 - 87{,}850}{4370.37}\right) = P(Z \le 2.78) = .9973.$$

86. The student will not be late if $X_1 + X_3 \le X_2$, i.e. if $X_1 - X_2 + X_3 \le 0$. This linear combination has mean –2, variance 4.25, and standard deviation 2.06, so
$$P(X_1 - X_2 + X_3 \le 0) = P\left(Z \le \frac{0-(-2)}{2.06}\right) = P(Z \le .97) = .8340.$$

87.

a. $P(12 < X < 15) = P\left(\frac{12-13}{4} < Z < \frac{15-13}{4}\right) = P(-0.25 < Z < 0.5) = .6915 - .4013 = .2092$.

b. Since individual times are normally distributed, $\overline{X}$ is also normal, with the same mean $\mu = 13$ but with standard deviation $\sigma_{\overline{X}} = \sigma/\sqrt{n} = 4/\sqrt{16} = 1$. Thus,
$$P(12 < \overline{X} < 15) = P\left(\frac{12-13}{1} < Z < \frac{15-13}{1}\right) = P(-1 < Z < 2) = .9772 - .1587 = .8185.$$

c. The mean is $\mu = 13$. A sample mean $\overline{X}$ based on $n = 16$ observation is likely to be closer to the true mean than is a single observation X. That's because both are "centered" at μ, but the decreased variability in $\overline{X}$ gives it less ability to vary significantly from μ.

d. $P(\overline{X} > 20) = 1 - \Phi(7) \approx 1 - 1 = 0$.

88. Follow the hint, and apply the multiplication rule:
$$p(x_1, x_2, x_3) = P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = P(X_1 = x_1, X_2 = x_2, X_3 = x_3, N = n)$$
$$= P(X_1 = x_1, X_2 = x_2, X_3 = x_3 \mid N = n) \cdot P(N = n)$$
Conditional on $N = n$, the X's follow a multinomial distribution with count n and probabilities (.5, .3, .2). Also, N is Poisson(μ) by assumption. Write out the two terms, and then re-arrange so they match the Poisson distributions suggested in the problem:
$$p(x_1, x_2, x_3) = \frac{n!}{x_1!x_2!x_3!}(.5)^{x_1}(.3)^{x_2}(.2)^{x_3} \cdot \frac{e^{-\mu}\mu^n}{n!} = \frac{(.5)^{x_1}(.3)^{x_2}(.2)^{x_3} \cdot e^{-\mu}\mu^{x_1+x_2+x_3}}{x_1!x_2!x_3!}$$
$$= \frac{e^{-.5\mu}(.5\mu)^{x_1}}{x_1!} \cdot \frac{e^{-.3\mu}(.3\mu)^{x_2}}{x_2!} \cdot \frac{e^{-.2\mu}(.2\mu)^{x_3}}{x_3!}$$

Notice that this last step works because $e^{-.5\mu}e^{-.3\mu}e^{-.2\mu} = e^{-\mu}$. Looking now at the joint pmf, we observe that it factors into separate functions of x_1, x_2, and x_3. This implies that the rvs X_1, X_2, and X_3 are independent; moreover, we recognize these factors (i.e., their marginal pmfs) as Poisson pmfs with parameters $.5\mu$, $.3\mu$, and $.2\mu$, respectively, as claimed.

89.

a. $V(aX+Y) = a^2\sigma_X^2 + 2a\text{Cov}(X,Y) + \sigma_Y^2 = a^2\sigma_X^2 + 2a\sigma_X\sigma_Y\rho + \sigma_Y^2$.

Substituting $a = \dfrac{\sigma_Y}{\sigma_X}$ yields $\sigma_Y^2 + 2\sigma_Y^2\rho + \sigma_Y^2 = 2\sigma_Y^2(1+\rho) \geq 0$. This implies $(1 + \rho) \geq 0$, or $\rho \geq -1$.

b. The same argument as in **a** yields $2\sigma_Y^2(1-\rho) \geq 0$, from which $\rho \leq 1$.

c. Suppose $\rho = 1$. Then $V(aX-Y) = 2\sigma_Y^2(1-\rho) = 0$, which implies that $aX - Y$ is a constant. Solve for Y and $Y = aX$ – (constant), which is of the form $aX + b$.

90. $E(X+Y-t)^2 = \int_0^1\int_0^1 (x+y-t)^2 \cdot f(x,y)dxdy$. To find the minimizing value of t, take the derivative with respect to t and equate it to 0:

$0 = 2\int_0^1\int_0^1 (x+y-t)(-1)f(x,y) = 0 \Rightarrow \int_0^1\int_0^1 (x+y)f(x,y)dxdy = \int_0^1\int_0^1 t \cdot f(x,y)dxdy$. The left-hand side is $E(X + Y)$, while the right-hand side is $t \cdot 1 = t$.
So, the best prediction is $t = E(X + Y)$ = the individual's expected score = 1.167.

91.

a. With $Y = X_1 + X_2$, $F_Y(y) = \int_0^y \left\{ \int_0^{y-x_1} \frac{1}{2^{\nu_1/2}\Gamma(\nu_1/2)} \cdot \frac{1}{2^{\nu_{21}/2}\Gamma(\nu_2/2)} \cdot x_1^{\frac{\nu_1}{2}-1} x_2^{\frac{\nu_2}{2}-1} e^{-\frac{x_1+x_2}{2}} dx_2 \right\} dx_1$. But the inner integral can be shown to be equal to $\dfrac{1}{2^{(\nu_1+\nu_2)/2}\Gamma((\nu_1+\nu_2)/2)} y^{[(\nu_1+\nu_2)/2]-1}e^{-y/2}$, from which the result follows.

b. By **a,** $Z_1^2 + Z_2^2$ is chi-squared with ν = 2, so $(Z_1^2 + Z_2^2) + Z_3^2$ is chi-squared with ν = 3, etc., until $Z_1^2 + \ldots + Z_n^2$ is chi-squared with ν = n.

c. $\dfrac{X_i - \mu}{\sigma}$ is standard normal, so $\left[\dfrac{X_i-\mu}{\sigma}\right]^2$ is chi-squared with ν = 1, so the sum is chi-squared with parameter ν = n.

92.

a. Cov(X, $Y + Z$) = E[X(Y + Z)] – E(X) · E(Y + Z)
= E(XY) + E(XZ) – E(X) · E(Y) – E(X) · E(Z)
= E(XY) – E(X) · E(Y) + E(XZ) – E(X) · E(Z)
= Cov(X, Y) + Cov(X, Z).

b. Cov($X_1 + X_2$, $Y_1 + Y_2$) = Cov(X_1 , Y_1) + Cov(X_1 ,Y_2) + Cov(X_2 , Y_1) + Cov(X_2 ,Y_2) by applying (a) twice, which equals 16.

93.

a. $V(X_1) = V(W + E_1) = \sigma_W^2 + \sigma_E^2 = V(W + E_2) = V(X_2)$ and $\text{Cov}(X_1, X_2) =$ $\text{Cov}(W + E_1, W + E_2) = \text{Cov}(W, W) + \text{Cov}(W, E_2) + \text{Cov}(E_1, W) + \text{Cov}(E_1, E_2) =$ Cov(*W*, *W*) + 0 + 0 + 0 = *V*(*W*) = σ_W^2 .

Thus, $\rho = \dfrac{\sigma_W^2}{\sqrt{\sigma_W^2 + \sigma_E^2} \cdot \sqrt{\sigma_W^2 + \sigma_E^2}} = \dfrac{\sigma_W^2}{\sigma_W^2 + \sigma_E^2}$.

b. $\rho = \dfrac{1}{1 + .0001} = .9999.$

94.

a. Cov(*X*, *Y*) = Cov(*A*+*D*, *B*+*E*) = Cov(*A*, *B*) + Cov(*D*, *B*) + Cov(*A*, *E*) + Cov(*D*, *E*) = Cov(*A*, *B*) + 0 + 0 + 0 = Cov(*A*, *B*). Thus

$$\text{Corr}(X, Y) = \frac{\text{Cov}(A, B)}{\sqrt{\sigma_A^2 + \sigma_D^2} \cdot \sqrt{\sigma_B^2 + \sigma_E^2}} = \frac{\text{Cov}(A, B)}{\sigma_A \sigma_B} \cdot \frac{\sigma_A}{\sqrt{\sigma_A^2 + \sigma_D^2}} \cdot \frac{\sigma_B}{\sqrt{\sigma_B^2 + \sigma_E^2}}$$

The first factor in this expression is Corr(*A*, *B*), and (by the result of exercise 91**a**) the second and third factors are the square roots of Corr(X_1, X_2) and Corr(Y_1, Y_2), respectively. Clearly, measurement error reduces the correlation, since both square-root factors are between 0 and 1.

b. $\sqrt{.8100} \cdot \sqrt{.9025} = .855$. This is disturbing, because measurement error substantially reduces the correlation.

95. $E(Y) \doteq h(\mu_1, \mu_2, \mu_3, \mu_4) = 120\left[\frac{1}{10} + \frac{1}{15} + \frac{1}{20}\right] = 26.$

The partial derivatives of $h(\mu_1, \mu_2, \mu_3, \mu_4)$ with respect to x_1, x_2, x_3, and x_4 are $-\frac{x_4}{x_1^2}$, $-\frac{x_4}{x_2^2}$, $-\frac{x_4}{x_3^2}$, and $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$, respectively. Substituting $x_1 = 10$, $x_2 = 15$, $x_3 = 20$, and $x_4 = 120$ gives –1.2, –.5333, –.3000, and .2167, respectively, so $V(Y) = (1)(-1.2)^2 + (1)(-.5333)^2 + (1.5)(-.3000)^2 + (4.0)(.2167)^2 = 2.6783$, and the approximate sd of *Y* is 1.64.

96. The four second order partials are $\frac{2x_4}{x_1^3}$, $\frac{2x_4}{x_2^3}$, $\frac{2x_4}{x_3^3}$, and 0 respectively. Substitution gives *E*(*Y*) = 26 + .1200 + .0356 + .0338 = 26.1894.

97. Since X and Y are standard normal, each has mean 0 and variance 1.

a. $\text{Cov}(X, U) = \text{Cov}(X, .6X + .8Y) = .6\text{Cov}(X, X) + .8\text{Cov}(X, Y) = .6V(X) + .8(0) = .6(1) = .6$.
The covariance of X and Y is zero because X and Y are independent.
Also, $V(U) = V(.6X + .8Y) = (.6)^2V(X) + (.8)^2V(Y) = (.36)(1) + (.64)(1) = 1$. Therefore,

$$\text{Corr}(X,U) = \frac{\text{Cov}(X,U)}{\sigma_X\sigma_U} = \frac{.6}{\sqrt{1}\sqrt{1}} = .6, \text{ the coefficient on } X.$$

b. Based on part **a**, for any specified ρ we want $U = \rho X + bY$, where the coefficient b on Y has the feature that $\rho^2 + b^2 = 1$ (so that the variance of U equals 1). One possible option for b is $b = \sqrt{1-\rho^2}$, from which $U = \rho X + \sqrt{1-\rho^2}\,Y$.

98. Let Y denote the maximum bid. Following the hint, the cdf of Y is
$F_Y(y) = P(Y \le y) = P(X_i \le y \text{ for } i = 1, \ldots, n) = P(X_1 \le y) \times \ldots \times P(X_n \le y)$ by independence
Each Xi has the uniform cdf on [100, 200]: $F_X(x) = (x - 100)/(200 - 100) = (x - 100)/100$. Substituting, $F_Y(y) = (y - 100)/100 \times \ldots \times (y - 100)/100 = (y - 100)^n/100^n$. To find the pdf, differentiate:
$f_Y(y) = n(y - 100)^{n-1}/100^n$. This pdf is valid on [100, 200], since Y must lie in this interval. Finally, to find the expected amount earned, $E(Y)$, evaluate an integral:

$$E(Y) = \int_{100}^{200} y \cdot f_Y(y)dy = \int_{100}^{200} y \cdot \frac{n(y-100)^{n-1}}{100^n}dy = \frac{n}{100^n}\int_{100}^{200} y(y-100)^{n-1}dy$$

$$= \frac{n}{100^n}\int_0^{100}(u+100)u^{n-1}du \qquad \text{substitute } u = y-100$$

$$= \frac{n}{100^n}\int_0^{100}(u^n+100u^{n-1})du = \frac{n}{100^n}\left[\frac{100^{n+1}}{n+1}+100\frac{100^n}{n}\right]$$

$$= 100\cdot\frac{2n+1}{n+1}$$

Notice that when $n = 1$ (single bidder), $E(Y) = 150$, the expected value of that single bid on [100, 200]. As n gets large, the fraction on the right converges to 2, and so $E(Y)$ gets ever closer to 200, the theoretical maximum possible bid.

CHAPTER 6

Section 6.1

1.

a. We use the sample mean, $\bar{x}$, to estimate the population mean μ. $\hat{\mu} = \bar{x} = \frac{\Sigma x_i}{n} = \frac{219.80}{27} = 8.1407.$

b. We use the sample median, $\tilde{x} = 7.7$ (the middle observation when arranged in ascending order).

c. We use the sample standard deviation, $s = \sqrt{s^2} = \sqrt{\frac{1860.94 - \frac{(219.8)^2}{27}}{26}} = 1.660.$

d. With "success" = observation greater than 10, x = # of successes = 4, and $\hat{p} = \frac{x}{n} = \frac{4}{27} = .1481.$

e. We use the sample (std dev)/(mean), or $\frac{s}{\bar{x}} = \frac{1.660}{8.1407} = .2039.$

2.

a. A sensible point estimate of the population mean μ is the sample mean, $\bar{x}$ = 49.95 mg/dl.

b. Of interest is the population median, $\tilde{\mu}$. The logical point estimate is the sample median, $\tilde{x}$, which is the average of the 10th and 11th ordered values: $\tilde{x}$ = 47.5 mg/dl.

c. The point estimate of the population sd, σ, is the sample standard deviation, s = 16.81 mg/dl.

d. The natural estimate of $p = P(X \geq 60)$ is the sample proportion of HDL observations that are at least 60. In this sample of n = 20 observations, 4 are 60 or higher, so the point estimate is $\hat{p}$ = 4/20 = .2.

3.

a. We use the sample mean, $\bar{x} = 1.3481.$

b. Because we assume normality, the mean = median, so we also use the sample mean $\bar{x} = 1.3481$. We could also easily use the sample median.

c. We use the 90th percentile of the sample: $\hat{\mu} + (1.28)\hat{\sigma} = \bar{x} + 1.28s = 1.3481 + (1.28)(.3385) = 1.7814$.

d. Since we can assume normality,

$$P(X < 1.5) \approx P\left(Z < \frac{1.5 - \bar{x}}{s}\right) = P\left(Z < \frac{1.5 - 1.3481}{.3385}\right) = P(Z < .45) = .6736.$$

e. The estimated standard error of $\bar{x} = \frac{\hat{\sigma}}{\sqrt{n}} = \frac{s}{\sqrt{n}} = \frac{.3385}{\sqrt{16}} = .0846.$

4.

a. $E\left(\overline{X}-\overline{Y}\right)=E\left(\overline{X}\right)-E\left(\overline{Y}\right)=\mu_1-\mu_2$; $\overline{x}-\overline{y}=8.141-8.575=-.434$.

b. $V\left(\overline{X}-\overline{Y}\right)=V\left(\overline{X}\right)+V\left(\overline{Y}\right)=\sigma^2_{\overline{X}}+\sigma^2_{\overline{Y}}=\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}$ $\sigma_{\overline{X}-\overline{Y}}=\sqrt{V\left(\overline{X}-\overline{Y}\right)}=\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}$. The estimate would be $s_{\overline{X}-\overline{Y}}=\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}=\sqrt{\frac{1.66^2}{27}+\frac{2.104^2}{20}}=.5687$.

c. $\frac{s_1}{s_2}=\frac{1.660}{2.104}=.7890.$

d. $V\left(X-Y\right)=V\left(X\right)+V\left(Y\right)=\sigma_1^2+\sigma_2^2=1.66^2+2.104^2=7.1824.$

5. Let θ = the total audited value. Three potential estimators of θ are $\hat{\theta}_1=N\overline{X}$, $\hat{\theta}_2=T-N\overline{D}$, and $\hat{\theta}_3=T\cdot\frac{\overline{X}}{\overline{Y}}$. From the data, $\overline{y}$ = 374.6, $\overline{x}$ = 340.6, and $\overline{d}$ = 34.0. Knowing N = 5,000 and T = 1,761,300, the three corresponding estimates are $\hat{\theta}_1=(5{,}000)(340.6)=1{,}703{,}000$, $\hat{\theta}_2=1{,}761{,}300-(5{,}000)(34.0)=1{,}591{,}300$, and $\hat{\theta}_3=1{,}761{,}300\left(\frac{340.6}{374.6}\right)=1{,}601{,}438.281$.

6.

a. Let $y_i=\ln(x_i)$ for i = 1, .., 40. Using software, the sample mean and sample sd of the y_is are $\overline{y}$ = 4.430 and s_y = 1.515. Using the sample mean and sample sd to estimate μ and σ, respectively, gives $\hat{\mu}$ = 4.430 and $\hat{\sigma}$ = 1.515 (whence $\hat{\sigma}^2=s_y^2$ = 2.295).

b. $E(X)\equiv\exp\left[\mu+\frac{\sigma^2}{2}\right]$. It is natural to estimate $E(X)$ by using $\hat{\mu}$ and $\hat{\sigma}^2$ in place of μ and σ^2 in this expression: $\widehat{E(X)}=\exp\left[4.430+\frac{2.295}{2}\right]$ = 264.4 μg.

7.

a. $\hat{\mu}=\overline{x}=\frac{\sum x_i}{n}=\frac{1206}{10}=120.6.$

b. $\hat{\tau}=10{,}000\ \hat{\mu}=1{,}206{,}000.$

c. 8 of 10 houses in the sample used at least 100 therms (the "successes"), so $\hat{p}=\frac{8}{10}=.80.$

d. The ordered sample values are 89, 99, 103, 109, 118, 122, 125, 138, 147, 156, from which the two middle values are 118 and 122, so $\hat{\tilde{\mu}}=\tilde{x}=\frac{118+122}{2}=120.0.$

8.

a. With p denoting the true proportion of *non*-defective components, $\hat{p} = \frac{80-12}{80} = .85$.

b. P(system works) = p^2 , so an estimate of this probability is $\hat{p}^2 = \left(\frac{68}{80}\right)^2 = .723$.

9.

a. $E(\overline{X}) = \mu = E(X)$, so $\overline{X}$ is an unbiased estimator for the Poisson parameter μ. Since n = 150,

$$\hat{\mu} = \overline{x} = \frac{\Sigma x_i}{n} = \frac{(0)(18)+(1)(37)+...+(7)(1)}{150} = \frac{317}{150} = 2.11.$$

b. $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{\mu}}{\sqrt{n}}$, so the estimated standard error is $\sqrt{\frac{\hat{\mu}}{n}} = \frac{\sqrt{2.11}}{\sqrt{150}} = .119$.

10.

a. The hint tells us that $E(\overline{X}^2) = V(\overline{X}) + [E(\overline{X})]^2$. We know that $E(\overline{X}) = \mu$ and $SD(\overline{X}) = \frac{\sigma}{\sqrt{n}}$, so

$E(\overline{X}^2) = \left(\frac{\sigma}{\sqrt{n}}\right)^2 + [\mu]^2 = \frac{\sigma^2}{n} + \mu^2$. Since $E(\overline{X}^2) \neq \mu^2$, we've discovered that $\overline{X}^2$ is <u>not</u> an unbiased estimator of μ^2 !

In fact, the bias equals $E(\overline{X}^2) - \mu^2 = \frac{\sigma^2}{n} > 0$, so $\overline{X}^2$ is *biased high*. It will tend to *over*estimate the true value of μ^2.

b. By linearity of expectation, $E(\overline{X}^2 - kS^2) = E(\overline{X}^2) - kE(S^2)$. The author proves in Section 6.1 that $E(S^2) = \sigma^2$, so $E(\overline{X}^2) - kE(S^2) = \frac{\sigma^2}{n} + \mu^2 - k\sigma^2$.

The goal is to find k so that $E(\overline{X}^2 - kS^2) = \mu^2$. That requires $\frac{\sigma^2}{n} - k\sigma^2 = 0$, or $k = \frac{1}{n}$.

11.

a. $E\left(\frac{X_1}{n_1} - \frac{X_2}{n_2}\right) = \frac{1}{n_1}E(X_1) - \frac{1}{n_2}E(X_2) = \frac{1}{n_1}(n_1 p_1) - \frac{1}{n_2}(n_2 p_2) = p_1 - p_2$.

b. $V\left(\frac{X_1}{n_1} - \frac{X_2}{n_2}\right) = V\left(\frac{X_1}{n_1}\right) + V\left(\frac{X_2}{n_2}\right) = \left(\frac{1}{n_1}\right)^2 V(X_1) + \left(\frac{1}{n_2}\right)^2 V(X_2) =$

$\frac{1}{n_1^2}(n_1 p_1 q_1) + \frac{1}{n_2^2}(n_2 p_2 q_2) = \frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}$, and the standard error is the square root of this quantity.

c. With $\hat{p}_1 = \frac{x_1}{n_1}$, $\hat{q}_1 = 1 - \hat{p}_1$, $\hat{p}_2 = \frac{x_2}{n_2}$, $\hat{q}_2 = 1 - \hat{p}_2$, the estimated standard error is $\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}$.

d. $\left(\hat{p}_1 - \hat{p}_2\right) = \frac{127}{200} - \frac{176}{200} = .635 - .880 = -.245$

e. $\sqrt{\frac{(.635)(.365)}{200} + \frac{(.880)(.120)}{200}} = .041$

12. $E\left[\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}\right] = \frac{(n_1-1)}{n_1+n_2-2}E(S_1^2) + \frac{(n_2-1)}{n_1+n_2-2}E(S_2^2) = \frac{(n_1-1)}{n_1+n_2-2}\sigma^2 + \frac{(n_2-1)}{n_1+n_2-2}\sigma^2 = \sigma^2$.

13. $\mu = E(X) = \int_{-1}^{1} x \cdot \tfrac{1}{2}(1+\theta x)dx = \left.\frac{x^2}{4} + \frac{\theta x^3}{6}\right|_{-1}^{1} = \frac{1}{3}\theta \Rightarrow \theta = 3\mu$

$\Rightarrow \hat{\theta} = 3\bar{X} \Rightarrow E(\hat{\theta}) = E(3\bar{X}) = 3E(\bar{X}) = 3\mu = 3\left(\frac{1}{3}\right)\theta = \theta.$

14.

a. min(x_i) = 202 and max(x_i) = 525, so the estimate of the number of planes manufactured is max(x_i) – min(x_i) + 1 = 525 – 202 + 1 = 324.

b. The estimate will equal the true number of planes manufactured iff min(x_i) = α and max(x_i) = β, i.e., iff the smallest serial number in the population and the largest serial number in the population both appear in the sample. The estimator is not unbiased. This is because max(x_i) never overestimates β and will usually underestimate it (unless max(x_i) = β) , so that *E*[max(X_i)] < β.
Similarly, *E*[min(X_i)] > α, so *E*[max(X_i) – min(X_i)] < β – α + 1; The estimate will usually be smaller than β – α + 1, and can never exceed it.

15.

a. $E(X^2) = 2\theta$ implies that $E\left(\frac{X^2}{2}\right) = \theta$. Consider $\hat{\theta} = \frac{\sum X_i^2}{2n}$. Then

$E\left(\hat{\theta}\right) = E\left(\frac{\sum X_i^2}{2n}\right) = \frac{\sum E\left(X_i^2\right)}{2n} = \frac{\sum 2\theta}{2n} = \frac{2n\theta}{2n} = \theta$, implying that $\hat{\theta}$ is an unbiased estimator for *θ*.

b. $\sum x_i^2 = 1490.1058$, so $\hat{\theta} = \frac{1490.1058}{20} = 74.505.$

16.

a. By linearity of expectation, $E(\hat{\mu}) = E(\delta\overline{X} + (1-\delta)\overline{Y}) = \delta E(\overline{X}) + (1-\delta)E(\overline{Y})$. We know that $E(\overline{X}) = \mu$ and $E(\overline{Y}) = \mu$, and we're told these are the same μ. Therefore, $E(\hat{\mu}) = \delta\mu + (1-\delta)\mu = \mu$. That proves $\hat{\mu}$ is unbiased for μ.

b. Assuming the samples are independent, $V(\hat{\mu}) = V(\delta\overline{X} + (1-\delta)\overline{Y}) = \delta^2 V(\overline{X}) + (1-\delta)^2 V(\overline{Y})$. We know the variance of a sample mean, so this continues $V(\hat{\mu}) = V(\delta\overline{X} + (1-\delta)\overline{Y}) = \delta^2\frac{\sigma_X^2}{m} + (1-\delta)^2\frac{\sigma_Y^2}{n} = \delta^2\frac{\sigma^2}{m} + (1-\delta)^2\frac{4\sigma^2}{n}$. To minimize this, take the derivative and set it equal to zero:

$$\frac{dV(\hat{\mu})}{d\delta} = 2\delta\frac{\sigma^2}{m} + 2(1-\delta)(-1)\frac{4\sigma^2}{n} = 0 \Rightarrow \delta = \frac{4m}{4m+n}.$$

In other words, that's the weight we should give the *X*-sample so that the variability in our estimator, $\hat{\mu}$, is minimized.

17.

a.
$$E(\hat{p}) = \sum_{x=0}^{\infty}\frac{r-1}{x+r-1}\cdot\binom{x+r-1}{x}\cdot p^r\cdot(1-p)^x$$
$$= p\sum_{x=0}^{\infty}\frac{(x+r-2)!}{x!(r-2)!}\cdot p^{r-1}\cdot(1-p)^x = p\sum_{x=0}^{\infty}\binom{x+r-2}{x}p^{r-1}(1-p)^x = p\sum_{x=0}^{\infty}nb(x;r-1,p) = p.$$

b. For the given sequence, $x = 5$, so $\hat{p} = \frac{5-1}{5+5-1} = \frac{4}{9} = .444$.

18.

a. $f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}$, so $f(\mu;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}$ and $\frac{1}{4n[[f(\mu)]^2} = \frac{2\pi\sigma^2}{4n} = \frac{\pi}{2}\cdot\frac{\sigma^2}{n}$.

Since $\frac{\pi}{2} > 1, V(\tilde{X}) > V(\overline{X})$.

b. $f(\mu) = \frac{1}{\pi}$, so $V(\tilde{X}) \approx \frac{\pi^2}{4n} = \frac{2.467}{n}$.

19.

a. $\lambda = .5p + .15 \Rightarrow 2\lambda = p + .3$, so $p = 2\lambda - .3$ and $\hat{p} = 2\hat{\lambda} - .3 = 2\left(\frac{Y}{n}\right) - .3$; the estimate is $2\left(\frac{20}{80}\right) - .3 = .2$.

b. $E(\hat{p}) = E\left(2\hat{\lambda} - .3\right) = 2E\left(\hat{\lambda}\right) - .3 = 2\lambda - .3 = p$, as desired.

c. Here $\lambda = .7p + (.3)(.3)$, so $p = \frac{10}{7}\lambda - \frac{9}{70}$ and $\hat{p} = \frac{10}{7}\left(\frac{Y}{n}\right) - \frac{9}{70}$.

Section 6.2

20.

a. To find the mle of p, we'll take the derivative of the log-likelihood function $\ell(p) = \ln\left[\binom{n}{x} p^x (1-p)^{n-x}\right] = \ln\binom{n}{x} + x\ln(p) + (n-x)\ln(1-p)$, set it equal to zero, and solve for p.
$\ell'(p) = \frac{d}{dp}\left[\ln\binom{n}{x} + x\ln(p) + (n-x)\ln(1-p)\right] = \frac{x}{p} - \frac{n-x}{1-p} = 0 \Rightarrow x(1-p) = p(n-x) \Rightarrow p = x/n$, so the mle of p is $\hat{p} = \frac{x}{n}$, which is simply the sample proportion of successes. For $n = 20$ and $x = 3$, $\hat{p} = \frac{3}{20} =$ .15.

b. Since X is binomial, $E(X) = np$, from which $E(\hat{p}) = E\left(\frac{X}{n}\right) = \frac{1}{n}E(X) = \frac{1}{n}(np) = p$; thus, $\hat{p}$ is an unbiased estimator of p.

c. By the invariance principle, the mle of $(1 - p)^5$ is just $(1 - \hat{p})^5$. For $n = 20$ and $x = 3$, we have $(1 - .15)^5 = .4437$.

21.

a. $E\left(X\right)=\beta\cdot\Gamma\left(1+\frac{1}{\alpha}\right)$ and $E\left(X^2\right)=V(X)+[E(X)]^2=\beta^2\Gamma\left(1+\frac{2}{\alpha}\right)$, so the moment estimators $\hat{\alpha}$ and $\hat{\beta}$ are the solution to $\bar{X}=\hat{\beta}\cdot\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)$, $\frac{1}{n}\sum X_i^2=\hat{\beta}^2\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)$. Thus $\hat{\beta}=\frac{\bar{X}}{\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)}$, so once $\hat{\alpha}$ has been determined $\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)$ is evaluated and $\hat{\beta}$ then computed. Since $\bar{X}^2=\hat{\beta}^2\cdot\Gamma^2\left(1+\frac{1}{\hat{\alpha}}\right)$, $\frac{1}{n}\sum\frac{X_i^2}{\bar{X}^2}=\frac{\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)}{\Gamma^2\left(1+\frac{1}{\hat{\alpha}}\right)}$, so this equation must be solved to obtain $\hat{\alpha}$.

b. From **a**, $\frac{1}{20}\left(\frac{16{,}500}{28.0^2}\right)=1.05=\frac{\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)}{\Gamma^2\left(1+\frac{1}{\hat{\alpha}}\right)}$, so $\frac{1}{1.05}=.95=\frac{\Gamma^2\left(1+\frac{1}{\hat{\alpha}}\right)}{\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)}$, and from the hint, $\frac{1}{\hat{\alpha}}=.2\Rightarrow\hat{\alpha}=5$. Then $\hat{\beta}=\frac{\bar{x}}{\Gamma(1.2)}=\frac{28.0}{\Gamma(1.2)}$.

22.

a. $E(X)=\int_0^1 x(\theta+1)x^\theta dx=\frac{\theta+1}{\theta+2}=1-\frac{1}{\theta+2}$, so the moment estimator $\hat{\theta}$ is the solution to $\bar{X}=1-\frac{1}{\hat{\theta}+2}$, yielding $\hat{\theta}=\frac{1}{1-\bar{X}}-2$. Since $\bar{x}=.80,\hat{\theta}=5-2=3.$

b. $f\left(x_1,...,x_n;\theta\right)=\left(\theta+1\right)^n\left(x_1x_2...x_n\right)^\theta$, so the log likelihood is $n\ln\left(\theta+1\right)+\theta\sum\ln\left(x_i\right)$. Taking $\frac{d}{d\theta}$ and equating to 0 yields $\frac{n}{\theta+1}=-\sum\ln(x_i)$, so $\hat{\theta}=-\frac{n}{\sum\ln(X_i)}-1$. Taking ln($x_i$) for each given x_i yields ultimately $\hat{\theta}=3.12$.

23. Determine the joint pdf (aka the likelihood function), take a logarithm, and then use calculus:

$$f(x_1,...,x_n\,|\,\theta)=\prod_{i=1}^n\frac{1}{\sqrt{2\pi\theta}}e^{-x_i^2/2\theta}=(2\pi\theta)^{-n/2}e^{-\sum x_i^2/2\theta}$$

$$\ell(\theta)=\ln[f(x_1,...,x_n\,|\,\theta)]=-\frac{n}{2}\ln(2\pi)-\frac{n}{2}\ln(\theta)-\sum x_i^2/2\theta$$

$$\ell'(\theta)=0-\frac{n}{2\theta}+\sum x_i^2/2\theta^2=0\Rightarrow-n\theta+\sum x_i^2=0$$

Solving for θ, the maximum likelihood estimator is $\hat{\theta}=\frac{1}{n}\sum X_i^2$.

24. The number of incorrect diagnoses, X, until r correct diagnoses are made has a negative binomial distribution: $X \sim \text{nb}(r, p)$. To find the mle of p, we'll take the derivative of the log-likelihood function $\ell(p) = \ln\left[\binom{x+r-1}{x} p^r (1-p)^x\right] = \ln\binom{x+r-1}{x} + r\ln(p) + x\ln(1-p)$, set it equal to zero, and solve for p.

$\ell'(p) = \dfrac{d}{dp}\left[\ln\binom{x+r-1}{x} + r\ln(p) + x\ln(1-p)\right] = \dfrac{r}{p} - \dfrac{x}{1-p} = 0 \Rightarrow r(1-p) = xp \Rightarrow p = r/(r+x)$, so the mle of p is $\hat{p} = \dfrac{r}{r+x}$. This is the number of successes over the total number of trials; with $r = 3$ and $x = 17$, $\hat{p} = .15$.

Yes, this is the same as the mle based on a random sample of 20 mechanics with 3 correct diagnoses — see the binomial mle for p in Exercise 6.20.

Both mles are equal to the fraction (number of successes) / (number of trials).

In contrast, the unbiased estimator from Exercise 6.17 is $\hat{p} = \dfrac{r-1}{r+x-1}$, which is not the same as the maximum likelihood estimator. (With $r = 3$ and $x = 17$, the calculated value of the unbiased estimator is 2/19, rather than 3/20.)

25.

a. $\hat{\mu} = \bar{x} = 384.4; s^2 = 395.16$, so $\frac{1}{n}\sum\left(x_i - \bar{x}\right)^2 = \hat{\sigma}^2 = \frac{9}{10}\left(395.16\right) = 355.64$ and $\hat{\sigma} = \sqrt{355.64} = 18.86$ (this is not s).

b. The 95th percentile is $\mu + 1.645\sigma$, so the mle of this is (by the invariance principle) $\hat{\mu} + 1.645\hat{\sigma} = 415.42$.

c. The mle of $P(X \le 400)$ is, by the invariance principle, $\Phi\left(\dfrac{400-\hat{\mu}}{\hat{\sigma}}\right) = \Phi\left(\dfrac{400-384.4}{18.86}\right) = \Phi(0.83) = .7967$.

26. $R_i \sim \text{Exponential}(\lambda)$ implies $Y_i = t_iR_i \sim \text{Exponential}(\lambda/t_i)$. Hence, the joint pdf of the Y's, aka the likelihood, is

$$L(\lambda) = f(y_1,...,y_n;\lambda) = (\lambda/t_1)e^{-(\lambda/t_1)y_1}\cdots(\lambda/t_n)e^{-(\lambda/t_n)y_n} = \frac{\lambda^n}{t_1\cdots t_n}e^{-\lambda\Sigma(y_i/t_i)}.$$

To determine the mle, find the log-likelihood, differentiate, and set equal to zero:

$$\ell(\lambda) = \ln[L(\lambda)] = n\ln(\lambda) - \ln(t_1\cdots t_n) - -\lambda\sum_{i=1}^{n}\frac{y_i}{t_i} \Rightarrow \ell'(\lambda) = \frac{n}{\lambda} - 0 - \sum_{i=1}^{n}\frac{y_i}{t_i} = 0 \Rightarrow \lambda = \frac{n}{\sum_{i=1}^{n}(y_i/t_i)}.$$

Therefore, the mle of λ under this model is $\hat{\lambda} = \dfrac{n}{\sum_{i=1}^{n}(Y_i/t_i)}$.

27.

a. $f\left(x_1,...,x_n;\alpha,\beta\right)=\dfrac{\left(x_1x_2...x_n\right)^{\alpha-1}e^{-\Sigma x_i/\beta}}{\beta^{n\alpha}\Gamma^n\left(\alpha\right)}$, so the log likelihood is $(\alpha-1)\sum\ln\left(x_i\right)-\dfrac{\sum x_i}{\beta}-n\alpha\ln\left(\beta\right)-n\ln\Gamma\left(\alpha\right)$. Equating both $\dfrac{d}{d\alpha}$ and $\dfrac{d}{d\beta}$ to 0 yields $\sum\ln\left(x_i\right)-n\ln\left(\beta\right)-n\dfrac{d}{d\alpha}\Gamma\left(\alpha\right)=0$ and $\dfrac{\sum x_i}{\beta^2}=\dfrac{n\alpha}{\beta}=0$, a very difficult system of equations to solve.

b. From the second equation in **a**, $\dfrac{\sum x_i}{\beta}=n\alpha\Rightarrow\overline{x}=\alpha\beta=\mu$, so the mle of μ is $\hat{\mu}=\overline{X}$.

28.

a. $\left(\dfrac{x_1}{\theta}\exp\left[-x_1^2/2\theta\right]\right)...\left(\dfrac{x_n}{\theta}\exp\left[-x_n^2/2\theta\right]\right)=\left(x_1...x_n\right)\dfrac{\exp\left[-\Sigma x_i^2/2\theta\right]}{\theta^n}$. The natural log of the likelihood function is $\ln\left(x_i...x_n\right)-n\ln\left(\theta\right)-\dfrac{\Sigma x_i^2}{2\theta}$. Taking the derivative with respect to θ and equating to 0 gives $-\dfrac{n}{\theta}+\dfrac{\Sigma x_i^2}{2\theta^2}=0$, so $n\theta=\dfrac{\Sigma x_i^2}{2}$ and $\theta=\dfrac{\Sigma x_i^2}{2n}$. The mle is therefore $\hat{\theta}=\dfrac{\Sigma X_i^2}{2n}$, which is identical to the unbiased estimator suggested in Exercise 15.

b. For $x > 0$ the cdf of X is $F(x;\ \theta) = P(X \le x) = 1-\exp\left[\dfrac{-x^2}{2\theta}\right]$. Equating this to .5 and solving for x gives the median in terms of θ: $.5=\exp\left[\dfrac{-x^2}{2\theta}\right]\Rightarrow x=\tilde{\mu}=\sqrt{-2\theta\ln(.5)}=\sqrt{1.3863\theta}$. The mle of $\tilde{\mu}$ is therefore $\sqrt{1.3863\hat{\theta}}$.

29.

a. The joint pdf (likelihood function) is

$$f\left(x_1,...,x_n;\lambda,\theta\right)=\begin{cases}\lambda^n e^{-\lambda\Sigma\left(x_i-\theta\right)} & x_1\ge\theta,...,x_n\ge\theta\\ 0 & \text{otherwise}\end{cases}$$

Notice that $x_1\ge\theta,...,x_n\ge\theta$ iff $\min\left(x_i\right)\ge\theta$, and that $-\lambda\Sigma\left(x_i-\theta\right)=-\lambda\Sigma x_i+n\lambda\theta$.

Thus likelihood = $\begin{cases}\lambda^n\exp\left(-\lambda\Sigma x_i\right)\exp\left(n\lambda\theta\right) & \min\left(x_i\right)\ge\theta\\ 0 & \min\left(x_i\right)<\theta\end{cases}$

Consider maximization with respect to θ. Because the exponent $n\lambda\theta$ is positive, increasing θ will increase the likelihood provided that $\min\left(x_i\right)\ge\theta$; if we make θ larger than $\min\left(x_i\right)$, the likelihood drops to 0. This implies that the mle of θ is $\hat{\theta}=\min\left(x_i\right)$. The log likelihood is now $n\ln(\lambda)-\lambda\Sigma\left(x_i-\hat{\theta}\right)$. Equating the derivative w.r.t. λ to 0 and solving yields $\hat{\lambda}=\dfrac{n}{\Sigma\left(x_i-\hat{\theta}\right)}=\dfrac{n}{\Sigma x_i-n\hat{\theta}}$.

b. $\hat{\theta}=\min\left(x_i\right)=.64,$ and $\Sigma x_i=55.80$, so $\hat{\lambda}=\dfrac{10}{55.80-6.4}=.202$

30. The likelihood is $f\left(y;n,p\right)=\binom{n}{y}p^y\left(1-p\right)^{n-y}$ where $p=P\left(X\geq 24\right)=1-\int_0^{24}\lambda e^{-\lambda x}dx=e^{-24\lambda}$. We know $\hat{p}=\frac{y}{n}$, so by the invariance principle $\hat{p}=e^{-24\hat{\lambda}}\Rightarrow\hat{\lambda}=-\frac{\ln\hat{p}}{24}=.0120$ for n = 20, y = 15.

Supplementary Exercises

31. Substitute $k=\varepsilon/\sigma_Y$ into Chebyshev's inequality to write $P(|Y-\mu_Y|\geq\varepsilon)\leq 1/(\varepsilon/\sigma_Y)^2=V(Y)/\varepsilon^2$. Since $E(\overline{X})=\mu$ and $V(\overline{X})=\sigma^2/n$, we may then write $P\left(\left|\overline{X}-\mu\right|\geq\varepsilon\right)\leq\frac{\sigma^2/n}{\varepsilon^2}$. As $n\to\infty$, this fraction converges to 0, hence $P\left(\left|\overline{X}-\mu\right|\geq\varepsilon\right)\to 0$, as desired.

32.

a. $F_Y\left(y\right)=P\left(Y\leq y\right)=P\left(X_1\leq y,...,X_n\leq y\right)=P\left(X_1\leq y\right)...P\left(X_n\leq y\right)=\left(\frac{y}{\theta}\right)^n$ for $0\leq y\leq\theta$, so $f_Y\left(y\right)=\frac{ny^{n-1}}{\theta^n}$.

b. $E(Y)=\int_0^\theta y\cdot\frac{ny^{n-1}}{n}dy=\frac{n}{n+1}\theta.$ While $\hat{\theta}=Y$ is not unbiased, $\frac{n+1}{n}Y$ is, since $E\left[\frac{n+1}{n}Y\right]=\frac{n+1}{n}E\left(Y\right)=\frac{n+1}{n}\cdot\frac{n}{n+1}\theta=\theta$.

33. Let x_1 = the time until the first birth, x_2 = the elapsed time between the first and second births, and so on. Then $f\left(x_1,...,x_n;\lambda\right)=\lambda e^{-\lambda x_1}\cdot\left(2\lambda\right)e^{-2\lambda x_2}...\left(n\lambda\right)e^{-n\lambda x_n}=n!\lambda^n e^{-\lambda\Sigma kx_k}$. Thus the log likelihood is $\ln\left(n!\right)+n\ln\left(\lambda\right)-\lambda\Sigma kx_k$. Taking $\frac{d}{d\lambda}$ and equating to 0 yields $\hat{\lambda}=\frac{n}{\Sigma kx_k}$.

For the given sample, n = 6, x_1 = 25.2, x_2 = 41.7 – 25.2 = 16.5, x_3 = 9.5, x_4 = 4.3, x_5 = 4.0, x_6 = 2.3; so $\sum_{k=1}^{6}kx_k=(1)(25.2)+(2)(16.5)+...+(6)(2.3)=137.7$ and $\hat{\lambda}=\frac{6}{137.7}=.0436$.

34. $\text{MSE}\left(KS^2\right)=V(KS^2)+\text{Bias}^2(KS^2)$. $\text{Bias}(KS^2)=E(KS^2)-\sigma^2=K\sigma^2-\sigma^2=\sigma^2\left(K-1\right)$, and

$$V(KS^2)=K^2V(S^2)=K^2\left(E\left[(S^2)^2\right]-\left[E(S^2)\right]^2\right)=K^2\left(\frac{(n+1)\sigma^4}{n-1}-\left(\sigma^2\right)^2\right)$$

$=\frac{2K^2\sigma^4}{n-1}\Rightarrow\text{MSE}=\left[\frac{2K^2}{n-1}+\left(K-1\right)^2\right]\sigma^4$. To find the minimizing value of K, take $\frac{d}{dK}$ and equate to 0; the result is $K=\frac{n-1}{n+1}$; thus the estimator which minimizes MSE is neither the unbiased estimator (K = 1) nor the mle ($K=\frac{n-1}{n}$).

35.

$x_i + x_j$	23.5	26.3	28.0	28.2	29.4	29.5	30.6	31.6	33.9	49.3
23.5	23.5	24.9	25.75	25.85	26.45	26.5	27.05	27.55	28.7	36.4
26.3		26.3	27.15	27.25	27.85	27.9	28.45	28.95	30.1	37.8
28.0			28.0	28.1	28.7	28.75	29.3	29.8	30.95	38.65
28.2				28.2	28.8	28.85	29.4	29.9	31.05	38.75
29.4					29.4	29.45	30.0	30.5	30.65	39.35
29.5						29.5	30.05	30.55	31.7	39.4
30.6							30.6	31.1	32.25	39.95
31.6								31.6	32.75	40.45
33.9									33.9	41.6
49.3										49.3

There are 55 averages, so the median is the 28th in order of increasing magnitude. Therefore, $\hat{\mu} = 29.5$.

36. With $\sum x = 555.86$ and $\sum x^2 = 15,490$, $s = \sqrt{2.1570} = 1.4687$. The $|x_i - \tilde{x}|$s are, in increasing order, .02, .02, .08, .22, .32, .42, .53, .54, .65, .81, .91, 1.15, 1.17, 1.30, 1.54, 1.54, 1.71, 2.35, 2.92, 3.50. The median of these values is $\frac{(.81+.91)}{2} = .86$. The estimate based on the resistant estimator is then $\frac{.86}{.6745} = 1.275$. This estimate is in reasonably close agreement with s.

37. Let $c = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\cdot\sqrt{\frac{2}{n-1}}}$. Then *E*(*cS*) = *cE*(*S*), and *c* cancels with the two Γ factors and the square root in *E*(*S*), leaving just *σ*. When *n* = 20, $c = \frac{\Gamma(9.5)}{\Gamma(10)\cdot\sqrt{\frac{2}{19}}} = \frac{(8.5)(7.5)\cdots(.5)\Gamma(.5)}{(10-1)!\sqrt{\frac{2}{19}}} = \frac{(8.5)(7.5)\cdots(.5)\sqrt{\pi}}{9!\sqrt{\frac{2}{19}}}$ = 1.0132.

38.

a. The likelihood is $\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu_i)}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i-\mu_i)}{2\sigma^2}} = \frac{1}{\left(2\pi\sigma^2\right)^n} e^{-\frac{\left(\Sigma(x_i-\mu_i)^2+\Sigma(y_i-\mu_i)^2\right)}{2\sigma^2}}$. The log likelihood is thus $-n\ln\left(2\pi\sigma^2\right) - \frac{\left(\Sigma(x_i-\mu_i)^2+\Sigma(y_i-\mu_i)^2\right)}{2\sigma^2}$. Taking $\frac{d}{d\mu_i}$ and equating to zero gives $\hat{\mu}_i = \frac{x_i + y_i}{2}$. Substituting these estimates of the $\hat{\mu}_i$s into the log likelihood gives

$$-n\ln\left(2\pi\sigma^2\right) - \tfrac{1}{2\sigma^2}\left(\sum\left(x_i - \frac{x_i + y_i}{2}\right)^2 + \sum\left(y_i - \frac{x_i + y_i}{2}\right)^2\right) = -n\ln\left(2\pi\sigma^2\right) - \tfrac{1}{2\sigma^2}\left(\tfrac{1}{2}\Sigma\left(x_i - y_i\right)^2\right).$$

Now taking $\frac{d}{d\sigma^2}$, equating to zero, and solving for σ^2 gives the desired result.

b. $E\left(\hat{\sigma}^2\right) = \frac{1}{4n}E\left(\Sigma\left(X_i - Y_i\right)^2\right) = \frac{1}{4n}\cdot\Sigma E\left(X_i - Y_i\right)^2$, but $E\left(X_i - Y_i\right)^2 = V\left(X_i - Y_i\right) + \left[E\left(X_i - Y_i\right)\right]^2 =$ $2\sigma^2 - [0]^2 = 2\sigma^2$. Thus $E\left(\hat{\sigma}^2\right) = \frac{1}{4n}\Sigma\left(2\sigma^2\right) = \frac{1}{4n}2n\sigma^2 = \frac{\sigma^2}{2}$, so the mle is definitely not unbiased—the expected value of the estimator is only half the value of what is being estimated!

CHAPTER 7

Section 7.1

1.

a. $z_{\alpha/2}$ = 2.81 implies that α/2 = 1 – Φ(2.81) = .0025, so α = .005 and the confidence level is 100(1–α)% = 99.5%.

b. $z_{\alpha/2}$ = 1.44 implies that α = 2[1 – Φ(1.44)] = .15, and the confidence level is 100(1–α)% = 85%.

c. 99.7% confidence implies that α = .003, α/2 = .0015, and $z_{.0015}$ = 2.96. (Look for cumulative area equal to 1 – .0015 = .9985 in the main body of table A.3.) Or, just use $z \approx 3$ by the empirical rule.

d. 75% confidence implies α = .25, α/2 = .125, and $z_{.125}$ = 1.15.

2.

a. The sample mean is the center of the interval, so $\bar{x} = \dfrac{114.4+115.6}{2} = 115$.

b. The interval (114.4, 115.6) has the 90% confidence level. The higher confidence level will produce a wider interval.

3.

a. A 90% confidence interval will be narrower. The z critical value for a 90% confidence level is 1.645, smaller than the z of 1.96 for the 95% confidence level, thus producing a narrower interval.

b. Not a correct statement. Once and interval has been created from a sample, the mean μ is either enclosed by it, or not. We have 95% confidence in the general procedure, under repeated and independent sampling.

c. Not a correct statement. The interval is an estimate for the population mean, not a boundary for population values.

d. Not a correct statement. In theory, if the process were repeated an infinite number of times, 95% of the intervals would contain the population mean μ. We *expect* 95 out of 100 intervals will contain μ, but we don't know this to be true.

4.

a. $58.3 \pm \frac{1.96(3)}{\sqrt{25}} = 58.3 \pm 1.18 = (57.1, 59.5)$.

b. $58.3 \pm \frac{1.96(3)}{\sqrt{100}} = 58.3 \pm .59 = (57.7, 58.9)$.

c. $58.3 \pm \frac{2.58(3)}{\sqrt{100}} = 58.3 \pm .77 = (57.5, 59.1)$.

d. 82% confidence ⇒ 1 – α = .02 ⇒ α = .18 ⇒ α/2 = .09, and $z_{.09}$ = 1.34. The interval is $58.3 \pm \frac{1.34(3)}{\sqrt{100}} = (57.9, 58.7)$.

e. $n = \left[\frac{2(2.58)3}{1}\right]^2 = 239.62 \nearrow 240$.

5.

a. $4.85 \pm \frac{(1.96)(.75)}{\sqrt{20}} = 4.85 \pm .33 =$ (4.52, 5.18).

b. $z_{\alpha/2}$ = z.01 = 2.33, so the interval is $4.56 \pm \frac{(2.33)(.75)}{\sqrt{16}} =$ (4.12, 5.00).

c. $n = \left[\frac{2(1.96)(.75)}{.40}\right]^2 = 54.02 \nearrow 55$.

d. Width w = 2(.2) = .4, so $n = \left[\frac{2(2.58)(.75)}{.4}\right]^2 = 93.61 \nearrow 94$.

6.

a. $8439 \pm \frac{(1.645)(100)}{\sqrt{25}} = 8439 \pm 32.9 =$ (8406.1, 8471.9).

b. $1 - \alpha = .92 \Rightarrow \alpha = .08 \Rightarrow \alpha / 2 = .04$ so $z_{\alpha/2} = z_{.04}$ = 1.75.

7. If $L = 2z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ and we increase the sample size by a factor of 4, the new length is $L' = 2z_{\alpha/2}\frac{\sigma}{\sqrt{4n}} = \left[2z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]\left(\frac{1}{2}\right) = \frac{L}{2}$. Thus halving the length requires n to be increased fourfold. If $n' = 25n$, then $L' = \frac{L}{5}$, so the length is decreased by a factor of 5.

8.

a. With probability 1 – α, $z_{\alpha_1} \le \left(\bar{X} - \mu\right) \div \left(\dfrac{\sigma}{\sqrt{n}}\right) \le z_{\alpha_2}$. These inequalities can be manipulated exactly as was done in the text to isolate μ; the result is $\bar{X} - z_{\alpha_2}\dfrac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\alpha_1}\dfrac{\sigma}{\sqrt{n}}$, so a 100(1–α)% confidence interval is $\left(\bar{X} - z_{\alpha_2}\dfrac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha_1}\dfrac{\sigma}{\sqrt{n}}\right)$.

b. The usual 95% confidence interval has length $3.92\dfrac{\sigma}{\sqrt{n}}$, while this interval will have length $\left(z_{\alpha_1} + z_{\alpha_2}\right)\dfrac{\sigma}{\sqrt{n}}$. With $z_{\alpha_1} = z_{.0125} = 2.24$ and $z_{\alpha_2} = z_{.0375} = 1.78$, the length is $\left(2.24 + 1.78\right)\dfrac{\sigma}{\sqrt{n}} = 4.02\dfrac{\sigma}{\sqrt{n}}$, which is longer.

9.

a. $\left(\bar{x} - 1.645\dfrac{\sigma}{\sqrt{n}}, \infty\right)$. From 5**a**, $\bar{x} = 4.85$, σ = .75, and n = 20; $4.85 - 1.645\dfrac{.75}{\sqrt{20}} = 4.5741$, so the interval is $\left(4.5741, \infty\right)$.

b. $\left(\bar{x} - z_\alpha\dfrac{\sigma}{\sqrt{n}}, \infty\right)$

c. $\left(-\infty, \bar{x} + z_\alpha\dfrac{\sigma}{\sqrt{n}}\right)$; From 4**a**, $\bar{x} = 58.3$, σ = 3.0, and n = 25; $58.3 + 2.33\dfrac{3}{\sqrt{25}} = 59.70$, so the interval is (–∞, 59.70).

10.

a. When n = 15, $2\lambda\sum X_i$ has a chi-squared distribution with 30 df. From the 30 df. row of Table A.6, the critical values that capture lower and upper tail areas of .025 (and thus a central area of .95) are 16.791 and 46.979. An argument parallel to that given in Example 7.5 gives $\left(\dfrac{2\sum x_i}{46.979}, \dfrac{2\sum x_i}{16.791}\right)$ as a 95% CI for $\mu = \dfrac{1}{\lambda}$. Since $\sum x_i = 63.2$ the interval is (2.69, 7.53).

b. A 99% confidence level requires using critical values that capture area .005 in each tail of the chi-squared curve with 30 df.; these are 13.787 and 53.672, which replace 16.791 and 46.979 in **a**.

c. $V\left(X\right) = \dfrac{1}{\lambda^2}$ when X has an exponential distribution, so the standard deviation is $\dfrac{1}{\lambda}$, the same as the mean. Thus the interval of **a** is also a 95% CI for the standard deviation of the lifetime distribution.

11. Y is a binomial rv with $n = 1000$ and $p = .95$, so $E(Y) = np = 950$, the expected number of intervals that capture μ, and $\sigma_Y = \sqrt{npq} = 6.892$. Using the normal approximation to the binomial distribution, $P(940 \le Y \le 960) = P(939.5 \le Y \le 960.5) \approx P(-1.52 \le Z \le 1.52) = .9357 - .0643 = .8714$.

Section 7.2

12.

a. Yes: even if the data implied a non-normal population distribution, the sample size ($n = 43$) is large enough that the "large-sample" confidence interval for μ would still be reasonable.

b. For this data set, the sample mean and standard deviation are $\bar{x} = 1191.6$ and $s = 506.6$. The z critical value for 99% confidence is $z.01/2 = z_{.005} \approx 2.58$. Thus, a 99% confidence interval for μ, the true average lifetime (days) subsequent to diagnosis of all patients with blood cancer is

$$\bar{x} \pm 2.58\frac{s}{\sqrt{n}} = 1191.6 \pm 2.58\frac{506.6}{\sqrt{43}} = 1191.6 \pm 199.3 = (992.3, 1390.9).$$

13.

a. $\bar{x} \pm z_{.025}\frac{s}{\sqrt{n}} = 654.16 \pm 1.96\frac{164.43}{\sqrt{50}} = (608.58, 699.74)$. We are 95% confident that the true average CO_2 level in this population of homes with gas cooking appliances is between 608.58ppm and 699.74ppm

b. $w = 50 = \frac{2(1.96)(175)}{\sqrt{n}} \Rightarrow \sqrt{n} = \frac{2(1.96)(175)}{50} = 13.72 \Rightarrow n = (13.72)^2 = 188.24$, which rounds up to 189.

14.

a. $\bar{x} \pm z_{.025}\frac{s}{\sqrt{n}} = 1427 \pm 1.96\frac{325}{\sqrt{514}} = 1427 \pm 28.1 = (1398.9, 1455.1)$. We are 95% confident that the true mean FEV_1 of all children living in coal-use homes is between 1398.9 ml and 1455.1 ml.
This interval is quite narrow relative to the scale of the data values themselves, so it could be argued that the mean μ has been precisely estimated. (This is really a judgment call.)

b. $n = \left[\frac{2z_{\alpha/2}s}{w}\right]^2 = \left[\frac{2(1.96)(320)}{50}\right]^2 = 629.4 \Rightarrow n$ must be at least 630.

15.

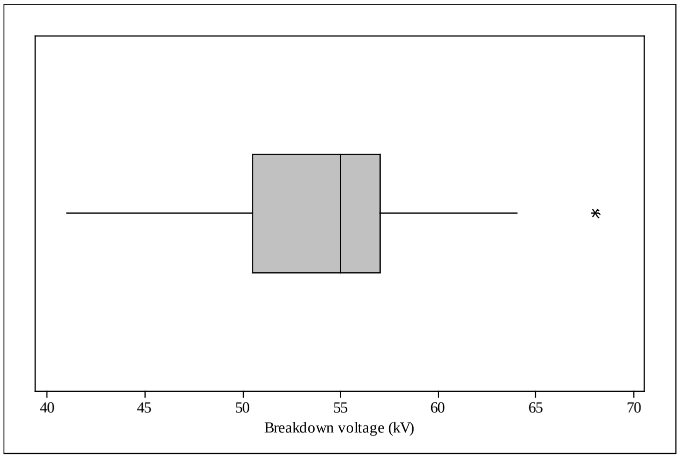
a. $z_\alpha = .84$, and $\Phi(.84) = .7995 \approx .80$, so the confidence level is 80%.

b. $z_\alpha = 2.05$, and $\Phi(2.05) = .9798 \approx .98$, so the confidence level is 98%.

c. $z_\alpha = .67$, and $\Phi(.67) = .7486 \approx .75$, so the confidence level is 75%.

16.

a. The boxplot shows a high concentration in the middle half of the data (narrow box width). There is a single outlier at the upper end, but this value is actually a bit closer to the median (55 kV) than is the smallest sample observation.



b. From software, $\bar{x}$ = 54.7 and s = 5.23. The 95% confidence interval is then

$$\bar{x} \pm 1.96\frac{s}{\sqrt{n}} = 54.7 \pm 1.96\frac{5.23}{\sqrt{48}} = 54.7 \pm 1.5 = (53.2,\ 56.2)$$

We are 95% confident that the true mean breakdown voltage under these conditions is between 53.2 kV and 56.2 kV. The interval is reasonably narrow, indicating that we have estimated μ fairly precisely.

c. A conservative estimate of standard deviation would be (70 – 40)/4 = 7.5. To achieve a margin of error of at most 1 kV with 95% confidence, we desire

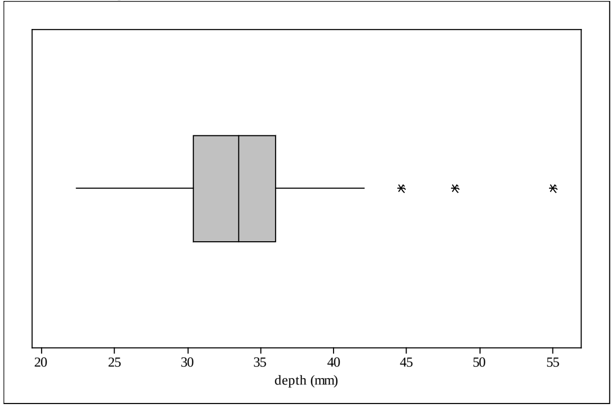
$$1.96\frac{s}{\sqrt{n}} \le 1 \Rightarrow n \ge \left[\frac{1.96s}{1}\right]^2 = \left[\frac{1.96(7.5)}{1}\right]^2 = 216.09.$$

Therefore, a sample of size at least n = 217 would be required.

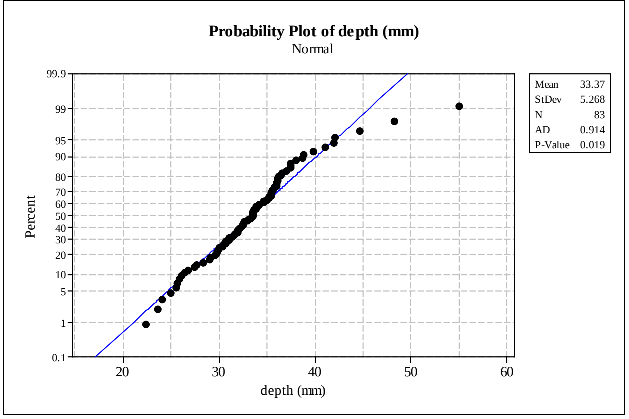
17. $\bar{x} - z_{.01}\frac{s}{\sqrt{n}} = 135.39 - 2.33\frac{4.59}{\sqrt{153}} = 135.39 - .865 = 134.53$. We are 99% confident that the true average ultimate tensile strength is greater than 134.53.

18.

a. The distribution of impression depths from these armor tests is roughly symmetric, with the exception of three high outliers: 44.6 mm, 48.3 mm, and 55.0 mm.



b. The accompanying normal probability plot shows a stark deviation from linearity, due almost entirely to the three large outliers. (Since a normal distribution is symmetric, a sample from a normal population shouldn't have so many outliers all to one side.) We conclude here that the population distribution of impression depths is *not* normal. However, since the sample size is quite large (n = 83), we can still use the "large-sample" inference procedures for the population mean μ.



c. From the output provided, $\bar{x}$ = 33.370 and $s/\sqrt{n}$ = 0.578. With $z_{.01}$ = 2.33, a 99% upper confidence bound for the true mean impression depth under these testing conditions is 33.370 + 2.33(0.578) = 34.72 mm.

19. $\hat{p} = \dfrac{201}{356} = .5646$; We calculate a 95% confidence interval for the proportion of all dies that pass the probe:

$$\frac{.5646 + \dfrac{(1.96)^2}{2(356)} \pm 1.96\sqrt{\dfrac{(.5646)(.4354)}{356} + \dfrac{(1.96)^2}{4(356)^2}}}{1 + \dfrac{(1.96)^2}{356}} = \frac{.5700 \pm .0518}{1.01079} = (.513, .615)$$

. The simpler CI formula (7.11) gives $.5646 \pm 1.96\sqrt{\dfrac{.5646(.4354)}{356}}$ = (.513, .616), which is almost identical.

20.

a. With the numbers provided, a 99% confidence interval for the parameter *p* is

$$\frac{.53+\frac{(2.58)^2}{2(2343)}\pm 2.58\sqrt{\frac{(.53)(.47)}{2343}+\frac{(2.58)^2}{4(2343)^2}}}{1+\frac{(2.58)^2}{2343}}$$ = (.503, .556). We are 99% confident that the proportion of all adult Americans who watched streamed programming was between .503 and .556.

b. $n = \left[\frac{2z_{\alpha/2}\hat{p}\hat{q}}{w}\right]^2 = \left[\frac{2(2.58)(.5)(.5)}{.05}\right]^2$ = 665.64, so *n* must be at least 666.

21. For a one-sided bound, we need $z_\alpha = z_{.05} = 1.645$; $\hat{p} = \frac{250}{1000} = .25$; and $\tilde{p} = \frac{.25+1.645^2/2000}{1+1.645^2/1000}$ = .2507. The resulting 95% upper confidence bound for *p*, the true proportion of such consumers who never apply for a rebate, is $.2507+\frac{1.645\sqrt{(.25)(.75)/1000+(1.645)^2/(4\cdot 1000^2)}}{1+(1.645)^2/1000}$ = .2507 + .0225 = .2732.

Yes, there is compelling evidence the true proportion is less than 1/3 (.3333), since we are 95% confident this true proportion is less than .2732.

22.

a. For a one-sided bound, we need $z_\alpha = z_{.05} = 1.645$; $\hat{p} = \frac{10}{143} = .07$; and $\tilde{p} = \frac{.07+1.645^2/(2\cdot 143)}{1+1.645^2/143}$ = .078. The resulting 95% lower confidence bound for *p*, the true proportion of such artificial hip recipients that experience squeaking, is $.078-\frac{1.645\sqrt{(.07)(.93)/143+(1.645)^2/(4\cdot 143^2)}}{1+(1.645)^2/143}$ = .078 – .036 = .042.

We are 95% confident that more than 4.2% of all such artificial hip recipients experience squeaking.

b. If we were to sample repeatedly, the calculation method in (a) is such that *p* will exceed the calculated lower confidence bound for 95% of all possible random samples of *n* = 143 individuals who received ceramic hips. (We hope that our sample is among that 95%!)

23.

a. With $\hat{p}$ = .25, *n* = 2003, and $z_{\alpha/2} = z_{.005} \approx 2.58$, the 99% confidence interval for *p* is

$$\frac{.25+\frac{(2.58)^2}{2(2003)}\pm 2.58\sqrt{\frac{(.25)(.75)}{2003}+\frac{(2.58)^2}{4(2003)^2}}}{1+\frac{(2.58)^2}{2003}}$$ = (.225, .275).

b. Using the "simplified" formula for sample size and $\hat{p} = \hat{q} = .5$,

$$n = \frac{4z^2\hat{p}\hat{q}}{w^2} = \frac{4(2.576)^2(.5)(.5)}{(.05)^2} = 2654.31$$

So, a sample of size at least 2655 is required. (We use $\hat{p} = \hat{q} = .5$ here, rather than the values from the sample data, so that our CI has the desired width irrespective of what the true value of *p* might be. See the textbook discussion toward the end of Section 7.2.)

24. $n = 56$, $\bar{x} = 8.17$, $s = 1.42$; for a 95% CI, $z_{\alpha/2} = 1.96$. The interval is $8.17 \pm 1.96\left(\frac{1.42}{\sqrt{56}}\right) = (7.798, 8.542)$. We make no assumptions about the distribution if percentage elongation.

25.

a. $n = \dfrac{2(1.96)^2(.25) - (1.96)^2(.01) \pm \sqrt{4(1.96)^4(.25)(.25-.01) + .01(1.96)^4}}{.01} \approx 381$

b. $n = \dfrac{2(1.96)^2\left(\frac{1}{3}\cdot\frac{2}{3}\right) - (1.96)^2(.01) \pm \sqrt{4(1.96)^4\left(\frac{1}{3}\cdot\frac{2}{3}\right)\left(\frac{1}{3}\cdot\frac{2}{3} - .01\right) + .01(1.96)^4}}{.01} \approx 339$

26. Using the quadratic equation to solve the limits $\dfrac{\bar{x}-\mu}{\sqrt{\mu/n}} = \pm z$ gives the solutions $\mu = \left(\bar{x} + \dfrac{z^2}{2n}\right) \pm z \cdot \dfrac{\sqrt{z^2 + 4n\bar{x}}}{2n}$. For the data provided, $n = 50$ and $\bar{x}$ =203/50 = 4.06. Substituting these and using $z = 1.96$, we are 95% confident that the true value of μ is in the interval $\left(4.06 + \dfrac{1.96^2}{2(50)}\right) \pm 1.96 \cdot \dfrac{\sqrt{1.96^2 + 4(50)(4.06)}}{2(50)}$ = 4.10 ± .56 = (3.54, 4.66).

Note that for large n, this is approximately equivalent to $\bar{x} \pm z\dfrac{\sqrt{\bar{x}}}{\sqrt{n}} = \hat{\mu} \pm z\dfrac{\hat{\sigma}}{\sqrt{n}}$ (since $\mu = \sigma^2$ for the Poisson distribution).

27. Note that the midpoint of the new interval is $\dfrac{x + z^2/2}{n + z^2}$, which is roughly $\dfrac{x+2}{n+4}$ with a confidence level of 95% and approximating $1.96 \approx 2$. The variance of this quantity is $\dfrac{np(1-p)}{(n+z^2)^2}$, or roughly $\dfrac{p(1-p)}{n+4}$. Now replacing p with $\dfrac{x+2}{n+4}$, we have $\left(\dfrac{x+2}{n+4}\right) \pm z_{\alpha/2}\sqrt{\dfrac{\left(\frac{x+2}{n+4}\right)\left(1 - \frac{x+2}{n+4}\right)}{n+4}}$. For clarity, let $x^* = x + 2$ and $n^* = n + 4$, then $\hat{p}^* = \dfrac{x^*}{n^*}$ and the formula reduces to $\hat{p}^* \pm z_{\alpha/2}\sqrt{\dfrac{\hat{p}^*\hat{q}^*}{n^*}}$, the desired conclusion. For further discussion, see the Agresti article.

Section 7.3

28.

a. 1.341

b. 1.753

c. 1.708

d. 1.684

e. 2.704

29.

a. $t_{.025,10} = 2.228$

b. $t_{.025,20} = 2.086$

c. $t_{.005,20} = 2.845$

d. $t_{.005,50} = 2.678$

e. $t_{.01,25} = 2.485$

f. $-t_{.025,5} = -2.571$

30.

a. $t_{.025,10} = 2.228$

b. $t_{.025,15} = 2.131$

c. $t_{.005,15} = 2.947$

d. $t_{.005,4} = 4.604$

e. $t_{.01,24} = 2.492$

f. $t_{.005,37} \approx 2.712$

31.

a. $t_{05,10} = 1.812$

b. $t_{.05,15} = 1.753$

c. $t_{.01,15} = 2.602$

d. $t_{.01,4} = 3.747$

e. $\approx t_{.025,24} = 2.064$

f. $t_{.01,37} \approx 2.429$

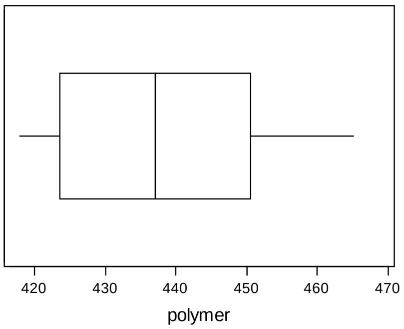
32. We have n = 20, $\bar{x}$ = 1584, and s = 607; the critical value is $t_{.005,20-1} = t_{.005,19} = 2.861$. The resulting 99% CI for μ is

$$1584 \pm 2.861\frac{607}{\sqrt{20}} = 1584 \pm 388.3 = (1195.7, 1972.3)$$

We are 99% confident that the true average number of cycles required to break this type of condom is between 1195.7 cycles and 1972.3 cycles.

33.

a. The boxplot indicates a very slight positive skew, with no outliers. The data appears to center near 438.



b. Based on a normal probability plot, it is reasonable to assume the sample observations came from a normal distribution.

c. With df = n – 1 = 16, the critical value for a 95% CI is $t_{.025,16} = 2.120$, and the interval is $438.29 \pm (2.120)\left(\frac{15.14}{\sqrt{17}}\right) = 438.29 \pm 7.785 = (430.51, 446.08)$. Since 440 is within the interval, 440 is a plausible value for the true mean. 450, however, is not, since it lies outside the interval.

34. $n = 14$, $\bar{x} = 8.48$, $s = .79$; $t_{.05,13} = 1.771$

a. A 95% lower confidence bound: $8.48 - 1.771\left(\frac{.79}{\sqrt{14}}\right) = 8.48 - .37 = 8.11$. With 95% confidence, the value of the true mean proportional limit stress of all such joints is greater than 8.11 MPa. We must assume that the sample observations were taken from a normally distributed population.

b. A 95% lower prediction bound: $8.48 - 1.771(.79)\sqrt{1 + \frac{1}{14}} = 8.48 - 1.45 = 7.03$. If this bound is calculated for sample after sample, in the long run 95% of these bounds will provide a lower bound for the corresponding future values of the proportional limit stress of a single joint of this type.

35. $n = 15$, $\bar{x} = 25.0$, $s = 3.5$; $t_{.025,14} = 2.145$

a. A 95% CI for the mean: $25.0 \pm 2.145\frac{3.5}{\sqrt{15}}$ = (23.06, 26.94).

b. A 95% prediction interval: $25.0 \pm 2.145(3.5)\sqrt{1 + \frac{1}{15}}$ = (17.25, 32.75). The prediction interval is about 4 times wider than the confidence interval.

36. $n = 26$, $\bar{x} = 370.69$, $s = 24.36$; $t_{.05,25} = 1.708$

a. A 95% upper confidence bound: $370.69 + (1.708)\left(\frac{24.36}{\sqrt{26}}\right) = 370.69 + 8.16 = 378.85$

b. A 95% upper prediction bound: $370.69 + 1.708(24.36)\sqrt{1 + \frac{1}{26}} = 370.69 + 42.45 = 413.14$

c. Following a similar argument as the one presented in the Prediction Interval section, we need to find the variance of $\bar{X} - \bar{X}_{\text{new}}$: $V\left(\bar{X} - \bar{X}_{\text{new}}\right) = V\left(\bar{X}\right) + V\left(\bar{X}_{\text{new}}\right) = V\left(\bar{X}\right) + V\left(\tfrac{1}{2}\left(X_{27} + X_{28}\right)\right)$

$= V\left(\bar{X}\right) + V\left(\tfrac{1}{2}X_{27}\right) + V\left(\tfrac{1}{2}X_{28}\right) = V\left(\bar{X}\right) + \tfrac{1}{4}V\left(X_{27}\right) + \tfrac{1}{4}V\left(X_{28}\right)$

$= \frac{\sigma^2}{n} + \frac{1}{4}\sigma^2 + \frac{1}{4}\sigma^2 = \sigma^2\left(\frac{1}{2} + \frac{1}{n}\right)$. We eventually arrive at $T = \frac{\bar{X} - \bar{X}_{\text{new}}}{s\sqrt{\frac{1}{2} + \frac{1}{n}}} \sim t$ distribution with n – 1 df, so the new prediction interval is $\bar{x} \pm t_{\alpha/2,n-1} \cdot s\sqrt{\tfrac{1}{2} + \tfrac{1}{n}}$. For this situation, we have

$370.69 \pm 1.708(24.36)\sqrt{\frac{1}{2} + \frac{1}{26}} = 370.69 \pm 30.53 = (340.16, 401.22)$.

37.

a. A 95% CI : $.9255 \pm 2.093(.0181) = .9255 \pm .0379 \Rightarrow (.8876, .9634)$

b. A 95% P.I. : $.9255 \pm 2.093(.0809)\sqrt{1+\frac{1}{20}} = .9255 \pm .1735 \Rightarrow (.7520, 1.0990)$

c. A tolerance interval is requested, with k = 99, confidence level 95%, and n = 20. The tolerance critical value, from Table A.6, is 3.615. The interval is $.9255 \pm 3.615(.0809) \Rightarrow (.6330, 1.2180)$.

38.

a. Maybe: A normal probability plot exhibits some curvature, though perhaps not enough for us to certainly declare the population non-normal. (With n = 5, it's honestly hard to tell.)

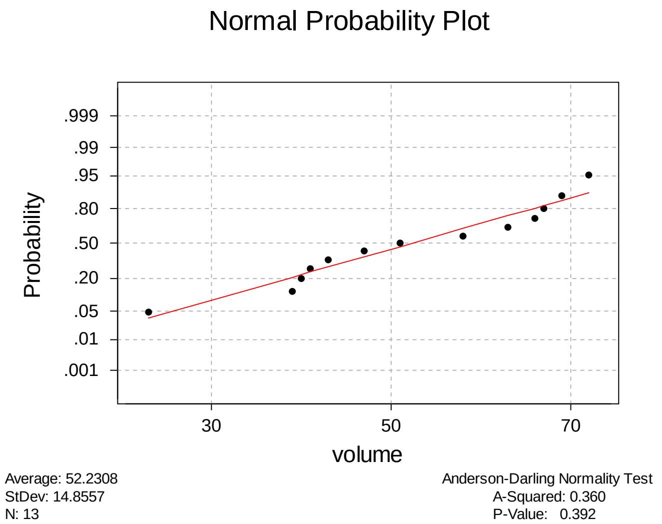
b. From the data provided, $\bar{x}$ = 107.78 and s = 1.076. The corresponding 95% CI for μ is $\bar{x} \pm t_{.025,5-1}\frac{s}{\sqrt{n}} = 107.78 \pm 2.776\frac{1.076}{\sqrt{5}}$ = (106.44, 109.12). The CI suggests that while 107 is a plausible value for μ (since it lies in the interval), 110 is not.

c. A 95% PI for a single future value is $107.78 \pm 2.776 \cdot 1.076\sqrt{1+\frac{1}{5}}$ = (104.51, 111.05). As is always the case, the prediction interval for a single future value is considerably wider (less precise) than the CI for the population mean.

d. Looking at Table A.6, the critical value for a 95% confidence tolerance interval capturing at least 95% of the population, based on n = 5, is k = 5.079. The resulting tolerance interval is $\bar{x} \pm k \cdot s$ = 107.78 ± 5.079(1.076) = (102.31, 113.25).

39.

a.



Based on the above plot, generated by Minitab, it is plausible that the population distribution is normal.

b. We require a tolerance interval. From table A.6, with 95% confidence, k = 95, and n=13, the tolerance critical value is 3.081. $\bar{x} \pm 3.081s = 52.231 \pm 3.081(14.856) = 52.231 \pm 45.771 \Rightarrow (6.460, 98.002)$.

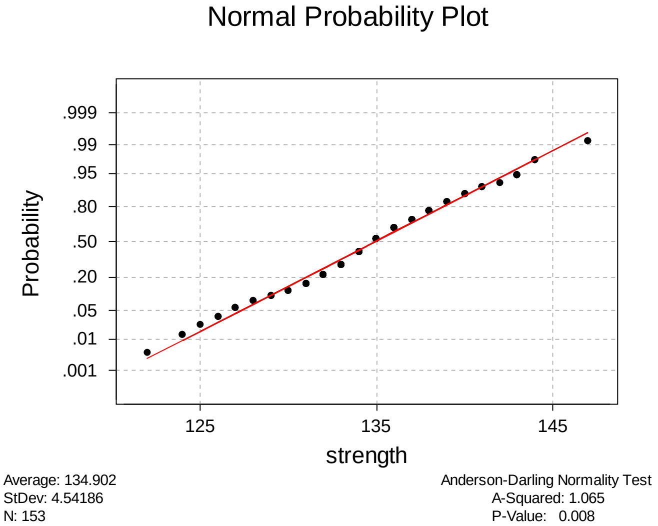
c. A prediction interval, with $t_{.025,12} = 2.179$:

$$52.231 \pm 2.179(14.856)\sqrt{1+\tfrac{1}{13}} = 52.231 \pm 33.593 \Rightarrow (18.638, 85.824)$$

40.

a. We need to assume the samples came from a normally distributed population.

b. A normal probability plot, generated by Minitab, appears below. The very small *P*-value indicates that the population distribution from which this data was taken is most likely not normal.



c. Despite the apparent lack of normality, we will proceed with a 95% lower prediction bound: with df = 153 – 1 = 152, we estimate t = –1.98: $135.39 - 1.98(4.59)\sqrt{1+\tfrac{1}{153}} = 135.39 - 9.12 = 126.27$.

41. The 20 df row of Table A.5 shows that 1.725 captures upper tail area .05 and 1.325 captures upper tail area .10 The confidence level for each interval is 100(central area)%.

For the first interval, central area = 1 – sum of tail areas = 1 – (.25 + .05) = .70, and for the second and third intervals the central areas are 1 – (.20 + .10) = .70 and 1 – (.15 + .15) = 70. Thus each interval has confidence level 70%. The width of the first interval is $\dfrac{s(.687+1.725)}{\sqrt{n}} = 2.412\dfrac{s}{\sqrt{n}}$, whereas the widths of the second and third intervals are 2.185 and 2.128 standard errors respectively. The third interval, with symmetrically placed critical values, is the shortest, so it should be used. This will always be true for a *t* interval.

Section 7.4

42.

a. $\chi^2_{.1,15} = 22.307$ (.1 column, 15 df row)

b. $\chi^2_{.1,25} = 34.381$

c. $\chi^2_{.01,25} = 44.313$

d. $\chi^2_{.005,25} = 46.925$

e. $\chi^2_{.99,25} = 11.523$ (.99 col., 25 df row)

f. $\chi^2_{.995,25} = 10.519$

43.

a. $\chi^2_{.05,10} = 18.307$

b. $\chi^2_{.95,10} = 3.940$

c. Since $10.987 = \chi^2_{.975,22}$ and $36.78 = \chi^2_{.025,22}$, $P\left(\chi^2_{.975,22} \le \chi^2 \le \chi^2_{.025,22}\right) = .95$.

d. Since $14.611 = \chi^2_{.95,25}$ and $37.652 = \chi^2_{.05,25}$, $P(\chi^2 < 14.611 \text{ or } \chi^2 > 37.652) =$
$1 - P(\chi^2 > 14.611) + P(\chi^2 > 37.652) = (1 - .95) + .05 = .10$.

44. $n - 1 = 8$, $\chi^2_{.025,8} = 17.534$, $\chi^2_{.975,8} = 2.180$, so the 95% interval for σ^2 is $\left(\frac{8(7.90)}{17.534}, \frac{8(7.90)}{2.180}\right) = (3.60, 28.98)$.
The 95% interval for σ is $\left(\sqrt{3.60}, \sqrt{28.98}\right) = (1.90, 5.38)$.

45. For the $n = 8$ observations provided, the sample standard deviation is $s = 8.2115$. A 99% CI for the population variance, σ^2, is given by
$\left((n-1)s^2 / \chi^2_{.005,n-1}, (n-1)s^2 / \chi^2_{.995,n-1}\right) = \left(7 \cdot 8.2115^2 / 20.276, 7 \cdot 8.2115^2 / 0.989\right)$ = (23.28, 477.25)
Taking square roots, a 99% CI for σ is (4.82, 21.85). Validity of this interval requires that coating layer thickness be (at least approximately) normally distributed.

46.

a. Using a normal probability plot, we ascertain that it is plausible that this sample was taken from a normal population distribution.

b. With $s = 1.579$, $n = 15$, and $\chi^2_{.95,14} = 6.571$, the 95% upper confidence bound for σ is

$$\sqrt{\frac{14(1.579)^2}{6.571}} = 2.305 .$$

Supplementary Exercises

47.

a. $n = 48$, $\bar{x} = 8.079a$, $s^2 = 23.7017$, and $s = 4.868$.
A 95% CI for μ = the true average strength is

$$\bar{x} \pm 1.96\frac{s}{\sqrt{n}} = 8.079 \pm 1.96\frac{4.868}{\sqrt{48}} = 8.079 \pm 1.377 = (6.702, 9.456).$$

b. $\hat{p} = \frac{13}{48} = .2708$. A 95% CI for p is

$$\frac{.2708 + \frac{1.96^2}{2(48)} \pm 1.96\sqrt{\frac{(.2708)(.7292)}{48} + \frac{1.96^2}{4(48)^2}}}{1 + \frac{1.96^2}{48}} = \frac{.3108 \pm .1319}{1.0800} = (.166, .410)$$

48.

a. With $n = 18$, $\bar{x} = 64.41$, $s = 10.32$, and $t_{.02/2,18-1} = t_{.01,17} = 2.567$, a 98% CI for the population mean μ is given by $64.41 \pm 2.567\frac{10.32}{\sqrt{18}}$ = (58.17, 70.65).

b. Notice the goal is to obtain a <u>lower</u> <u>prediction</u> bound for a single future compressive strength measurement. This requires determining $\bar{x} - t_{\alpha,n-1}s\sqrt{1+\frac{1}{n}} = 64.41 - 2.224(10.32)\sqrt{1+\frac{1}{18}}$ = 64.41 – 23.58 = 40.83 MPa.

49. The sample mean is the midpoint of the interval: $\bar{x} = \frac{60.2 + 70.6}{2}$ = 65.4 N. The 95% confidence margin of error for the mean must have been ± 5.2, so $t \cdot s/\sqrt{n}$ = 5.2. The 95% confidence margin of error for a prediction interval (i.e., an individual) is $t \cdot s\sqrt{1+\frac{1}{n}}$ = $\sqrt{n+1} \cdot t \cdot s/\sqrt{n} = \sqrt{11+1}(5.2)$ = 18.0. Thus, the 95% PI is 65.4 ± 18.0 = (47.4 N, 83.4 N). (You could also determine t from n and α, then s separately.)

50. $\bar{x}$ = the midpoint of the interval = $\frac{229.764 + 233.504}{2} = 231.634$. To find s we use width $= 2t_{.025,4}\left(\frac{s}{\sqrt{n}}\right)$, and solve for s. Here, $n = 5$, $t_{.025,4} = 2.776$, and width = upper limit – lower limit = 3.74.

$3.74 = 2(2776)\frac{s}{\sqrt{5}} \Rightarrow s = \frac{\sqrt{5}(3.74)}{2(2.776)} = 1.5063$. So for a 99% CI, $t_{.005,4} = 4.604$, and the interval is

$$231.634 \pm 4.604\frac{1.5063}{\sqrt{5}} = 213.634 \pm 3.101 = (228.533, 234.735).$$

51.

a. With $\hat{p} = 31/88 = .352$, $\tilde{p} = \frac{.352 + 1.96^2/2(88)}{1 + 1.96^2/88}$ = .358, and the CI is

$.358 \pm 1.96\frac{\sqrt{(.352)(.648) + 1.96^2/4(88)^2}}{1 + 1.96^2/88}$ = (.260, .456). We are 95% confident that between 26.0% and 45.6% of <u>all</u> athletes under these conditions have an exercise-induced laryngeal obstruction.

b. Using the "simplified" formula, $n=\frac{4z^2\hat{p}\hat{q}}{w^2}=\frac{4(1.96)^2(.5)(.5)}{(.04)^2}=2401$. So, roughly 2400 people should be surveyed to assure a width no more than .04 with 95% confidence. Using Equation (7.12) gives the almost identical n = 2398.

c. No. The upper bound in (a) uses a z-value of 1.96 = $z_{.025}$. So, if this is used as an upper bound (and hence .025 equals α rather than $\alpha/2$), it gives a (1 – .025) = 97.5% upper bound. If we want a 95% confidence upper bound for p, 1.96 should be replaced by the critical value $z_{.05}$ = 1.645.

52. n = 5, $\bar{x}$ = 24.3, s = 4.1

a. $t_{.025,4}$ = 2.776: $24.3\pm2.776\frac{4.1}{\sqrt{5}}=24.3\pm5.09=$ (19.21, 29.39). We are 95% confident that the true average arsenic concentration in all such water specimens is between 19.21 μg/L and 29.39 μg/L.

b. A 90% upper bound for σ, with $\chi^2_{.90,4}=1.064$, is $\sqrt{\frac{4(4.1)^2}{1.064}}$ = 7.95 μg/L

c. A 95% prediction interval is $24.3\pm2.776(4.1)\sqrt{1+\frac{1}{5}}$ = (11.83, 36.77).

53. With $\hat{\theta}=\frac{1}{3}\left(\bar{X}_1+\bar{X}_2+\bar{X}_3\right)-\bar{X}_4$, $\sigma^2_{\hat{\theta}}=\frac{1}{9}V\left(\bar{X}_1+\bar{X}_2+\bar{X}_3\right)+V\left(\bar{X}_4\right)$ = $\frac{1}{9}\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}+\frac{\sigma_3^2}{n_3}\right)+\frac{\sigma_4^2}{n_4}$; $\hat{\sigma}_{\hat{\theta}}$ is obtained by replacing each σ_i^2 by s_i^2 and taking the square root. The large-sample interval for θ is then

$$\tfrac{1}{3}\left(\bar{x}_1+\bar{x}_2+\bar{x}_3\right)-\bar{x}_4\pm z_{\alpha/2}\sqrt{\frac{1}{9}\left(\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}+\frac{s_3^2}{n_3}\right)+\frac{s_4^2}{n_4}}.$$

For the given data, $\hat{\theta}=-.50$ and $\hat{\sigma}_{\hat{\theta}}=.1718$, so the interval is $-.50\pm1.96(.1718)=(-.84,-.16)$.

54. $\hat{p}=\frac{11}{55}=.2\Rightarrow$ a 90% CI is $$\frac{.2+\frac{1.645^2}{2(55)}\pm1.645\sqrt{\frac{(.2)(.8)}{55}+\frac{1.645^2}{4(55)^2}}}{1+\frac{1.645^2}{55}}=\frac{.2246\pm.0887}{1.0492}=(.1295,.2986).$$

55. The specified condition is that the interval be length .2, so $n=\left[\frac{2(1.96)(.8)}{.2}\right]^2=245.86\nearrow246$.

56. From the data provided, n = 16, $\bar{x}$ = 7.1875, and s = 1.9585.

a. A 99% CI for μ, the true average crack initiation depth, is

$$\bar{x}\pm t_{.005,n-1}\frac{s}{\sqrt{n}}=7.1875\pm2.947\frac{1.9585}{\sqrt{16}}\ \text{= (6.144, 8.231)}$$

b. The corresponding 99% PI is $7.1875\pm2.947\cdot1.9585\sqrt{1+\frac{1}{16}}$ = (1.238, 13.134).

c. If we were to take repeated samples of size n = 16 from this population, and from each we constructed a 99% prediction interval, then in the long run 99% of all such samples would generate a PI that would contain the numerical value of a single future crack initiation depth, X.

57. Proceeding as in Example 7.5 with T_r replacing ΣX_i, the CI for $\frac{1}{\lambda}$ is $\left(\frac{2t_r}{\chi^2_{1-\alpha/2,2r}}, \frac{2t_r}{\chi^2_{\alpha/2,2r}}\right)$ where $t_r = y_1 + ... + y_r + (n-r)y_r$. In Example 6.7, n = 20, r = 10, and t_r = 1115. With df = 20, the necessary critical values are 9.591 and 34.170, giving the interval (65.3, 232.5). This is obviously an extremely wide interval. The censored experiment provides less information about $\frac{1}{\lambda}$ than would an uncensored experiment with n = 20.

58.

a. $P(\min(X_i) \le \tilde{\mu} \le \max(X_i)) = 1 - P(\tilde{\mu} < \min(X_i) \text{ or } \max(X_i) < \tilde{\mu})$
$= 1 - P(\tilde{\mu} < \min(X_i)) - P(\max(X_i) < \tilde{\mu}) = 1 - P(\tilde{\mu} < X_1, ..., \tilde{\mu} < X_n) - P(X_1 < \tilde{\mu}, ..., X_n < \tilde{\mu})$
$= 1 - (.5)^n - (.5)^n = 1 - (.5)^{n-1}$, as desired.

b. Since $\min(x_i) = 1.44$ and $\max(x_i) = 3.54$, the CI is (1.44, 3.54).

c. $P(X_{(2)} \le \tilde{\mu} \le X_{(n-1)}) = 1 - P(\tilde{\mu} < X_{(2)}) - P(X_{(n-1)} < \tilde{\mu})$
= 1 – P(at most one X_i is below $\tilde{\mu}$) – P(at most one X_i exceeds $\tilde{\mu}$)
$1 - (.5)^n - \binom{n}{1}(.5)^1(.5)^{n-1} - (.5)^n - \binom{n}{1}(.5)^{n-1}(.5) = 1 - 2(n+1)(.5)^n = 1 - (n+1)(.5)^{n-1}$. Thus the confidence coefficient is $1 - (n+1)(.5)^{n-1}$; i.e., we have a $100\left(1 - (n+1)(.5)^{n-1}\right)$% confidence interval.

59.

a. $\int_{(\alpha/2)^{1/n}}^{(1-\alpha/2)^{1/n}} nu^{n-1}du = u^n \Big]_{(\alpha/2)^{1/n}}^{(1-\alpha/2)^{1/n}} = 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$. From the probability statement, $\frac{(\alpha/2)^{1/n}}{\max(X_i)} \le \frac{1}{\theta} \le \frac{(1-\alpha/2)^{1/n}}{\max(X_i)}$ with probability 1 – α, so taking the reciprocal of each endpoint and interchanging gives the CI $\left(\frac{\max(X_i)}{(1-\alpha/2)^{1/n}}, \frac{\max(X_i)}{(\alpha/2)^{1/n}}\right)$ for θ.

b. $\alpha^{1/n} \le \frac{\max(X_i)}{\theta} \le 1$ with probability 1 – α, so $1 \le \frac{\theta}{\max(X_i)} \le \frac{1}{\alpha^{1/n}}$ with probability 1 – α, which yields the interval $\left(\max(X_i), \frac{\max(X_i)}{\alpha^{1/n}}\right)$.

c. It is easily verified that the interval of **b** is shorter — draw a graph of $f_U(u)$ and verify that the shortest interval which captures area 1 – α under the curve is the rightmost such interval, which leads to the CI of **b**. With α = .05, n = 5, and max(x_i)=4.2, this yields (4.2, 7.65).

60. The length of the interval is $\left(z_\gamma + z_{\alpha-\gamma}\right)\frac{s}{\sqrt{n}}$, which is minimized when $z_\gamma + z_{\alpha-\gamma}$ is minimized, i.e. when $\Phi^{-1}\left(1-\gamma\right)+\Phi^{-1}\left(1-\alpha+\gamma\right)$ is minimized. Taking $\frac{d}{d\gamma}$ and equating to 0 yields $\frac{1}{\phi\left(1-\gamma\right)}=\frac{1}{\phi\left(1-\alpha+\gamma\right)}$ where $\phi(\ )$ is the standard normal pdf. Since the normal pdf if symmetric about zero, this equation is true iff $1-\gamma=\pm(1-\alpha+\gamma)$, whence $\gamma=\frac{\alpha}{2}$.

61. $\tilde{x}=76.2$, the lower and upper fourths are 73.5 and 79.7, respectively, and f_s = 6.2. The robust interval is $76.2\pm\left(1.93\right)\left(\frac{6.2}{\sqrt{22}}\right)=76.2\pm2.6=\left(73.6,78.8\right)$.

$\bar{x}=77.33$, s = 5.037, and $t_{.025,21}=2.080$, so the t interval is $77.33\pm\left(2.080\right)\left(\frac{5.037}{\sqrt{22}}\right)=77.33\pm2.23=\left(75.1,79.6\right)$. The t interval is centered at $\bar{x}$, which is pulled out to the right of $\tilde{x}$ by the single mild outlier 93.7; the interval widths are comparable.

62.

a. Since $2\lambda\Sigma X_i$ has a chi-squared distribution with 2n df and the area under this chi-squared curve to the right of $\chi^2_{.95,2n}$ is .95, $P\left(\chi^2_{.95,2n}<2\lambda\Sigma X_i\right)=.95$. This implies that $\frac{\chi^2_{.95,2n}}{2\Sigma X_i}$ is a lower confidence bound for λ with confidence coefficient 95%. Table A.7 gives the chi-squared critical value for 20 df as 10.851, so the bound is $\frac{10.851}{2\left(550.87\right)}=.0098$. We can be 95% confident that λ exceeds .0098.

b. Arguing as in **a**, $P\left(2\lambda\Sigma X_i<\chi^2_{.05,2n}\right)=.95$. The following inequalities are equivalent to the one in parentheses:

$$\lambda<\frac{\chi^2_{.05,2n}}{2\Sigma X_i}\Rightarrow-\lambda t<\frac{-t\chi^2_{.05,2n}}{2\Sigma X_i}\Rightarrow\exp(-\lambda t)<\exp\left[\frac{-t\chi^2_{.05,2n}}{2\Sigma X_i}\right].$$

Replacing ΣX_i by Σx_i in the expression on the right hand side of the last inequality gives a 95% lower confidence bound for $e^{-\lambda t}$. Substituting t = 100, $\chi^2_{.05,20}=31.410$ and $\Sigma x_i=550.87$ gives .058 as the lower bound for the probability that time until breakdown exceeds 100 minutes.

CHAPTER 8

Section 8.1

1.

a. Yes. It is an assertion about the value of a parameter.

b. No. The sample median $\tilde{x}$ is not a parameter.

c. No. The sample standard deviation s is not a parameter.

d. Yes. The assertion is that the standard deviation of population #2 exceeds that of population #1.

e. No. $\overline{X}$ and $\overline{Y}$ are statistics rather than parameters, so they cannot appear in a hypothesis.

f. Yes. H is an assertion about the value of a parameter.

2.

a. These hypotheses comply with our rules.

b. H_a cannot include equality (i.e. $\sigma = 20$), so these hypotheses are not in compliance.

c. H_0 should contain the equality claim, whereas H_a does here, so these are not legitimate.

d. The asserted value of $\mu_1 - \mu_2$ in H_0 should also appear in H_a. It does not here, so our conditions are not met.

e. Each S^2 is a statistic and so does not belong in a hypothesis.

f. We are not allowing both H_0 and H_a to be equality claims (though this is allowed in more comprehensive treatments of hypothesis testing).

g. These hypotheses comply with our rules.

h. These hypotheses comply with our rules.

3. We reject H_0 iff P-value $\leq \alpha = .05$.

a. Reject H_0 **b.** Reject H_0 **c.** Do not reject H_0 **d.** Reject H_0 **e.** Do not reject H_0

4. We reject H_0 iff P-value $\leq \alpha$.

a. Do not reject H_0, since .084 > .05.

b. Do not reject H_0, since .003 > .001.

c. Do not reject H_0, since .498 > .05.

d. Reject H_0, since .084 ≤ .10.

e. Do not reject H_0, since .039 > .01.

f. Do not reject H_0, since .218 > .10.

5. In this formulation, H_0 states the welds do not conform to specification. This assertion will not be rejected unless there is strong evidence to the contrary. Thus the burden of proof is on those who wish to assert that the specification is satisfied. Using H_a: $\mu < 100$ results in the welds being believed in conformance unless proved otherwise, so the burden of proof is on the non-conformance claim.

6. When the alternative is H_a: $\mu < 5$, the formulation is such that the water is believed unsafe until proved otherwise. A type I error involved deciding that the water is safe (rejecting H_0) when it isn'*t* (H_0 is true). This is a very serious error, so a test which ensures that this error is highly unlikely is desirable. A type II error involves judging the water unsafe when it is actually safe. Though a serious error, this is less so than the type I error. It is generally desirable to formulate so that the type I error is more serious, so that the probability of this error can be explicitly controlled. Using H_a: $\mu > 5$, the type II error (now stating that the water is safe when it isn'*t*) is the more serious of the two errors.

7. Let σ denote the population standard deviation. The appropriate hypotheses are H_0: $\sigma = .05$ v. H_a: $\sigma < .05$. With this formulation, the burden of proof is on the data to show that the requirement has been met (the sheaths will not be used unless H_0 can be rejected in favor of H_a. Type I error: Conclude that the standard deviation is < .05 mm when it is really equal to .05 mm. Type II error: Conclude that the standard deviation is .05 mm when it is really < .05.

8. H_0: $\mu = 40$ v. H_a: $\mu \neq 40$, where μ is the true average burn-out amperage for this type of fuse. The alternative reflects the fact that a departure from $\mu = 40$ in either direction is of concern. A type I error would say that one of the two concerns exists (either $\mu < 40$ or $\mu > 40$) when, in fact, the fuses are perfectly compliant. A type II error would be to fail to detect either of these concerns when one exists.

9. A type I error here involves saying that the plant is not in compliance when in fact it is. A type II error occurs when we conclude that the plant is in compliance when in fact it isn'*t*. Reasonable people may disagree as to which of the two errors is more serious. If in your judgment it is the type II error, then the reformulation H_0: $\mu = 150$ v. H_a: $\mu < 150$ makes the type I error more serious.

10. Let μ_1 = the average amount of warpage for the regular laminate, and μ_2 = the analogous value for the special laminate. Then the hypotheses are H_0: $\mu_1 = \mu_2$ v. H_a: $\mu_1 > \mu_2$. Type I error: Conclude that the special laminate produces less warpage than the regular, when it really does not. Type II error: Conclude that there is no difference in the two laminates when in reality, the special one produces less warpage.

11.

a. A type I error consists of judging one of the two companies favored over the other when in fact there is a 50-50 split in the population. A type II error involves judging the split to be 50-50 when it is not.

b. We expect 25(.5) = 12.5 "successes" when H_0 is true. So, any X-values less than 6 are at least as contradictory to H_0 as $x = 6$. But since the alternative hypothesis states $p \neq .5$, X-values that are just as far away on the <u>high</u> side are equally contradictory. Those are 19 and above.
So, values at least as contradictory to H_0 as $x = 6$ are {0,1,2,3,4,5,6,19,20,21,22,23,24,25}.

c. When H_0 is true, X has a binomial distribution with $n = 25$ and $p = .5$.
From part (b), P-value = $P(X \leq 6 \text{ or } X \geq 19) = B(6; 25,.5) + [1 - B(18; 25,.5)] = .014$.

d. Looking at Table A.1, a two-tailed P-value of .044 (2 × .022) occurs when $x = 7$. That is, saying we'll reject H_0 iff P-value $\leq .044$ must be equivalent to saying we'll reject H_0 iff $X \leq 7$ or $X \geq 18$ (the same distance from 12.5, but on the high side). Therefore, for any value of $p \neq .5$, $\beta(p) = P$(do not reject H_0 when $X \sim \text{Bin}(25, p)) = P(7 < X < 18 \text{ when } X \sim \text{Bin}(25, p)) = B(17; 25, p) - B(7; 25, p)$.
$\beta(.4) = B(17; 25,.4) - B(7, 25,.4) = .845$, while $\beta(.3) = B(17; 25, .3) - B(7; 25, .3) = .488$.
By symmetry (or re-computation), $\beta(.6) = .845$ and $\beta(.7) = .488$.

e. From part (c), the P-value associated with $x = 6$ is .014. Since $.014 \leq .044$, the procedure in (d) leads us to reject H_0.

12.

a. H_0: $\mu = 1300$ v. H_a: $\mu > 1300$.

b. When H_0 is true, $\overline{X}$ is normally distributed with mean $E(\overline{X}) = \mu = 1300$ and standard deviation $\dfrac{\sigma}{\sqrt{n}} = \dfrac{60}{\sqrt{10}} = 18.97$. Values more contradictory to H_0 (more indicative of H_a) than $\bar{x} = 1340$ would be anything above 1340. Thus, the P-value is

P-value = $P(\overline{X} \geq 1340 \text{ when } H_0 \text{ is true}) = P\left(Z \geq \dfrac{1340-1300}{18.97}\right) = 1 - \Phi(2.11) = .0174$.

In particular, since .0174 > .01, H_0 would <u>not</u> be rejected at the $\alpha = .01$ significance level.

c. When $\mu = 1350$, $\overline{X}$ has a normal distribution with mean 1350 and standard deviation 18.97.

To determine $\beta(1350)$, we first need to figure out the threshold between P-value $\leq \alpha$ and P-value $> \alpha$ in terms of $\bar{x}$. Parallel to part (b), proceed as follows:

$.01 = P(\text{reject } H_0 \text{ when } H_0 \text{ is true}) = P(\overline{X} \geq \bar{x} \text{ when } H_0 \text{ is true}) = 1 - \Phi\left(\dfrac{\bar{x}-1300}{18.97}\right) \Rightarrow$

$\Phi\left(\dfrac{\bar{x}-1300}{18.97}\right) = .99 \Rightarrow \dfrac{\bar{x}-1300}{18.97} = 2.33 \Rightarrow \bar{x} = 1344.21$. That is, we'd reject H_0 at the $\alpha = .01$ level iff the observed value of $\overline{X}$ is ≥ 1344.21.

Finally, $\beta(1350) = P(\text{do not reject } H_0 \text{ when } \mu = 1350) = P(\overline{X} < 1344.21 \text{ when } \mu = 1350) =$
$P\left(Z < \dfrac{1344.21-1350}{18.97}\right) \approx \Phi(-.31) = .3783$.

13.

a. H_0: $\mu = 10$ v. H_a: $\mu \neq 10$.

b. Since the alternative is two-sided, values at least as contradictory to H_0 as $\bar{x} = 9.85$ are not only those less than 9.85 but also those equally far from $\mu = 10$ on the high side: i.e., $\bar{x}$ values ≥ 10.15.

When H_0 is true, $\bar{X}$ has a normal distribution with mean $\mu = 10$ and sd $\frac{\sigma}{\sqrt{n}} = \frac{.200}{\sqrt{25}} = .04$. Hence,

P-value = $P(\bar{X} \leq 9.85$ or $\bar{X} \geq 10.15$ when H_0 is true) = $2P(\bar{X} \leq 9.85$ when H_0 is true) by symmetry $= 2P\left(Z < \frac{9.85-10}{.04}\right) = 2\Phi(-3.75) \approx 0$. (Software gives the more precise P-value .00018.)

In particular, since P-value $\approx 0 < \alpha = .01$, we reject H_0 at the .01 significance level and conclude that the true mean measured weight differs from 10 kg.

c. To determine $\beta(\mu)$ for any $\mu \neq 10$, we must first find the threshold between P-value $\leq \alpha$ and P-value > α in terms of $\bar{x}$. Parallel to part (b), proceed as follows:

$.01 = P(\text{reject } H_0 \text{ when } H_0 \text{ is true}) = 2P(\bar{X} \leq \bar{x} \text{ when } H_0 \text{ is true}) = 2\Phi\left(\frac{\bar{x}-10}{.04}\right) \Rightarrow$

$\Phi\left(\frac{\bar{x}-10}{.04}\right) = .005 \Rightarrow \frac{\bar{x}-10}{.04} = -2.58 \Rightarrow \bar{x} = 9.8968$. That is, we'd reject H_0 at the $\alpha = .01$ level iff the observed value of $\bar{X}$ is ≤ 9.8968 — or, by symmetry, $\geq 10 + (10 - 9.8968) = 10.1032$. Equivalently, we do not reject H_0 at the $\alpha = .01$ level if $9.8968 < \bar{X} < 10.1032$.

Now we can determine the chance of a type II error:

$\beta(10.1) = P(9.8968 < \bar{X} < 10.1032$ when $\mu = 10.1) = P(-5.08 < Z < .08) = .5319$.

Similarly, $\beta(9.8) = P(9.8968 < \bar{X} < 10.1032$ when $\mu = 9.8) = P(2.42 < Z < 7.58) = .0078$.

14.

a. Let μ = true average braking distance for the new design at 40 mph.
The hypotheses are H_0: $\mu = 120$ v. H_a: $\mu < 120$.

b. Values less than 120 are more contradictory to H_0 (more indicative of H_a). In particular, values of $\bar{X}$ at least as contradictory to H_0 as $\bar{x} = 117.2$ are $\bar{X} \leq 117.2$.

When H_0 is true, $\bar{X}$ has a normal distribution with mean $\mu = 120$ and sd $\frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{36}} = \frac{5}{3}$. Therefore,

P-value = $P(\bar{X} \leq 117.2$ when H_0 is true) = $P\left(Z \leq \frac{117.2-120}{5/3}\right) = \Phi(-1.68) = .0465$.

In particular, since $.0465 \leq .10$, we reject H_0 at the .10 significance level and conclude that the new design does have a mean braking distance less than 120 feet at 40 mph.

c. Similar to part (c) in the last two exercises, we must find the threshold value of $\bar{x}$ corresponding to a probability of .10:

$.10 = P(\bar{X} \leq \bar{x} \text{ when } H_0 \text{ is true}) = \Phi\left(\frac{\bar{x}-120}{5/3}\right) \Rightarrow \ldots \Rightarrow \bar{x} = 117.867$. That is, we reject H_0 if and only if $\bar{X} \leq 117.867$. When $\mu = 115$, $\bar{X}$ is normal with mean 115 and sd 5/3. Thus,

P(new design not implemented when $\mu = 115$) = P(do not reject H_0 when $\mu = 115$) = $\beta(115)$
$= P(\bar{X} > 117.867$ when $\mu = 115) = P(Z > 1.72) = 1 - \Phi(1.72) = .0427$.

Section 8.2

15. In each case, the direction of H_a indicates that the *P*-value is $P(Z \geq z) = 1 - \Phi(z)$.

a. *P*-value = $1 - \Phi(1.42) = .0778$.

b. *P*-value = $1 - \Phi(0.90) = .1841$.

c. *P*-value = $1 - \Phi(1.96) = .0250$.

d. *P*-value = $1 - \Phi(2.48) = .0066$.

e. *P*-value = $1 - \Phi(-.11) = .5438$.

16. The implicit hypotheses are H_0: $\mu = 30$ and H_a: $\mu \neq 30$ ("whether μ differs from the target value"). So, in each case, the *P*-value is $2 \cdot P(Z \geq |z|) = 2 \cdot [1 - \Phi(|z|)]$.

a. *P*-value = $2 \cdot [1 - \Phi(|2.10|)] = .0358$.

b. *P*-value = $2 \cdot [1 - \Phi(|-1.75|)] = .0802$.

c. *P*-value = $2 \cdot [1 - \Phi(|-0.55|)] = .5824$.

d. *P*-value = $2 \cdot [1 - \Phi(|1.41|)] = .1586$.

e. *P*-value = $2 \cdot [1 - \Phi(|-5.3|)] \approx 0$.

17.

a. $z = \dfrac{30,960 - 30,000}{1500/\sqrt{16}} = 2.56$, so *P*-value = $P(Z \geq 2.56) = 1 - \Phi(2.56) = .0052$.
Since $.0052 < \alpha = .01$, reject H_0.

b. $z_\alpha = z_{.01} = 2.33$, so $\beta(30500) = \Phi\left(2.33 + \dfrac{30000 - 30500}{1500/\sqrt{16}}\right) = \Phi(1.00) = .8413$.

c. $z_\alpha = z_{.01} = 2.33$ and $z_\beta = z_{.05} = 1.645$. Hence, $n = \left[\dfrac{1500(2.33 + 1.645)}{30,000 - 30,500}\right]^2 = 142.2$, so use $n = 143$.

d. From (a), the *P*-value is .0052. Hence, the smallest α at which H_0 can be rejected is .0052.

18.

a. $\dfrac{72.3 - 75}{1.8} = -1.5$ so 72.3 is 1.5 SDs (of $\bar{x}$) below 75.

b. *P*-value = $P(\bar{X} \leq 72.3) = P(Z \leq -1.5) = .0668$. Since $.0668 > .002$, don't reject H_0.

c. $z_\alpha = z_{.002} = 2.88$, so $\beta(70) = 1 - \Phi\left(-2.88 + \dfrac{75 - 70}{9/\sqrt{25}}\right) = 1 - \Phi(-0.10) = .5398$.

d. $z_\beta = z_{.01} = 2.33$. Hence, $n = \left[\frac{9(2.88+2.33)}{75-70}\right]^2 = 87.95$, so use $n = 88$.

e. Zero. By definition, a type I error can only occur when H_0 is true, but $\mu = 76$ means that H_0 is actually false.

19.

a. Since the alternative hypothesis is two-sided, *P*-value = $2 \cdot \left[1-\Phi\left(\left|\frac{94.32-95}{1.20/\sqrt{16}}\right|\right)\right]$ = 2 · [1 – Φ(2.27)] = 2(.0116) = .0232. Since .0232 > α = .01, we do not reject H_0 at the .01 significance level.

b. $z_{\alpha/2} = z_{.005} = 2.58$, so $\beta(94) = \Phi\left(2.58+\frac{95-94}{1.20/\sqrt{16}}\right) - \Phi\left(-2.58+\frac{95-94}{1.20/\sqrt{16}}\right)$ = Φ(5.91) – Φ(0.75) = .2266.

c. $z_\beta = z_{.1} = 1.28$. Hence, $n = \left[\frac{1.20(2.58+1.28)}{95-94}\right]^2 = 21.46$, so use $n = 22$.

20. The hypotheses are H_0: $\mu = 750$ and H_a: $\mu < 750$. With a *P*-value of .016, we reject H_0 if $\alpha = .05$ (because $.016 \le .05$) but we do not reject H_0 if $\alpha = .01$ (since .016 > .01). Thus, we do not make the purchase if we use $\alpha = .05$ (the "significant evidence" was observed), and we proceed with the purchase if $\alpha = .01$.

In this context, a type I error is to reject these new light bulbs when they're actually as good as advertised, while a type II error is to fail to recognize that the new light bulbs underperform (i.e., purchase them even though their mean lifetime is less than 750 hours).

Though it's certainly debatable, given the favorable price a type I error (here, an opportunity loss) might be considered more serious. In that case, the lower α level, .01, should be used, and we proceed with the purchase. (You could certainly also argue that, price notwithstanding, buying a product that's less good than advertised is ill-advised. In that case, a type II error would be deemed worse, we should use the higher α level of .05, and based on the observed data we do not purchase the light bulbs.)

21. The hypotheses are H_0: $\mu = 5.5$ v. H_a: $\mu \neq 5.5$.

a. The *P*-value is $2 \cdot \left[1-\Phi\left(\left|\frac{5.25-5.5}{.3/\sqrt{16}}\right|\right)\right]$ = 2 · [1 – Φ(3.33)] = .0008. Since the *P*-value is smaller than any reasonable significance level (.1, .05, .01, .001), we reject H_0.

b. The chance of detecting that H_0 is false is the complement of the chance of a type II error. With $z_{\alpha/2} = z_{.005} = 2.58$, $1-\beta(5.6) = 1-\left[\Phi\left(2.58+\frac{5.5-5.6}{.3/\sqrt{16}}\right) - \Phi\left(-2.58+\frac{5.5-5.6}{.3/\sqrt{16}}\right)\right]$ = 1 – Φ(1.25) + Φ(3.91) = .1056.

c. $n = \left[\frac{.3(2.58+2.33)}{5.5-5.6}\right]^2 = 216.97$, so use $n = 217$.

22. Let μ denote the true average corrosion penetration under these settings. The hypotheses are H_0: $\mu = 50$ (really, $\mu \le 50$) versus H_a: $\mu > 50$. Using the large-sample z test and the information provided,

P-value = $P\left(Z \ge \frac{52.7-50}{4.8/\sqrt{45}}\right) = 1 - \Phi(3.77) \approx .0001$.

Since this P-value is less than any reasonable significance level, we reject H_0 and conclude that the true mean corrosion penetration definitely exceeds 50 mils, and so these conduits should not be used.

23.

a. Using software, $\bar{x} = 0.75$, $\tilde{x} = 0.64$, $s = .3025$, $f_s = 0.48$. These summary statistics, as well as a box plot (not shown) indicate substantial positive skewness, but no outliers.

b. No, it is not plausible from the results in part **a** that the variable ALD is normal. However, since n = 49, normality is not required for the use of z inference procedures.

c. We wish to test H_0: $\mu = 1.0$ versus H_a: $\mu < 1.0$. The test statistic is $z = \frac{0.75-1.0}{.3025/\sqrt{49}} = -5.79$, and so the P-value is $P(Z \le -5.79) \approx 0$. At any reasonable significance level, we reject the null hypothesis. Therefore, yes, the data provides strong evidence that the true average ALD is less than 1.0.

d. $\bar{x} + z_{.05}\frac{s}{\sqrt{n}} = 0.75 + 1.645\frac{.3025}{\sqrt{49}} = 0.821$

24. Let μ denote the true average estimated calorie content of this 153-calorie beer. The hypotheses of interest are H_0: $\mu = 153$ v. H_a: $\mu > 153$. Using z-based inference with the data provided, the P-value of the test is $P\left(Z \ge \frac{191-153}{89/\sqrt{58}}\right) = 1 - \Phi(3.25) = .0006$. At any reasonable significance level, we reject the null hypothesis. Therefore, yes, there is evidence that the true average estimated calorie content of this beer exceeds the actual calorie content.

25. Let μ denote the true average task time. The hypotheses of interest are H_0: $\mu = 2$ v. H_a: $\mu < 2$. Using z-based inference with the data provided, the P-value of the test is $P\left(Z \le \frac{1.95-2}{.20/\sqrt{52}}\right) = \Phi(-1.80) = .0359$. Since $.0359 > .01$, at the $\alpha = .01$ significance level we do not reject H_0. At the .01 level, we do not have sufficient evidence to conclude that the true average task time is less than 2 seconds.

26. The parameter of interest is μ = the true average dietary intake of zinc among males aged 65-74 years. The hypotheses are H_0: $\mu = 15$ versus H_a: $\mu < 15$.
Since the sample size is large, we'll use a z-procedure here; with no significance level specified, we'll default to $\alpha = .05$.

From the summary statistics provided, $z = \frac{11.3-15}{6.43/\sqrt{115}} = -6.17$, and so P-value = $P(Z \le -6.17) \approx 0$. Hence, we reject H_0 at the $\alpha = .05$ level; in fact, with a test statistic that large, -6.17, we would reject H_0 at any reasonable significance level. There is convincing evidence that average daily intake of zinc for males aged 65-74 years falls below the recommended daily allowance of 15 mg/day.

27. $\beta(\mu_0-\Delta)=\Phi\left(z_{\alpha/2}+\Delta\sqrt{n}/\sigma\right)-\Phi\left(-z_{\alpha/2}+\Delta\sqrt{n}/\sigma\right)=1-\Phi(-z_{\alpha/2}-\Delta\sqrt{n}/\sigma)-\left[1-\Phi(z_{\alpha/2}-\Delta\sqrt{n}/\sigma)\right]=$ $\Phi\left(z_{\alpha/2}-\Delta\sqrt{n}/\sigma\right)-\Phi\left(-z_{\alpha/2}-\Delta\sqrt{n}/\sigma\right)=\beta(\mu_0+\Delta)$.

28. For an upper-tailed test, $=\beta(\mu)=\Phi\left(z_\alpha+\sqrt{n}(\mu_0-\mu)/\sigma\right)$. Since in this case we are considering $\mu>\mu_0$, $\mu_0-\mu$ is negative so $\sqrt{n}(\mu_0-\mu)/\sigma\to-\infty$ as $n\to\infty$. The desired conclusion follows since $\Phi(-\infty)=0$. The arguments for a lower-tailed and two-tailed test are similar.

Section 8.3

29. The hypotheses are H_0: $\mu = .5$ versus H_a: $\mu \neq .5$. Since this is a two-sided test, we must double the one-tail area in each case to determine the *P*-value.

a. $n = 13 \Rightarrow$ df = 13 – 1 = 12. Looking at column 12 of Table A.8, the area to the right of $t = 1.6$ is .068. Doubling this area gives the two-tailed *P*-value of 2(.068) = .134. Since .134 > α = .05, we do not reject H_0.

b. For a two-sided test, observing $t = –1.6$ is equivalent to observing $t = 1.6$. So, again the *P*-value is 2(.068) = .134, and again we do not reject H_0 at α = .05.

c. df = n – 1 = 24; the area to the left of –2.6 = the area to the right of 2.6 = .008 according to Table A.8. Hence, the two-tailed *P*-value is 2(.008) = .016. Since .016 > .01, we do not reject H_0 in this case.

d. Similar to part (c), Table A.8 gives a one-tail area of .000 for $t = \pm3.9$ at df = 24. Hence, the two-tailed *P*-value is 2(.000) = .000, and we reject H_0 at any reasonable α level.

30. The hypotheses are H_0: $\mu = 7.0$ versus H_a: $\mu < 7.0$. In each case, we want the one-tail area to the left of the observed test statistic.

a. $n = 6 \Rightarrow$ df = 6 – 1 = 5. From Table A.8, $P(T \le –2.3$ when $T \sim t_5) = P(T \ge 2.3$ when $T \sim t_5)$ = .035. Since .035 ≤ .05, we reject H_0 at the α = .05 level.

b. Similarly, *P*-value = $P(T \ge 3.1$ when $T \sim t_{14})$ = .004. Since .004 < .01, reject H_0.

c. Similarly, *P*-value = $P(T \ge 1.3$ when $T \sim t_{11})$ = .110. Since .110 ≥ .05, do not reject H_0.

d. Here, *P*-value = $P(T \le .7$ when $T \sim t_5)$ because it's a lower tailed test, and this is 1 – $P(T > .7$ when $T \sim t_5)$ = 1 – .258 = .742. Since .742 > .05, do not reject H_0. (Note: since the sign of the *t*-statistic contradicted H_a, we know immediately not to reject H_0.)

e. The observed value of the test statistic is $t=\dfrac{\bar{x}-\mu_0}{s/\sqrt{n}}=\dfrac{6.68-7.0}{.0820}$ = –3.90. From this, similar to parts (a)-(c), *P*-value = $P(T \ge 3.90$ when $T \sim t_5)$ = .006 according to Table A.8. We would reject H_0 for any significance level at or above .006.

31. This is an upper-tailed test, so the *P*-value in each case is $P(T \geq \text{observed } t)$.

a. *P*-value = $P(T \geq 3.2$ with df = 14) = .003 according to Table A.8. Since .003 ≤ .05, we reject H_0.

b. *P*-value = $P(T \geq 1.8$ with df = 8) = .055. Since .055 > .01, do not reject H_0.

c. *P*-value = $P(T \geq -.2$ with df = 23) = 1 – $P(T \geq .2$ with df = 23) by symmetry = 1 – .422 = .578. Since .578 is quite large, we would not reject H_0 at any reasonable α level. (Note that the sign of the observed *t* statistic contradicts H_a, so we know immediately not to reject H_0.)

32. With H_0: $\mu = .60$ v. H_a: $\mu \neq .60$ and a two-tailed *P*-value of .0711, we fail to reject H_0 at levels .01 and .05 (thus concluding that the amount of impurities need not be adjusted) , but we would reject H_0 at level .10 (and conclude that the amount of impurities does need adjusting).

33.

a. It appears that the true average weight could be significantly off from the production specification of 200 lb per pipe. Most of the boxplot is to the right of 200.

b. Let μ denote the true average weight of a 200 lb pipe. The appropriate null and alternative hypotheses are H_0: $\mu = 200$ and H_a: $\mu \neq 200$. Since the data are reasonably normal, we will use a one-sample *t* procedure. Our test statistic is $t = \frac{206.73 - 200}{6.35/\sqrt{30}} = \frac{6.73}{1.16} = 5.80$, for a *P*-value of ≈ 0. So, we reject H_0.
At the 5% significance level, the test appears to substantiate the statement in part **a**.

34. The data provided has $n = 6$, $\bar{x} = 31.233$, and $s = 0.689$.

a. The hypotheses are H_0: $\mu = 30$ versus H_a: $\mu > 30$. The one-sample *t* statistic is $t = \frac{31.233 - 30}{0.689/\sqrt{6}} = 4.38$.
At df = 6 – 1 = 5, the *P*-value is $P(T \geq 4.38) = .004$ from software. Since .004 ≤ .01, we reject H_0 at the $\alpha = .01$ level and conclude the true average stopping distance does exceed 30 ft.

b. Using statistical software, the power of the test (with $n = 6$, $\alpha = .01$, and $\sigma = .65$) is .660 when $\mu = 31$ and .998 when $\mu = 32$. The corresponding chances of a type II error are $\beta(31) = 1 - .660 = .340$ and $\beta(32) = 1 - .998 = .002$.

c. Changing σ from .65 to .80 results in $\beta(31) = 1 - .470 = .530$ and $\beta(32) = 1 - .975 = .025$. It's still the case, as it should be, that type II error is less likely when the true value of μ is farther from μ_0. But, with the larger standard deviation, random sampling variation makes it more likely for the sample meant to deviate substantially from μ and, in particular, trick us into thinking H_0 might be plausible (even though it's false in both cases). We see β increase from .340 to .530 when $\mu = 31$ and from .002 to .025 when $\mu = 32$.

d. According to statistical software, $n = 9$ will suffice to achieve $\beta = .1$ (power = .9) when $\alpha = .01$, $\mu = 31$, and $\sigma = .65$.

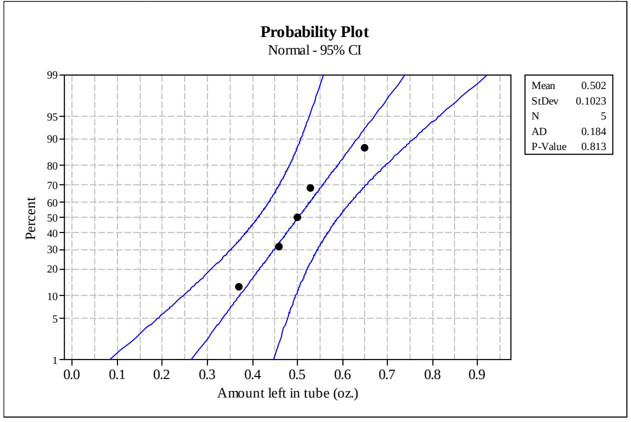
35.

a. The hypotheses are H_0: $\mu = 200$ versus H_a: $\mu > 200$. With the data provided, $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{249.7-200}{145.1/\sqrt{12}} = 1.2$; at df = 12 – 1 = 11, *P*-value = .128. Since .128 > .05, H_0 is not rejected at the $\alpha = .05$ level. We have insufficient evidence to conclude that the true average repair time exceeds 200 minutes.

b. With $d = \frac{|\mu_0-\mu|}{\sigma} = \frac{|200-300|}{150} = 0.67$, df = 11, and $\alpha = .05$, software calculates power ≈ .70, so $\beta(300) \approx .30$.

36. 10% of 6 oz. is 0.6 oz. So, the hypotheses of interest are H_0: $\mu = 0.6$ versus H_a: $\mu < 0.6$, where μ denotes the true average amount left in a 6 oz. tube of toothpaste.

a. The 5 tubes were randomly selected. Although we don't have much power with $n = 5$ to detect departures from normality, the probability plot below suggest the data are consistent with a normally distributed population. So, we are comfortable proceeding with the *t* procedure.

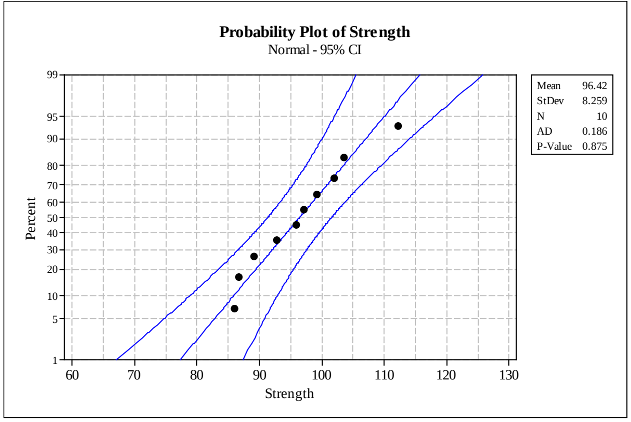


b. The test statistic is $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{.502-.6}{.1023/\sqrt{5}} = -2.14$. At df = 5 – 1 = 4, *P*-value = $P(T \le -2.14$ at df=4) = $P(T \ge 2.14$ at df = 4) ≈ .049 from software. We (barely) reject H_0 at the $\alpha = .05$ significance level, because $.049 \le .05$; however, we fail to reject H_0 at the $\alpha = .01$ level, because .049 > .01. So, we have evidence that the true average content is less than 10% of the original 6 oz at the 5% significance level, but not at the 1% significance level.

c. In this context, a Type I error would be to conclude that less than 10% of the tube's contents remain after squeezing, on average, when in fact 10% (or more) actually remains. When we rejected H_0 at the 5% level, we may have committed a Type I error. A Type II error occurs if we fail to recognize that less than 10% of a tube's contents remains, on average, when that's actually true (i.e., we fail to reject the false null hypothesis of $\mu = 0.6$ oz). When we failed to reject H_0 at the 1% level, we may have committed a Type II error.

37.

a. The accompanying normal probability plot is acceptably linear, which suggests that a normal population distribution is quite plausible.



b. The parameter of interest is μ = the true average compression strength (MPa) for this type of concrete. The hypotheses are H_0: $\mu = 100$ versus H_a: $\mu < 100$.
Since the data come from a plausibly normal population, we will use the *t* procedure. The test statistic is $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} = \dfrac{96.42 - 100}{8.26/\sqrt{10}} = -1.37$. The corresponding one-tailed *P*-value, at df = 10 – 1 = 9, is $P(T \le -1.37) \approx .102$.
The *P*-value slightly exceeds .10, the largest α level we'd consider using in practice, so the null hypothesis H_0: $\mu = 100$ should not be rejected. This concrete should be used.

38. μ = the true average percentage of organic matter in this type of soil, and the hypotheses are H_0: $\mu = 3$ versus H_a: $\mu \neq 3$. With $n = 30$, and assuming normality, we use the *t* test:
$t = \dfrac{\bar{x} - 3}{s/\sqrt{n}} = \dfrac{2.481 - 3}{.295} = \dfrac{-.519}{.295} = -1.759$. At df = 30 – 1 = 29, *P*-value = $2P(T > 1.759) = 2(.041) = .082$.
At significance level .10, since $.082 \le .10$, we would reject H_0 and conclude that the true average percentage of organic matter in this type of soil is something other than 3. At significance level .05, we would not have rejected H_0.

39. Software provides $\bar{x} = 1.243$ and $s = 0.448$ for this sample.

a. The parameter of interest is μ = the population mean expense ratio (%) for large-cap growth mutual funds. The hypotheses are H_0: $\mu = 1$ versus H_a: $\mu > 1$.
We have a random sample, and a normal probability plot is reasonably linear, so the assumptions for a *t* procedure are met.
The test statistic is $t = \dfrac{1.243 - 1}{0.448/\sqrt{20}} = 2.43$, for a *P*-value of $P(T \ge 2.43 \text{ at df} = 19) \approx .013$. Hence, we (barely) fail to reject H_0 at the .01 significance level. There is insufficient evidence, at the $\alpha = .01$ level, to conclude that the population mean expense ratio for large-cap growth mutual funds exceeds 1%.

b. A Type I error would be to incorrectly conclude that the population mean expense ratio for large-cap growth mutual funds exceeds 1% when, in fact the mean is 1%. A Type II error would be to fail to recognize that the population mean expense ratio for large-cap growth mutual funds exceeds 1% when that's actually true.
Since we failed to reject H_0 in (a), we potentially committed a Type II error there. If we later find out that, in fact, $\mu = 1.33$, so H_a was actually true all along, then yes we have committed a Type II error.

c. With $n = 20$ so df = 19, $d = \dfrac{1.33-1}{.5} = .66$, and $\alpha = .01$, software provides power ≈ .66. (Note: it's purely a coincidence that power and d are the same decimal!) This means that if the true values of μ and σ are $\mu = 1.33$ and $\sigma = .5$, then there is a 66% probability of correctly rejecting H_0: $\mu = 1$ in favor of H_a: $\mu > 1$ at the .01 significance level based upon a sample of size $n = 20$.

40. The hypotheses are H_0: $\mu = 48$ versus H_a: $\mu > 48$. Using the one-sample t procedure, the test statistic and P-value are $t = \dfrac{51.3-48}{1.2/\sqrt{10}}$ = 8.7 and $P(T \geq 8.7$ when df = 9$) \approx 0$. Hence, we reject H_0 at any reasonable significance level and conclude that the true average strength for the WSF/cellulose composite definitely exceeds 48 MPa.

41. μ = true average reading, H_0: $\mu = 70$ v. H_a: $\mu \neq 70$, and $t = \dfrac{\bar{x}-70}{s/\sqrt{n}} = \dfrac{75.5-70}{7/\sqrt{6}} = \dfrac{5.5}{2.86} = 1.92$.

From table A.8, df = 5, P-value = $2[P(T > 1.92)] \approx 2(.058) = .116$. At significance level .05, there is not enough evidence to conclude that the spectrophotometer needs recalibrating.

Section 8.4

42. In each instance, the P-value must be calculated in accordance with the inequality in H_a.

a. Upper-tailed test: P-value = $P(Z \geq 1.47) = 1 - \Phi(1.47) = .0708$.

b. Lower-tailed test: P-value = $P(Z \leq -2.70) = \Phi(-2.70) = .0035$.

c. Two-tailed test: P-value = $2 \cdot P(Z \geq |-2.70|) = 2(.0035) = .0070$.

d. Lower-tailed test: P-value = $P(Z \leq 0.25) = \Phi(0.25) = .5987$.

43.

a. The parameter of interest is p = the proportion of the population of female workers that have BMIs of at least 30 (and, hence, are obese). The hypotheses are H_0: $p = .20$ versus H_a: $p > .20$.
With $n = 541$, $np_0 = 541(.2) = 108.2 \geq 10$ and $n(1 - p_0) = 541(.8) = 432.8 \geq 10$, so the "large-sample" z procedure is applicable.
From the data provided, $\hat{p} = \dfrac{120}{541} = .2218$, so $z = \dfrac{\hat{p} - p_0}{\sqrt{p_0(1-p_0)/n}} = \dfrac{.2218 - .20}{\sqrt{.20(.80)/541}} = 1.27$ and P-value $= P(Z \geq 1.27) = 1 - \Phi(1.27) = .1020$. Since .1020 > .05, we fail to reject H_0 at the $\alpha = .05$ level. We do not have sufficient evidence to conclude that more than 20% of the population of female workers is obese.

b. A Type I error would be to incorrectly conclude that more than 20% of the population of female workers is obese, when the true percentage is 20%. A Type II error would be to fail to recognize that more than 20% of the population of female workers is obese when that's actually true.

c. The question is asking for the chance of committing a Type II error when the true value of p is .25, i.e. $\beta(.25)$. Using the textbook formula,

$$\beta(.25) = \Phi\left[\frac{.20 - .25 + 1.645\sqrt{.20(.80)/541}}{\sqrt{.25(.75)/541}}\right] = \Phi(-1.166) \approx .121.$$

44.

a. Let p = true proportion of all nickel plates that blister under the given circumstances. The hypotheses are H_0: p = .10 versus H_a: p > .10. Using the one-proportion z procedure, the test statistic is $z = \frac{14/100 - .10}{\sqrt{.10(.90)/100}} = 1.33$ and the P-value is $P(Z \geq 1.33) = 1 - \Phi(1.33) = .0918$. Since .0918 > .05, we fail to Reject H_0. The data does not give compelling evidence for concluding that more than 10% of all plates blister under the circumstances.

The possible error we could have made is a Type II error: failing to reject the null hypothesis when it is actually true.

b. $\beta(.15) = \Phi\left[\frac{.10 - .15 + 1.645\sqrt{.10(.90)/100}}{\sqrt{.15(.85)/100}}\right] = \Phi(-.02) = .4920$. When n = 200,

$$\beta(.15) = \Phi\left[\frac{.10 - .15 + 1.645\sqrt{.10(.90)/200}}{\sqrt{.15(.85)/200}}\right] = \Phi(-.60) = .2743$$

c. $n = \left[\frac{1.645\sqrt{.10(.90)} + 1.28\sqrt{.15(.85)}}{.15 - .10}\right]^2 = 19.01^2 = 361.4$, so use n = 362.

45. Let p = true proportion of all donors with type A blood. The hypotheses are H_0: p = .40 versus H_a: $p \neq .40$. Using the one-proportion z procedure, the test statistic is $z = \frac{82/150 - .40}{\sqrt{.40(.60)/150}} = \frac{.147}{.04} = 3.667$, and the corresponding P-value is $2P(Z \geq 3.667) \approx 0$. Hence, we reject H_0. The data does suggest that the percentage of all donors with type A blood differs from 40%. (at the .01 significance level). Since the P-value is also less than .05, the conclusion would not change.

46.

a. Let X = the number of couples who lean more to the right when they kiss. If n = 124 and p = 2/3, then $E[X]$ = 124(2/3) = 82.667. The researchers observed x = 80, for a difference of 2.667. The probability in question is $P(|X - 82.667| \geq 2.667) = P(X \leq 80 \text{ or } X \geq 85.33) = P(X \leq 80) + [1 - P(X \leq 85)] = B(80;124,2/3) + [1 - B(85;124,2/3)] = 0.634$. (Using a large-sample z-based calculation gives a probability of 0.610.)

b. We wish to test H_0: p = 2/3 v. H_a: $p \neq 2/3$. From the data, $\hat{p} = \frac{80}{124} = .645$, so our test statistic is $z = \frac{.645 - .667}{\sqrt{.667(.333)/124}} = -0.51$. We would fail to reject H_0 even at the α = .10 level, since the two-tailed P-value is quite large. There is no statistically significant evidence to suggest the p = 2/3 figure is implausible for right-leaning kissing behavior.

47.

a. The parameter of interest is p = the proportion of all wine customers who would find screw tops acceptable. The hypotheses are H_0: $p = .25$ versus H_a: $p < .25$.
With $n = 106$, $np_0 = 106(.25) = 26.5 \geq 10$ and $n(1 - p_0) = 106(.75) = 79.5 \geq 10$, so the "large-sample" z procedure is applicable.
From the data provided, $\hat{p} = \frac{22}{106} = .208$, so $z = \frac{.208 - .25}{\sqrt{.25(.75)/106}} = -1.01$ and P-value = $P(Z \leq -1.01)$ = $\Phi(-1.01) = .1562$.
Since .1562 > .10, we fail to reject H_0 at the $\alpha = .10$ level. We do not have sufficient evidence to suggest that less than 25% of all customers find screw tops acceptable. Therefore, we recommend that the winery <u>should</u> switch to screw tops.

b. A Type I error would be to incorrectly conclude that less than 25% of all customers find screw tops acceptable, when the true percentage is 25%. Hence, we'd recommend not switching to screw tops when there use is actually justified. A Type II error would be to fail to recognize that less than 25% of all customers find screw tops acceptable when that's actually true. Hence, we'd recommend (as we did in (a)) that the winery switch to screw tops when the switch is not justified. Since we failed to reject H_0 in (a), we may have committed a Type II error.

48.

a. The parameter of interest is p = the proportion of all households with Chinese drywall that have electrical/environmental problems. The hypotheses are H_0: $p = .5$, H_a: $p > .5$.
With $n = 51$, $np_0 = n(1 - p_0) = 51(.5) = 25.5 \geq 10$, so the "large-sample" z procedure is applicable.
From the data provided, $\hat{p} = \frac{41}{51} = .804$, so $z = \frac{.804 - .5}{\sqrt{.5(.5)/51}} = 4.34$ and P-value = $P(Z \geq 4.34) \approx 0$.

Thus, we reject H_0 at the $\alpha = .01$ level. We have compelling evidence that more than 50% of all households with Chinese drywall that have electrical/environmental problems.

b. From Chapter 7, a 99% lower confidence bound for p is

$$\frac{\hat{p} + \frac{z_\alpha}{2n} - z_\alpha\sqrt{\hat{p}\hat{q}/n + z_\alpha^2/4n^2}}{1 + z_\alpha^2/n} = \frac{.804 + \frac{2.33}{2(51)} - 2.33\sqrt{(.804)(.196)/51 + (2.33)^2/4(51)^2}}{1 + (2.33)^2/51} = .648.$$

That is, we're 99% confident that more than 64.8% of all homes with Chinese drywall have electrical/environmental problems.

c. The goal is to find the chance of a Type II error when the actual value of p is .80; i.e., we wish to find $\beta(.80)$. Using the textbook formula,

$$\beta(.80) = \Phi\left[\frac{.50 - .80 + 2.33\sqrt{.50(.50)/51}}{\sqrt{.80(.20)/51}}\right] = \Phi(-2.44) \approx .007.$$

49.

a. Let p = true proportion of current customers who qualify. The hypotheses are H_0: p = .05 v. H_a: $p \neq .05$. The test statistic is $z = \frac{.08-.05}{\sqrt{.05(.95)/n}}$ = 3.07, and the P-value is 2 · $P(Z \geq 3.07)$ = 2(.0011) = .0022.

Since $.0022 \leq \alpha = .01$, H_0 is rejected. The company's premise is not correct.

b. $$\beta(.10) = \Phi\left[\frac{.05-.10+2.58\sqrt{.05(.95)/500}}{\sqrt{.10(.90)/500}}\right] - \Phi\left[\frac{.05-.10-2.58\sqrt{.05(.95)/500}}{\sqrt{.10(.90)/500}}\right]$$

$$\approx \Phi(-1.85) - 0 = .0332$$

50. Notice that with the relatively small sample size, we should use a binomial model here.

a. The alternative of interest here is H_a: $p > .50$ (which states that more than 50% of all enthusiasts prefer gut). So, we'll reject H_0 in favor of H_a when the observed value of X is quite large (much more than 10). Suppose we reject H_0 when $X \geq x$; then $\alpha = P(X \geq x$ when H_0 is true$) = 1 - B(x-1; 20, .5)$, since $X \sim \text{Bin}(20, .5)$ when H_0 is true.

By trial and error, α = .058 if x = 14 and α = .021 if x = 15. Therefore, a significance level of exactly α = .05 is not possible, and the largest possible value less than .05 is α = .021 (occurring when we elect to reject H_0 iff $X \geq 15$).

b. $\beta(.6) = P(\text{do not reject } H_0 \text{ when } p = .6) = P(X < 15 \text{ when } X \sim \text{Bin}(20,.6)) = B(14; 20, .6) = .874$. Similarly, $\beta(.8) = B(14; 20, .8) = .196$.

c. No. Since 13 is not $\geq$ 15, we would not reject H_0 at the α = .021 level. Equivalently, the P-value for that observed count is $P(X \geq 13 \text{ when } p = .5) = 1 - P(X \leq 12 \text{ when } X \sim \text{Bin}(20,.5)) = .132$. Since .132 > .021, we do not reject H_0 at the .021 level (or at the .05 level, for that matter).

51. The hypotheses are H_0: p = .10 v. H_a: $p > .10$, and we reject H_0 iff $X \geq c$ for some unknown c. The corresponding chance of a type I error is $\alpha = P(X \geq c \text{ when } p = .10) = 1 - B(c-1; 10, .1)$, since the rv X has a Binomial(10, .1) distribution when H_0 is true.
The values n = 10, c = 3 yield $\alpha = 1 - B(2; 10, .1) = .07$, while $\alpha > .10$ for c = 0, 1, 2. Thus c = 3 is the best choice to achieve $\alpha \leq .10$ and simultaneously minimize β. However, $\beta(.3) = P(X < c \text{ when } p = .3) = B(2; 10, .3) = .383$, which has been deemed too high. So, the desired α and β levels cannot be achieved with a sample size of just n = 10.
The values n = 20, c = 5 yield $\alpha = 1 - B(4; 20, .1) = .043$, but again $\beta(.3) = B(4; 20, .3) = .238$ is too high.
The values n = 25, c = 5 yield $\alpha = 1 - B(4; 25, .1) = .098$ while $\beta(.3) = B(4; 25, .3) = .090 \leq .10$, so n = 25 should be used. In that case and with the rule that we reject H_0 iff $X \geq 5$, α = .098 and $\beta(.3)$ = .090.

52. Let p denote the proportion of all students that do <u>not</u> share this belief (i.e., that do <u>not</u> think the software unfairly targets students. The hypotheses of interest are H_0: p = .5 v. H_a: $p > .5$ (majority).
The sample proportion of students that do not share this belief is $\hat{p}$ = (171 – 58)/171 = .661. The test statistic is $z = \frac{.661-.5}{\sqrt{.5(.5)/171}}$ = 4.21, and the P-value is $P(Z \geq 4.21) \approx 0$. Thus, we strongly reject H_0 and conclude that a majority of all students do <u>not</u> share the belief that the plagiarism-detection software unfairly targets students.

Section 8.5

53.

a. The formula for β is $1-\Phi\left(-2.33+\frac{\sqrt{n}}{9}\right)$, which gives .8888 for n = 100, .1587 for n = 900, and .0006 for n = 2500.

b. Z = –5.3, which is "off the z table," so P-value < .0002; this value of z is quite statistically significant.

c. No. Even when the departure from H_0 is insignificant from a practical point of view, a statistically significant result is highly likely to appear; the test is too likely to detect small departures from H_0.

54.

a. Here $\beta=\Phi\left(\frac{-.01+.9320/\sqrt{n}}{.4073/\sqrt{n}}\right)=\Phi\left(\frac{-.01\sqrt{n}+.9320}{.4073}\right)$ = .9793, .8554, .4325, .0944, and 0 for n = 100, 2500, 10,000, 40,000, and 90,000, respectively.

b. Here $z=.025\sqrt{n}$ which equals .25, 1.25, 2.5, and 5 for the four n's, whence P-value = .4213, .1056, .0062, .0000, respectively.

c. No. Even when the departure from H_0 is insignificant from a practical point of view, a statistically significant result is highly likely to appear; the test is too likely to detect small departures from H_0.

55.

a. The chance of committing a type I error on a single test is .01. Hence, the chance of committing at least one type I error among m tests is P(at least on error) = 1 – P(no type I errors) = 1 –[P(no type I error)]^m by independence = 1 – .99^m. For m = 5, the probability is .049; for m = 10, it's .096.

b. Set the answer from (a) to .5 and solve for m: $1 - .99^m \geq .5 \Rightarrow .99^m \leq .5 \Rightarrow m \geq \log(.5)/\log(.99) = 68.97$. So, at least 69 tests must be run at the α = .01 level to have a 50-50 chance of committing at least one type I error.

56.

a. A two-sided 95% confidence interval is equivalent to a two-sided hypothesis test with significance level α = 100% – 95% = .05. In particular, since the null value of 2 is not in the 95% CI for μ, we conclude at the .05 level that H_0 should be rejected, i.e., H_a: $\mu \neq 2$ is concluded.

b. We now need a 99% CI for μ (which might turn out to contain 2). The sample mean must be the midpoint of the interval: $\bar{x}$ = 1.88. The margin of error is $t_{.025,14} \cdot se$ = .07, so se = .07/2.145. Thus, a 99% CI for μ is 1.88 ± $t_{.005,14} \cdot se$ = 1.88 ± 2.977(.07/2.145) = 1.88 ± .097 = (1.783, 1.977). Since this 99% CI for μ does not contain 2, we still reject H_0: μ = 2 and conclude that $\mu \neq 2$.

Alternatively, use the mean and standard error above to perform the actual test. The test statistic is $t=\frac{\bar{x}-\mu_0}{se}=\frac{1.88-2}{(.07/2.145)}$ = –3.7. At 14 df, the two-tailed P-value is 2(.001) = .002 from Table A.8. Since this P-value is less that α = .01, we reject H_0.

Supplementary Exercises

57. Because $n = 50$ is large, we use a z test here. The hypotheses are H_0: $\mu = 3.2$ versus H_a: $\mu \neq 3.2$. The computed z value is $z = \frac{3.05 - 3.20}{.34/\sqrt{50}} = -3.12$, and the P-value is $2\,P(Z \geq |-3.12|) = 2(.0009) = 0018$. Since $.0018 < .05$, H_0 should be rejected in favor of H_a.

58. Here we assume that thickness is normally distributed, so that for any n a t test is appropriate, and use Table A.17 to determine n. We wish $\beta(3) = .05$ when $d = \frac{|3.2 - 3|}{.3} = .667$. By inspection, df = 29 ($n = 30$) satisfies this requirement, so $n = 50$ is unnecessarily large.

59.

a. H_0: $\mu = .85$ v. H_a: $\mu \neq .85$

b. With a P-value of .30, we would reject the null hypothesis at any reasonable significance level, which includes both .05 and .10.

60.

a. H_0: $\mu = 2150$ v. H_a: $\mu > 2150$

b. $t = \frac{\bar{x} - 2150}{s/\sqrt{n}}$

c. $t = \frac{2160 - 2150}{30/\sqrt{16}} = \frac{10}{7.5} = 1.33$

d. At 15df, P-value = $P(T > 1.33) = .107$ (approximately)

e. From **d**, P-value > .05, so H_0 cannot be rejected at this significance level. The mean tensile strength for springs made using roller straightening is not significantly greater than 2150 N/mm^2.

61.

a. The parameter of interest is μ = the true average contamination level (Total Cu, in mg/kg) in this region. The hypotheses are H_0: $\mu = 20$ versus H_a: $\mu > 20$. Using a one-sample t procedure, with $\bar{x}$ = 45.31 and SE($\bar{x}$) = 5.26, the test statistic is $t = \frac{45.31 - 20}{5.26} = 3.86$. That's a very large t-statistic; however, at df = 3 – 1 = 2, the P-value is $P(T \geq 3.86) \approx .03$. (Using the tables with $t = 3.9$ gives a P-value of $\approx .02$.) Since the P-value exceeds .01, we would fail to reject H_0 at the $\alpha = .01$ level.
This is quite surprising, given the large t-value (45.31 greatly exceeds 20), but it's a result of the very small n.

b. We want the probability that we fail to reject H_0 in part (a) when $n = 3$ and the true values of μ and σ are $\mu = 50$ and $\sigma = 10$, i.e. $\beta(50)$. Using software, we get $\beta(50) \approx .57$.

62. The data provide $n = 8, \bar{x} = 30.7875, s = 6.5300$. The parameter of interest is μ = true average heat-flux of plots covered with coal dust, and the hypotheses are H_0: $\mu = 29.0$ versus H_a: $\mu > 29.0$. The test statistic equals $t = \dfrac{30.7875 - 29.0}{6.53/\sqrt{8}} = .7742$; at df = 8 – 1 = 7, the P-value is roughly $P(T \geq .8) = .225$. Since .225 > .05, we fail to reject H_0. The data does not indicate the mean heat-flux for pots covered with coal dust is greater than for plots covered with grass.

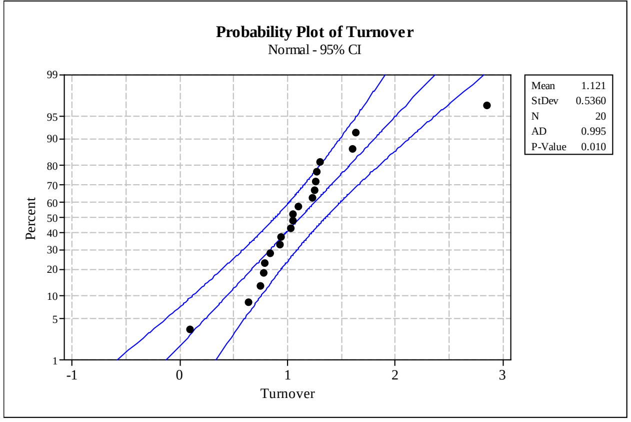
63. $n = 47$, $\bar{x} = 215$ mg, $s = 235$ mg, scope of values = 5 mg to 1,176 mg

a. No, the distribution does not appear to be normal. It appears to be skewed to the right, since 0 is less than one standard deviation below the mean. It is not necessary to assume normality if the sample size is large enough due to the central limit theorem. This sample size is large enough so we can conduct a hypothesis test about the mean.

b. The parameter of interest is μ = true daily caffeine consumption of adult women, and the hypotheses are H_0: $\mu = 200$ versus H_a: $\mu > 200$. The test statistic (using a z test) is $z = \dfrac{215 - 200}{235/\sqrt{47}} = .44$ with a corresponding P-value of $P(Z \geq .44) = 1 - \Phi(.44) = .33$. We fail to reject H_0, because .33 > .10. The data do not provide convincing evidence that daily consumption of all adult women exceeds 200 mg.

64.

a. No. The accompanying normal probability plot shows a substantial departure from linearity, so it would be unrealistic to assume the population of turnover ratios has a normal distribution. Moreover, since the sample size is small ($n = 20$), we cannot invoke a central limit theorem argument in order to use z or t procedures.



b. Let μ denote the mean (and median) of ln(X), so e^{μ} is the median of X. The median turnover is 100% = 1, i.e. $e^{\mu} = 1$, iff $\mu = \ln(1) = 0$. So, we'll use the 20 ln(x) data values to test the hypotheses H_0: $\mu = 0$ versus H_a: $\mu < 0$.
Let $y = \ln(x)$. From software, $\bar{y} = -0.026$ and $s_y = 0.652$, resulting in a test statistic of

$t = \dfrac{\bar{y} - \mu_0}{s_y/\sqrt{n}} = \dfrac{-0.026 - 0}{0.652/\sqrt{20}}$ = –0.18. At df = 20 – 1 = 19, this gives a P-value of .43, so we'd fail to reject H_0 at any reasonable significance level. There is no statistically significant evidence to suggest the true median turnover for this type of fund is less than 100%.

65.

a. From Table A.17, when $\mu = 9.5$, $d = .625$, and df = 9, $\beta \approx .60$.
When $\mu = 9.0$, $d = 1.25$, and df = 9, $\beta \approx .20$.

b. From Table A.17, when $\beta = .25$ and $d = .625$, $n \approx 28$.

66. A normality plot reveals that these observations could have come from a normally distributed population, therefore a *t*-test is appropriate. The relevant hypotheses are H_0: $\mu = 9.75$ v. H_a: $\mu > 9.75$. Summary statistics are $n = 20$, $\bar{x} = 9.8525$, and $s = .0965$, which leads to a test statistic $t = \dfrac{9.8525 - 9.75}{.0965/\sqrt{20}} = 4.75$, from which the *P*-value ≈ 0. With such a small *P*-value, the data strongly supports the alternative hypothesis. The condition is not met.

67.

a. With H_0: $p = 1/75$ v. H_a: $p \neq 1/75$, $\hat{p} = \dfrac{16}{800} = .02$, $z\dfrac{.02 - .01333}{\sqrt{\dfrac{.01333(.98667)}{800}}} = 1.645$, and *P*-value = .10, we fail to reject the null hypothesis at the $\alpha = .05$ level. There is no significant evidence that the incidence rate among prisoners differs from that of the adult population.

The possible error we could have made is a type II.

b. P–value $= 2[1 - \Phi(1.645)] = 2[.05] = .10$. Yes, since .10 < .20, we could reject H_0.

68. A *t* test is appropriate. H_0: $\mu = 1.75$ is rejected in favor of H_a: $\mu \neq 1.75$ if the *P*-value < .05. The computed test statistic is $t = \dfrac{1.89 - 1.75}{.42/\sqrt{26}} = 1.70$. Since the *P*-value is $2P(T > 1.7) = 2(.051) = .102 > .05$, do not reject H_0; the data does not contradict prior research.
We assume that the population from which the sample was taken was approximately normally distributed.

69. Even though the underlying distribution may not be normal, a *z* test can be used because *n* is large. The null hypothesis H_0: $\mu = 3200$ should be rejected in favor of H_a: $\mu < 3200$ if the *P*-value is less than .001. The computed test statistic is $z = \dfrac{3107 - 3200}{188/\sqrt{45}} = -3.32$ and the *P*-value is $\Phi(-3.32) = .0005 < .001$, so H_0 should be rejected at level .001.

70. Let *p* = the true proportion of all American adults that want to see the BCS replaced by a playoff system.
The hypotheses of interest are H_0: $p = .5$ versus H_a: $p > .5$.
With $n = 948$, $np_0 = n(1 - p_0) = 948(.5) = 474 \geq 10$, so the "large-sample" *z* procedure is applicable.
From the data provided, $\hat{p} = \dfrac{597}{948} = .6297$, so $z = \dfrac{.6297 - .5}{\sqrt{.5(1-.5)/948}} = 7.99$. The corresponding upper-tailed *P*-value is $P(Z \geq 7.99) = 1 - \Phi(7.99) \approx 1 - 1 = 0$. That is, assuming exactly 50% of the population wants to replace BCS by a playoff system, there is almost no chance of getting a sample proportion as large as ~63% in a sample of 948 people.
Therefore, we strongly reject H_0. There is compelling evidence to suggest that a majority of American adults want to switch the BCS for a playoff system.

71. We wish to test H_0: $\mu = 4$ versus H_a: $\mu > 4$ using the test statistic $z = \dfrac{\bar{x} - 4}{\sqrt{4/n}}$. For the given sample, $n = 36$ and $\bar{x} = \dfrac{160}{36} = 4.444$, so $z = \dfrac{4.444 - 4}{\sqrt{4/36}} = 1.33$.
The *P*-value is $P(Z \geq 1.33) = 1 - \Phi(1.33) = .0918$. Since $.0918 > .02$, H_0 should not be rejected at this level. We do not have significant evidence at the .02 level to conclude that the true mean of this Poisson process is greater than 4.

72. *Note*: It is not reasonable to use a *z* test here, since the values of *p* are so small.

a. Let p = the proportion of all mis-priced purchases at all California Wal-Mart stores. We wish to test the hypotheses H_0: $p = .02$ v. H_a: $p > .02$.
Let X = the number of mis-priced items in 200, so $X \sim \text{Bin}(200,.02)$ under the null hypothesis. The test procedure should reject H_0 iff X is too large, i.e., $X \geq x$ for some "critical value" x.

The value x should satisfy $P(X \geq x) = .05$ as closely as possible; $P(X \geq 8) = .0493$, so we will use $x = 8$ and reject H_0 iff $X \geq 8$. For our data, the observed value of X is $.083(200) = 16.6 \approx 17$, so we clearly reject H_0 here and conclude that the NIST benchmark is not satisfied.

b. If $p = 5\%$ in fact, so that $X \sim \text{Bin}(200,.05)$, then $P(\text{Reject } H_0) = P(X \geq 8) = 1 - P(X \leq 7) = 1 - B(7;200,.05) = .7867$. We have decent power to detect $p = 5\%$.

73. The parameter of interest is p = the proportion of all college students who have maintained lifetime abstinence from alcohol. The hypotheses are H_0: $p = .1$, H_a: $p > .1$.
With $n = 462$, $np_0 = 462(.1) = 46.2 \geq 10$ $n(1 - p_0) = 462(.9) = 415.8 \geq 10$, so the "large-sample" *z* procedure is applicable.
From the data provided, $\hat{p} = \dfrac{51}{462} = .1104$, so $z = \dfrac{.1104 - .1}{\sqrt{.1(.9)/462}} = 0.74$.
The corresponding one-tailed *P*-value is $P(Z \geq 0.74) = 1 - \Phi(0.74) = .2296$.
Since $.2296 > .05$, we fail to reject H_0 at the $\alpha = .05$ level (and, in fact, at any reasonable significance level). The data does not give evidence to suggest that more than 10% of all college students have completely abstained from alcohol use.

74. By guessing alone, the taster has a 1/3 chance of selecting the "different" wine. Hence, we wish to test H_0: $p = 1/3$ v. H_a: $p > 1/3$. With $\hat{p} = \dfrac{346}{855} = .4047$, our test statistic is $z = \dfrac{.4047 - .3333}{\sqrt{.3333(.6667)/855}} = 4.43$, and the corresponding *P*-value is $P(Z \geq 4.43) \approx 0$. Hence, we strongly reject the null hypothesis at any reasonable significance level and conclude that the population of wine tasters have the ability to distinguish the "different" wine out of three more than 1/3 of the time.

75. Since *n* is large, we'll use the one-sample *z* procedure. With μ = population mean Vitamin D level for infants, the hypotheses are H_0: $\mu = 20$ v. H_a: $\mu > 20$. The test statistic is $z = \dfrac{21 - 20}{11/\sqrt{102}} = 0.92$, and the upper-tailed *P*-value is $P(Z \geq 0.92) = .1788$. Since $.1788 > .10$, we fail to reject H_0. It cannot be concluded that $\mu > 20$.

76. The hypotheses of interest are H_0: $\sigma = .50$ versus H_a: $\sigma > .50$. The test statistic value is $(10 - 1)(.58)^2/(.5)^2 = 12.11$. The P-value is the area under the χ^2 curve with 9 df to the right of 12.11; with the aid of software, that's roughly .2. Because .2 > .01, H_0 cannot be rejected. The uniformity specification is not contradicted.

77. The 20 df row of Table A.7 shows that $\chi^2_{.99,20} = 8.26 < 8.58$ (H_0 not rejected at level .01) and $8.58 < 9.591 = \chi^2_{.975,20}$ (H_0 rejected at level .025). Thus .01 < P-value < .025, and H_0 cannot be rejected at level .01 (the P-value is the smallest α at which rejection can take place, and this exceeds .01).

78.

a. $E\left(\bar{X} + 2.33S\right) = E(\bar{X}) + 2.33E(S) \approx \mu + 2.33\sigma$, so $\hat{\theta} = \bar{X} + 2.33S$ is approximately unbiased.

b. $V\left(\bar{X} + 2.33S\right) = V(\bar{X}) + 2.33^2V(S) \approx \dfrac{\sigma^2}{n} + 5.4289\dfrac{\sigma^2}{2n}$. The estimated standard error (standard deviation) is $1.927\dfrac{s}{\sqrt{n}}$.

c. More than 99% of all soil samples have pH less than 6.75 iff the 99th percentile is less than 6.75. Thus we wish to test H_0: $\mu + 2.33\sigma = 6.75$ versus H_a: $\mu + 2.33\sigma < 6.75$.

Since $z = \dfrac{(\bar{x} + 2.33s) - 6.75}{1.927s/\sqrt{n}} = \dfrac{-.047}{.0385} = -1.22$ and P-value = $P(Z \le -1.22) = .1112 > .01$, H_0 is not rejected at the .01 level. The 99th percentile does not significantly exceed 6.75.

79.

a. When H_0 is true, $2\lambda_0\Sigma X_i = \dfrac{2}{\mu_0}\sum X_i$ has a chi-squared distribution with df = $2n$. If the alternative is H_a: $\mu < \mu_0$, then we should reject H_0 in favor of H_a when the sample mean $\bar{x}$ is small. Since $\bar{x}$ is small exactly when Σx_i is small, we'll reject H_0 when the test statistic is small. In particular, the P-value should be the area to the <u>left</u> of the observed value $\dfrac{2}{\mu_0}\sum x_i$.

b. The hypotheses are H_0: $\mu = 75$ versus H_a: $\mu < 75$. The test statistic value is $\dfrac{2}{\mu_0}\sum x_i = \dfrac{2}{75}(737) = 19.65$. At df = 2(10) = 20, the P-value is the area to the left of 19.65 under the χ^2_{20} curve. From software, this is about .52, so H_0 clearly should not be rejected (the P-value is very large). The sample data do not suggest that true average lifetime is less than the previously claimed value.

80.

a. For testing H_0: $p = .2$ v. H_a: $p > .2$, an upper-tailed test is appropriate. The computed Z is $z = .97$, so the P-value = $1 - \Phi(.97) = .166$. Because the P-value is rather large, H_0 would not be rejected at any reasonable α (it can't be rejected for any $\alpha < .166$), so no modification appears necessary.

b. With $p = .5$, $1 - \beta(.5) = 1 - \Phi\left[\left(-.3 + 2.33(.0516)\right)/.0645\right] = 1 - \Phi(-2.79) = .9974$.

CHAPTER 9

Section 9.1

1.

a. $E\left(\overline{X}-\overline{Y}\right)=E\left(\overline{X}\right)-E\left(\overline{Y}\right)=4.1-4.5=-.4$, irrespective of sample sizes.

b. $V\left(\overline{X}-\overline{Y}\right)=V\left(\overline{X}\right)+V\left(\overline{Y}\right)=\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}=\frac{\left(1.8\right)^2}{100}+\frac{\left(2.0\right)^2}{100}=.0724$, and the SD of $\overline{X}-\overline{Y}$ is $\overline{X}-\overline{Y}=\sqrt{.0724}=.2691$.

c. A normal curve with mean and sd as given in **a** and **b** (because $m = n = 100$, the CLT implies that both $\overline{X}$ and $\overline{Y}$ have approximately normal distributions, so $\overline{X}-\overline{Y}$ does also). The shape is not necessarily that of a normal curve when $m = n = 10$, because the CLT cannot be invoked. So if the two lifetime population distributions are not normal, the distribution of $\overline{X}-\overline{Y}$ will typically be quite complicated.

2.

a. With large sample sizes, a 95% confidence interval for the difference of population means, $\mu_1 - \mu_2$, is $(\overline{x}-\overline{y})\pm z_{\alpha/2}\sqrt{\frac{s_1^2}{m}+\frac{s_2^2}{n}}=(\overline{x}-\overline{y})\pm 1.96\sqrt{[SE(\overline{x})]^2+[SE(\overline{y})]^2}$. Using the values provided, we get $(64.9-63.1)\pm 1.96\sqrt{(.09)^2+(.11)^2}$ = 1.8 ± .28 = (1.52, 2.08). Therefore, we are 95% confident that the difference in the true mean heights for younger and older women (as defined in the exercise) is between 1.52 inches and 2.08 inches.

b. The null hypothesis states that the true mean height for younger women is 1 inch higher than for older women, i.e. $\mu_1 = \mu_2 + 1$. The alternative hypothesis states that the true mean height for younger women is more than 1 inch higher than for older women.
The test statistic, z, is given by

$$z=\frac{(\overline{x}-\overline{y})-\Delta_0}{SE(\overline{x}-\overline{y})}=\frac{1.8-1}{\sqrt{(.09)^2+(.11)^2}}=5.63$$

The P-value is $P(Z \geq 5.63) = 1 - \Phi(5.63) \approx 1 - 1 = 0$. Hence, we reject H_0 and conclude that the true mean height for younger women is indeed more than 1 inch higher than for older women.

c. From the calculation above, P-value = $P(Z \geq 5.63) = 1 - \Phi(5.63) \approx 1 - 1 = 0$. Therefore, yes, we would reject H_0 at any reasonable significance level (since the P-value is lower than any reasonable value for α).

d. The actual hypotheses of (b) have not been changed, but the subscripts have been reversed. So, the relevant hypotheses are now H_0: $\mu_2 - \mu_1 = 1$ versus H_a: $\mu_2 - \mu_1 > 1$. Or, equivalently, H_0: $\mu_1 - \mu_2 = -1$ versus H_a: $\mu_1 - \mu_2 < -1$.

3. Let μ_1 = the population mean pain level under the control condition and μ_2 = the population mean pain level under the treatment condition.

a. The hypotheses of interest are H_0: $\mu_1 - \mu_2 = 0$ versus H_a: $\mu_1 - \mu_2 > 0$. With the data provided, the test statistic value is $z = \frac{(5.2-3.1)-0}{\sqrt{\frac{2.3^2}{43}+\frac{2.3^2}{43}}}$ = 4.23. The corresponding P-value is $P(Z \geq 4.23) = 1 - \Phi(4.23) \approx 0$.

Hence, we reject H_0 at the $\alpha = .01$ level (in fact, at any reasonable level) and conclude that the average pain experienced under treatment is less than the average pain experienced under control.

b. Now the hypotheses are H_0: $\mu_1 - \mu_2 = 1$ versus H_a: $\mu_1 - \mu_2 > 1$. The test statistic value is $z = \frac{(5.2-3.1)-1}{\sqrt{\frac{2.3^2}{43}+\frac{2.3^2}{43}}}$ = 2.22, and the P-value is $P(Z \geq 2.22) = 1 - \Phi(2.22) = .0132$. Thus we would reject H_0 at the $\alpha = .05$ level and conclude that mean pain under control condition exceeds that of treatment condition by more than 1 point. However, we would not reach the same decision at the $\alpha = .01$ level (because $.0132 \leq .05$ but $.0132 > .01$).

4.

a. A 95% CI for the population mean PEF for children in biomass households is $3.30 \pm 1.96\frac{1.20}{\sqrt{755}}$ = (3.21, 3.39). Similarly, a 95% CI for the population mean PEF for children in LPG households is $4.25 \pm 1.96\frac{1.75}{\sqrt{750}}$ = (4.12, 4.38).

Assuming the two samples are independent, the simultaneous confidence level of the two 95% intervals is (.95)(.95) = 90.25%.

b. Let μ_1 and μ_2 denote the two relevant population means (1 = biomass, 2 = LPG). The hypotheses of interest are H_0: $\mu_1 - \mu_2 = 0$ versus H_a: $\mu_1 - \mu_2 < 0$. The test statistic value is $z = \frac{(3.30-4.25)-0}{\sqrt{\frac{1.20^2}{755}+\frac{1.75^2}{750}}}$ = $\frac{-0.95}{.0774}$ = –12.2. The P-value is $P(Z \leq -12.2) \approx 0$. Hence, we strongly reject H_0 and conclude that the population mean PEF is definitely lower for children in biomass households than LPG households.

c. No. The two CIs in **a** were from two independent random samples, and so the confidence levels could be multiplied: $P(A \cap B) = P(A)P(B)$. However, two variables (PEF and FEV_1) collected on the same group of 755 children do not constitute independent random samples. Hence, we cannot say that the two resulting CIs for that population have simultaneous confidence of 90.25%. (Using Bonferroni's inequality, we can say the simultaneous confidence level is between 90% and 95%.)

5.

a. H_a says that the average calorie output for sufferers is more than 1 cal/cm^2/min below that for non-sufferers. $\sqrt{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}} = \sqrt{\frac{(.2)^2}{10}+\frac{(.4)^2}{10}} = .1414$, so $z = \frac{(.64-2.05)-(-1)}{.1414} = -2.90$. The P-value for this one-sided test is $P(Z \leq -2.90) = .0019 < .01$. So, at level .01, H_0 is rejected.

b. $z_\alpha = z_{.01} = 2.33$, and so $\beta(-1.2) = 1 - \Phi\left(-2.33 - \frac{-1.2+1}{.1414}\right) = 1 - \Phi(-.92) = .8212$.

c. $m = n = \frac{.2(2.33+1.28)^2}{(-.2)^2} = 65.15$, so use 66.

6.

a. Since $z = \frac{(18.12 - 16.87)}{\sqrt{\frac{2.56}{40} + \frac{1.96}{32}}} = 3.53$ and P-value = $P(Z \le -3.53) \approx .0001 < .01$, H_0 should be rejected at level .01.

b. $\beta(1) = \Phi\left(2.33 - \frac{1-0}{.3539}\right) = \Phi(-.50) = .3085$.

c. $\frac{2.56}{40} + \frac{1.96}{n} = \frac{1}{(1.645+1.28)^2} = .1169 \Rightarrow \frac{1.96}{n} = .0529 \Rightarrow n = 37.06$, so use $n = 38$.

d. Since $n = 32$ is not a large sample, it would no longer be appropriate to use the large sample z test of Section 9.1. A small sample t procedure should be used (Section 9.2), and the appropriate conclusion would follow. Note, however, that the test statistic of 3.53 would not change, and thus it shouldn't come as a surprise that we would still reject H_0 at the .01 significance level.

7. Let μ_1 denote the true mean course GPA for all courses taught by full-time faculty, and let μ_2 denote the true mean course GPA for all courses taught by part-time faculty. The hypotheses of interest are H_0: $\mu_1 = \mu_2$ versus H_a: $\mu_1 \ne \mu_2$; or, equivalently, H_0: $\mu_1 - \mu_2 = 0$ v. H_a: $\mu_1 - \mu_2 \ne 0$.

The large-sample test statistic is $z = \frac{(\bar{x} - \bar{y}) - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} = \frac{(2.7186 - 2.8639) - 0}{\sqrt{\frac{(.63342)^2}{125} + \frac{(.49241)^2}{88}}} = -1.88$. The corresponding two-tailed P-value is $P(|Z| \ge |-1.88|) = 2[1 - \Phi(1.88)] = .0602$.
Since the P-value exceeds $\alpha = .01$, we fail to reject H_0. At the .01 significance level, there is insufficient evidence to conclude that the true mean course GPAs differ for these two populations of faculty.

8.

a. The parameter of interest is $\mu_1 - \mu_2$ = the true difference of mean tensile strength of the 1064-grade and the 1078-grade wire rod. The hypotheses are H_0: $\mu_1 - \mu_2 = -10$ versus H_a: $\mu_1 - \mu_2 < -10$. The calculated test statistic and P-value are $z = \frac{(107.6 - 123.6) - (-10)}{\sqrt{\frac{1.3^2}{129} + \frac{2.0^2}{129}}} = \frac{-6}{.210} = -28.57$ and P-value = $\Phi(-28.57) \approx 0$.
This is less than any reasonable α, so reject H_0. There is very compelling evidence that the mean tensile strength of the 1078 grade exceeds that of the 1064 grade by more than 10.

b. The requested information can be provided by a 95% confidence interval for $\mu_1 - \mu_2$:
$$(\bar{x} - \bar{y}) \pm 1.96\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}} = (-16) \pm 1.96(.210) = (-16.412, -15.588)\ .$$

9.

a. Point estimate $\bar{x} - \bar{y} = 19.9 - 13.7 = 6.2$. It appears that there could be a difference.

b. H_0: $\mu_1 - \mu_2 = 0$, H_a: $\mu_1 - \mu_2 \neq 0$, $z = \frac{(19.9 - 13.7)}{\sqrt{\frac{39.1^2}{60} + \frac{15.8^2}{60}}} = \frac{6.2}{5.44} = 1.14$, and the P-value = $2[P(Z > 1.14)]$ = 2(.1271) = .2542. The P-value is larger than any reasonable α, so we do not reject H_0. There is no statistically significant difference.

c. No. With a normal distribution, we would expect most of the data to be within 2 standard deviations of the mean, and the distribution should be symmetric. Two sd's above the mean is 98.1, but the distribution stops at zero on the left. The distribution is positively skewed.

d. We will calculate a 95% confidence interval for μ, the true average length of stays for patients given the treatment. $19.9 \pm 1.96 \frac{39.1}{\sqrt{60}} = 19.9 \pm 9.9 = (10.0, 21.8)$.

10.

a. The hypotheses are H_0: $\mu_1 - \mu_2 = 5$ and H_a: $\mu_1 - \mu_2 > 5$. Since $z = \frac{(65.6 - 59.8) - 5}{.2272} = 2.89$, the P-value is $P(Z \geq 2.89) = 1 - \Phi(.9981) = .0019$. At the $\alpha = .001$ level, H_0 cannot be rejected in favor of H_a at this level, so the use of the high purity steel cannot be justified.

b. $z_{.001} = 3.08$ and $\mu_1 - \mu_2 - \Delta_0 = 1$, so $\beta = \Phi\left(3.08 - \frac{1}{.2272}\right) = \Phi(-.53) = .2891$.

11. $(\bar{x} - \bar{y}) \pm z_{\alpha/2}\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}} = (\bar{x} - \bar{y}) \pm z_{\alpha/2}\sqrt{(SE_1)^2 + (SE_2)^2}$. Using $\alpha = .05$ and $z_{\alpha/2} = 1.96$ yields $(5.5 - 3.8) \pm 1.96\sqrt{(0.3)^2 + (0.2)^2} = (0.99, 2.41)$. We are 95% confident that the true average blood lead level for male workers is between 0.99 and 2.41 higher than the corresponding average for female workers.

12. The CI is $(\bar{x} - \bar{y}) \pm 2.58\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}} = (5.10 - 5.55) \pm 2.58\sqrt{\frac{1.07^2}{88} + \frac{1.10^2}{93}} = -.45 \pm .416 = (-.866, -.034)$.

With 99% confidence, the mean total cholesterol level for vegans is between .034 and .866 mmol/l lower than for omnivores.

13. $\sigma_1 = \sigma_2 = .05$, $d = .04$, $\alpha = .01$, $\beta = .05$, and the test is one-tailed ⇒

$$n = \frac{(.0025 + .0025)(2.33 + 1.645)^2}{.0016} = 49.38$$, so use $n = 50$.

14. The appropriate hypotheses are H_0: $\theta = 0$ v. H_a: $\theta < 0$, where $\theta = 2\mu_1 - \mu_2$. ($\theta < 0$ is equivalent to $2\mu_1 < \mu_2$, so normal is more than twice schizophrenic) The estimator of θ is $\hat{\theta} = 2\bar{X} - \bar{Y}$, with $V(\hat{\theta}) = 4V(\bar{X}) + V(\bar{Y}) = \frac{4\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$, $\sigma_{\hat{\theta}} = \sqrt{V(\hat{\theta})}$, and $\hat{\sigma}_{\hat{\theta}}$ is obtained by replacing each σ_i^2 with s_i^2. The test statistic is then $\frac{\hat{\theta} - 0}{\hat{\sigma}_{\hat{\theta}}}$ (since $\theta_0 = 0$), and H_0 is rejected if $z \le -2.33$. With $\hat{\theta} = 2(2.69) - 6.35 = -.97$ and $\hat{\sigma}_{\hat{\theta}} = \sqrt{\frac{4(2.3)^2}{43} + \frac{(4.03)^2}{45}} = .9236$, $z = \frac{-.97 - 0}{.9236} = -1.05$; Because –1.05 > –2.33, H_0 is not rejected.

15.

a. As either m or n increases, SD decreases, so $\frac{\mu_1 - \mu_2 - \Delta_0}{SD}$ increases (the numerator is positive), so $\left(z_\alpha - \frac{\mu_1 - \mu_2 - \Delta_0}{SD}\right)$ decreases, so $\beta = \Phi\left(z_\alpha - \frac{\mu_1 - \mu_2 - \Delta_0}{SD}\right)$ decreases.

b. As β decreases, z_β increases, and since z_β is the numerator of n, n increases also.

16. $z = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_1^2}{n} + \frac{s_2^2}{n}}} = \frac{.2}{\sqrt{\frac{2}{n}}}$. For $n = 100$, $z = 1.41$ and P-value = $2\left[1 - \Phi(1.41)\right] = .1586$.

For $n = 400$, $z = 2.83$ and P-value = .0046. From a practical point of view, the closeness of $\bar{x}$ and $\bar{y}$ suggests that there is essentially no difference between true average fracture toughness for type 1 and type 2 steels. The very small difference in sample averages has been magnified by the large sample sizes — statistical rather than practical significance. The P-value by itself would not have conveyed this message.

Section 9.2

17.

a. $$\nu = \frac{\left(5^2/10 + 6^2/10\right)^2}{\left(5^2/10\right)^2/9 + \left(6^2/10\right)^2/9} = \frac{37.21}{.694 + 1.44} = 17.43 \approx 17.$$

b. $$\nu = \frac{\left(5^2/10 + 6^2/15\right)^2}{\left(5^2/10\right)^2/9 + \left(6^2/15\right)^2/14} = \frac{24.01}{.694 + .411} = 21.7 \approx 21.$$

c. $$\nu = \frac{\left(2^2/10 + 6^2/15\right)^2}{\left(2^2/10\right)^2/9 + \left(6^2/15\right)^2/14} = \frac{7.84}{.018 + .411} = 18.27 \approx 18.$$

d. $$\nu = \frac{\left(5^2/12 + 6^2/24\right)^2}{\left(5^2/12\right)^2/11 + \left(6^2/24\right)^2/23} = \frac{12.84}{.395 + .098} = 26.05 \approx 26.$$

18.

a. Let μ_1 and μ_2 denote true mean CO_2 loss with a traditional pour and a slanted pour, respectively. The hypotheses of interest are H_0: $\mu_1 - \mu_2 = 0$ v. H_a: $\mu_1 - \mu_2 \neq 0$. We'll apply the two-sample *t* procedure, with $\nu = \dfrac{(.5^2/4+.3^2/4)^2}{(.5^2/4)^2/(4-1)+(.3^2/4)^2/(4-1)}$ = 4.91 → 4. The test statistic is $t = \dfrac{(4.0-3.7)-0}{\sqrt{\frac{.5^2}{4}+\frac{.3^2}{4}}}$ = 1.03, with a two-sided *P*-value of roughly 2(.187) = .374 from Table A.8. [Software provides the more accurate *P*-value of .362.] Hence, we fail to reject H_0 at any reasonable significance level; we conclude that there is no statistically significant difference in mean "bubble" loss between traditional and slanted champagne pouring, when the temperature is 18°C.

b. Repeating the process of **a** at 12°C, we have $\nu \approx 5$, $t = 7.21$, *P*-value ≈ 2(0) = 0. [Software gives P = .001]. Hence, we reject H_0 at any reasonable significance level; we conclude that there is a statistically significant difference in mean "bubble" loss between traditional and slanted champagne pouring, when the temperature is 12°C.

19. For the given hypotheses, the test statistic is $t = \dfrac{115.7-129.3+10}{\sqrt{\frac{5.03^2}{6}+\frac{5.38^2}{6}}} = \dfrac{-3.6}{3.007} = -1.20$, and the df is

$\nu = \dfrac{\left(4.2168+4.8241\right)^2}{\dfrac{\left(4.2168\right)^2}{5}+\dfrac{\left(4.8241\right)^2}{5}} = 9.96$, so use df = 9. The *P*-value is $P(T \leq -1.20$ when $T \sim t_9) \approx .130$.

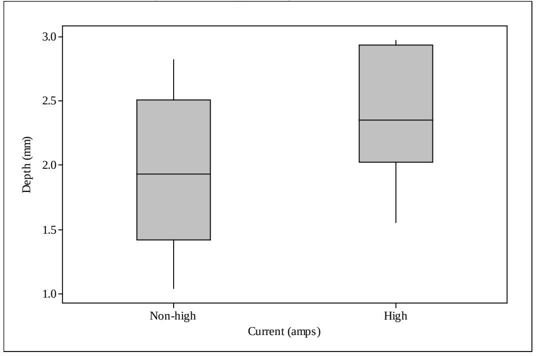
Since .130 > .01, we don't reject H_0.

20. We want a 95% confidence interval for $\mu_1 - \mu_2$. $t_{.025,9} = 2.262$, so the interval is $-13.6 \pm 2.262\left(3.007\right) = \left(-20.40,-6.80\right)$. Because the interval is so wide, it does not appear that precise information is available.

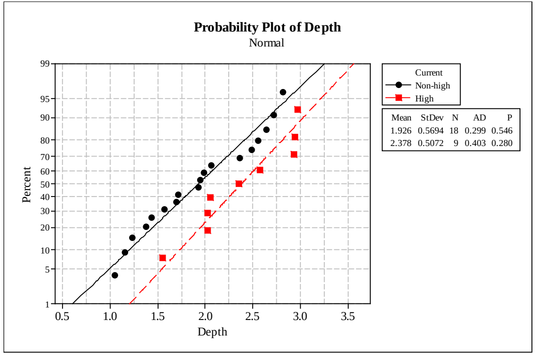
21. Let μ_1 = the true average gap detection threshold for normal subjects, and μ_2 = the corresponding value for CTS subjects. The relevant hypotheses are H_0: $\mu_1 - \mu_2 = 0$ v. H_a: $\mu_1 - \mu_2 < 0$, and the test statistic is $t = \dfrac{1.71-2.53}{\sqrt{.0351125+.07569}} = \dfrac{-.82}{.3329} = -2.46$. Using df $\nu = \dfrac{\left(.0351125+.07569\right)^2}{\dfrac{\left(.0351125\right)^2}{7}+\dfrac{\left(.07569\right)^2}{9}} = 15.1$, or 15, the *P*-value is $P(T \leq -2.46$ when $T \sim t_{15}) \approx .013$. Since .013 > .01, we fail to reject H_0 at the $\alpha = .01$ level. We have insufficient evidence to claim that the true average gap detection threshold for CTS subjects exceeds that for normal subjects.

22.

a. According to the boxplots, HAZ depth measurements are generally somewhat larger when the current is set at the higher amperage than at the lower amperage.



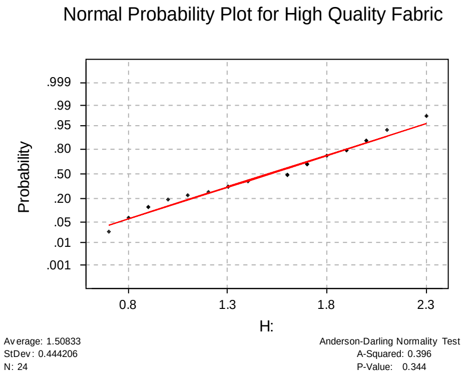
b. Yes, it would be reasonable to apply the two-sample *t* procedure here. The accompanying normal probability plots do not exhibit a strong lack of linearity, meaning that the assumption of normally distributed depth populations is at least plausible. (Of course, with $m = 18$ and $n = 9$, it's hard to detect such deviations.)

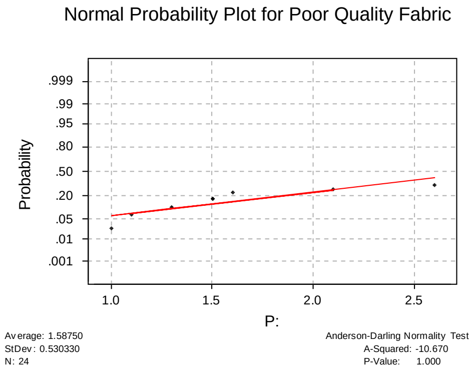


c. Let μ_1 = the true average depth under the high current setting and let μ_2 = the true average depth under the non-high current setting. We test H_0: $\mu_1 - \mu_2 = 0$ versus H_a: $\mu_1 - \mu_2 > 0$. [The order of the two groups is arbitrary; we just need the direction of the alternative to be consistent with our research question.] Using software, the test statistic is $t = \frac{(2.378 - 1.926) - 0}{\sqrt{0.507^2/9 + 0.569^2/18}}$ = 2.09, which we'll compare to a *t* distribution with 17 df (again, using software). The corresponding one-tailed *P*-value is .025 from Table A.8 (or .026 from software). Since .025 > .01, at the .01 significance level we fail to reject H_0. At the 1% level, we do not have sufficient evidence to conclude that the true mean HAZ depth is larger when the current setting is higher. (Notice, though, that even with the small sample sizes we do have enough evidence to reject H_0 at a .05 significance level.)

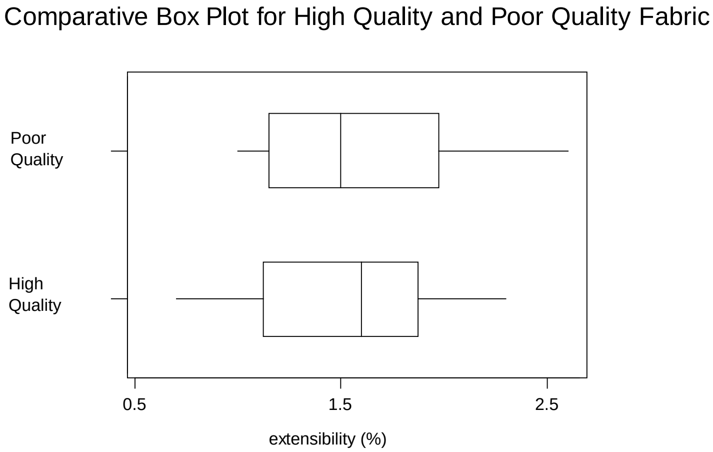
23.

a. Using Minitab to generate normal probability plots, we see that both plots illustrate sufficient linearity. Therefore, it is plausible that both samples have been selected from normal population distributions.





b. The comparative boxplot does not suggest a difference between average extensibility for the two types of fabrics.



c. We test $H_0 : \mu_1 - \mu_2 = 0$ v. $H_a : \mu_1 - \mu_2 \neq 0$. With degrees of freedom $\nu = \frac{(.0433265)^2}{.00017906} = 10.5$ (which we round down to 10) and test statistic is $t = \frac{-.08}{\sqrt{(.0433265)}} = -.38 \approx -0.4$, the *P*-value is 2(.349) = .698. Since the *P*-value is very large, we do not reject H_0. There is insufficient evidence to claim that the true average extensibility differs for the two types of fabrics.

24.

a. 95% upper confidence bound: $\bar{x} + t_{.05,65-1}SE = 13.4 + 1.671(2.05) = 16.83$ seconds

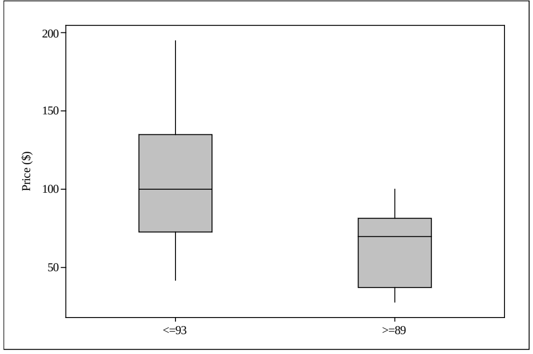
b. Let μ_1 and μ_2 represent the true average time spent by blackbirds at the experimental and natural locations, respectively. We wish to test H_0: $\mu_1 - \mu_2 = 0$ v. H_a: $\mu_1 - \mu_2 > 0$. The relevant test statistic is $t = \dfrac{13.4 - 9.7}{\sqrt{2.05^2 + 1.76^2}}$ = 1.37, with estimated df = $\dfrac{(2.05^2 + 1.76^2)^2}{\dfrac{2.05^4}{64} + \dfrac{1.76^4}{49}}$ ≈ 112.9. Rounding to $t = 1.4$ and df = 120, the tabulated P-value is very roughly .082. Hence, at the 5% significance level, we fail to reject the null hypothesis. The true average time spent by blackbirds at the experimental location is not statistically significantly higher than at the natural location.

c. 95% CI for silvereyes' average time – blackbirds' average time at the natural location: (38.4 – 9.7) ± $(2.00)\sqrt{1.76^2 + 5.06^2}$ = (17.96 sec, 39.44 sec). The t-value 2.00 is based on estimated df = 55.

25.

a. Normal probability plots of both samples (not shown) exhibit substantial linear patterns, suggesting that the normality assumption is reasonable for both populations of prices.

b. The comparative boxplots below suggest that the average price for a wine earning a ≥ 93 rating is much higher than the average price earning a ≤ 89 rating.



c. From the data provided, $\bar{x} = 110.8$, $\bar{y} = 61.7$, $s_1 = 48.7$, $s_2 = 23.8$, and $\nu \approx 15$. The resulting 95% CI for the difference of population means is $(110.8 - 61.7) \pm t_{.025,15}\sqrt{\dfrac{48.7^2}{12} + \dfrac{23.8^2}{14}}$ = (16.1, 82.0). That is, we are 95% confident that wines rated ≥ 93 cost, on average, between \$16.10 and \$82.00 more than wines rated ≤ 89. Since the CI does not include 0, this certainly contradicts the claim that price and quality are unrelated.

26. Let μ_1 = the true average potential drop for alloy connections and let μ_2 = the true average potential drop for EC connections. Since we are interested in whether the potential drop is higher for alloy connections, an upper tailed test is appropriate. We test H_0: $\mu_1 - \mu_2 = 0$ v. H_a: $\mu_1 - \mu_2 > 0$. Using the SAS output provided, the test statistic, when assuming unequal variances, is $t = 3.6362$, the corresponding df is 37.5, and the P-value for our upper tailed test would be ½(two-tailed P-value) = ½(.0008) = .0004. Our P-value of .0004 is less than the significance level of .01, so we reject H_0. We have sufficient evidence to claim that the true average potential drop for alloy connections is higher than that for EC connections.

27.

a. Let's construct a 99% CI for μ_{AN}, the true mean intermuscular adipose tissue (IAT) under the described AN protocol. Assuming the data comes from a normal population, the CI is given by $\bar{x} \pm t_{\alpha/2,n-1}\dfrac{s}{\sqrt{n}} = .52 \pm t_{.005,15}\dfrac{.26}{\sqrt{16}} = .52 \pm 2.947\dfrac{.26}{\sqrt{16}}$ = (.33, .71). We are 99% confident that the true mean IAT under the AN protocol is between .33 kg and .71 kg.

b. Let's construct a 99% CI for $\mu_{\text{AN}} - \mu_{\text{C}}$, the difference between true mean AN IAT and true mean control IAT. Assuming the data come from normal populations, the CI is given by $(\bar{x}-\bar{y}) \pm t_{\alpha/2,\nu}\sqrt{\dfrac{s_1^2}{m}+\dfrac{s_2^2}{n}} = (.52-.35) \pm t_{.005,21}\sqrt{\dfrac{(.26)^2}{16}+\dfrac{(.15)^2}{8}} = .17 \pm 2.831\sqrt{\dfrac{(.26)^2}{16}+\dfrac{(.15)^2}{8}}$ = (–.07, .41). Since this CI includes zero, it's plausible that the difference between the two true means is zero (i.e., $\mu_{\text{AN}} - \mu_{\text{C}} = 0$). [Note: the df calculation $\nu = 21$ comes from applying the formula in the textbook.]

28. We will test the hypotheses: H_0: $\mu_1 - \mu_2 = 10$ v. H_a: $\mu_1 - \mu_2 > 10$. The test statistic is

$$t = \frac{(\bar{x}-\bar{y})-10}{\sqrt{\left(\frac{2.75^2}{10}+\frac{4.44^2}{5}\right)}} = \frac{4.5}{2.17} = 2.08 \text{ with df = } \nu = \frac{\left(\frac{2.75^2}{10}+\frac{4.44^2}{5}\right)^2}{\frac{\left(\frac{2.75^2}{10}\right)^2}{9}+\frac{\left(\frac{4.44^2}{5}\right)^2}{4}} = \frac{22.08}{3.95} = 5.59 \searrow 5$$

, and the P-value from Table A.8 is ≈ .045, which is < .10 so we reject H_0 and conclude that the true average lean angle for older females is more than 10 degrees smaller than that of younger females.

29. Let μ_1 = the true average compression strength for strawberry drink and let μ_2 = the true average compression strength for cola. A lower tailed test is appropriate. We test H_0: $\mu_1 - \mu_2 = 0$ v. H_a: $\mu_1 - \mu_2 < 0$. The test statistic is $t = \dfrac{-14}{\sqrt{29.4+15}} = -2.10$; $\nu = \dfrac{(44.4)^2}{\frac{(29.4)^2}{14}+\frac{(15)^2}{14}} = \dfrac{1971.36}{77.8114} = 25.3$, so use df=25.

The P-value ≈ $P(t < -2.10) = .023$. This P-value indicates strong support for the alternative hypothesis. The data does suggest that the extra carbonation of cola results in a higher average compression strength.

30.

a. We desire a 99% confidence interval. First we calculate the degrees of freedom:

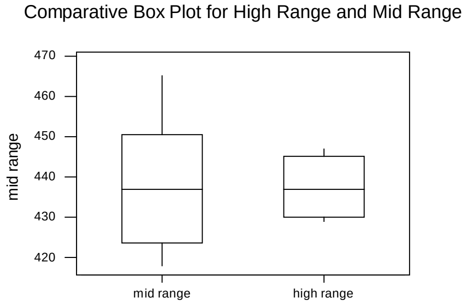
$$\nu = \frac{\left(\frac{2.2^2}{26}+\frac{4.3^2}{26}\right)^2}{\frac{\left(\frac{2.2^2}{26}\right)^2}{26}+\frac{\left(\frac{4.3^2}{26}\right)^2}{26}} = 37.24 \searrow 37$$

, but there is no df = 37 row in Table A.5. Using 36 degrees of freedom (a more conservative choice), $t_{.005,36} = 2.719$, and the 99% CI is $(33.4-42.8) \pm 2.719\sqrt{\frac{2.2^2}{26}+\frac{4.3^2}{26}} = -9.4 \pm 2.576 = (-11.98, -6.83)$. We are 99% confident that the true average load for carbon beams exceeds that for fiberglass beams by between 6.83 and 11.98 kN.

b. The upper limit of the interval in part **a** does not give a 99% upper confidence bound. The 99% upper bound would be $-9.4 + 2.434(.9473) = -7.09$, meaning that the true average load for carbon beams exceeds that for fiberglass beams by at least 7.09 kN.

31.

a. The most notable feature of these boxplots is the larger amount of variation present in the mid-range data compared to the high-range data. Otherwise, both look reasonably symmetric with no outliers present.



b. Using df = 23, a 95% confidence interval for $\mu_{\text{mid-range}} - \mu_{\text{high-range}}$ is

$\left(438.3-437.45\right)\pm 2.069\sqrt{\frac{15.1^2}{17}+\frac{6.83^2}{11}} = .85\pm 8.69 = \left(-7.84, 9.54\right)$. Since plausible values for $\mu_{\text{mid-range}} - \mu_{\text{high-range}}$ are both positive and negative (i.e., the interval spans zero) we would conclude that there is not sufficient evidence to suggest that the average value for mid-range and the average value for high-range differ.

32.

a. Let μ_1 denote the true average stance duration among elderly individuals. Using the summary information provided, a 99% CI for μ_1 is given by

$\bar{x} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} = 801 \pm t_{.005,27}\frac{117}{\sqrt{28}} = 801 \pm 2.771\frac{117}{\sqrt{28}}$ = (739.7, 862.3). We're 99% confident that the true average stance duration among elderly individuals is between 739.7 ms and 862.3 ms.

b. Let μ_2 denote the true average stance duration among younger individuals. We want to test the hypotheses H_0: $\mu_1 - \mu_2 = 0$ versus H_a: $\mu_1 - \mu_2 > 0$. Assuming that both stance duration distributions are normal, we'll use a two-sample *t* test; the test statistic is

$t = \frac{(\bar{x}-\bar{y})-\Delta_0}{\sqrt{\frac{s_1^2}{m}+\frac{s_2^2}{n}}} = \frac{(801-780)-0}{\sqrt{\frac{(117)^2}{28}+\frac{(72)^2}{16}}}$ = 0.74. Degrees of freedom are ν = 41 using the book's formula.

From Table A.8 (ν = 40 column, t = 0.7), the *P*-value is roughly .244. [Software gives .233.] Since .233 > .05, we fail to reject H_0 at the α = .05 level. (In fact, with such a small test statistic value, we would fail to reject H_0 at any reasonable significance level.) At the .05 level, there is not sufficient evidence to conclude that the true average stance duration is larger among elderly individuals than it is among younger individuals.

33. Let μ_1 and μ_2 represent the true mean body mass decrease for the vegan diet and the control diet, respectively. We wish to test the hypotheses H_0: $\mu_1 - \mu_2 \leq 1$ v. H_a: $\mu_1 - \mu_2 > 1$. The relevant test statistic is $t = \dfrac{(5.8-3.8)-1}{\sqrt{\dfrac{3.2^2}{32}+\dfrac{2.8^2}{32}}}$ = 1.33, with estimated df = 60 using the formula. Rounding to t = 1.3, Table A.8 gives a one-sided P-value of .098 (a computer will give the more accurate P-value of .094).
Since our P-value > α = .05, we fail to reject H_0 at the 5% level. We do not have statistically significant evidence that the true average weight loss for the vegan diet exceeds the true average weight loss for the control diet by more than 1 kg.

34.

a. Following the usual format for most confidence intervals: statistic ± (critical value)(standard error), a pooled variance confidence interval for the difference between two means is
$(\bar{x}-\bar{y}) \pm t_{\alpha/2,m+n-2} \cdot s_p\sqrt{\frac{1}{m}+\frac{1}{n}}$.

b. The sample means and standard deviations of the two samples are $\bar{x} = 13.90$, $s_1 = 1.225$, $\bar{y} = 12.20$, $s_2 = 1.010$. The pooled variance estimate is s_p^2 =
$\left(\dfrac{m-1}{m+n-2}\right)s_1^2 + \left(\dfrac{n-1}{m+n-2}\right)s_2^2 = \left(\dfrac{4-1}{4+4-2}\right)(1.225)^2 + \left(\dfrac{4-1}{4+4-2}\right)(1.010)^2$ = 1.260, so s_p = 1.1227.
With df = m + n – 1 = 6 for this interval, $t_{.025,6} = 2.447$ and the desired interval is
$(13.90-12.20) \pm (2.447)(1.1227)\sqrt{\frac{1}{4}+\frac{1}{4}}$ = 1.7 ± 1.945 = (–.24, 3.64). This interval contains 0, so it does not support the conclusion that the two population means are different.

c. Using the two-sample t interval discussed earlier, we use the CI as follows: First, we need to calculate the degrees of freedom. $\nu = \dfrac{\left(\frac{1.225^2}{4}+\frac{1.01^2}{4}\right)^2}{\dfrac{\left(\frac{1.225^2}{4}\right)^2}{3}+\dfrac{\left(\frac{1.01^2}{4}\right)^2}{3}} = \dfrac{.3971}{.0686} = 5.78 \searrow 5$ and $t_{.025,5} = 2.571$. Then the interval is $(13.9-12.2) \pm 2.571\sqrt{\frac{1.225^2}{4}+\frac{1.01^2}{4}} = 1.70 \pm 2.571(.7938) = (-.34, 3.74)$. This interval is slightly wider, but it still supports the same conclusion.

35. There are two changes that must be made to the procedure we currently use. First, the equation used to compute the value of the t test statistic is: $t = \dfrac{(\bar{x}-\bar{y})-\Delta}{s_p\sqrt{\dfrac{1}{m}+\dfrac{1}{n}}}$ where s_p is defined as in Exercise 34. Second, the degrees of freedom = m + n – 2. Assuming equal variances in the situation from Exercise 33, we calculate s_p as follows: $s_p = \sqrt{\left(\dfrac{7}{16}\right)(2.6)^2 + \left(\dfrac{9}{16}\right)(2.5)^2} = 2.544$. The value of the test statistic is, then, $t = \dfrac{(32.8-40.5)-(-5)}{2.544\sqrt{\dfrac{1}{8}+\dfrac{1}{10}}} = -2.24 \approx -2.2$ with df = 16, and the P-value is $P(T < -2.2)$ = .021. Since .021 > .01, we fail to reject H_0.

Section 9.3

36. From the data provided, $\bar{d} = 7.25$ and $s_D = 11.8628$. The parameter of interest: μ_D = true average difference of breaking load for fabric in unabraded or abraded condition. The hypotheses are H_0: $\mu_D = 0$ versus H_a: $\mu_D > 0$. The calculated test statistic is $t = \dfrac{7.25 - 0}{11.8628/\sqrt{8}} = 1.73$; at 7 df, the *P*-value is roughly .065. Since .065 > .01, we fail to reject H_0 at the $\alpha = .01$ level. The data do not indicate a significant mean difference in breaking load for the two fabric load conditions.

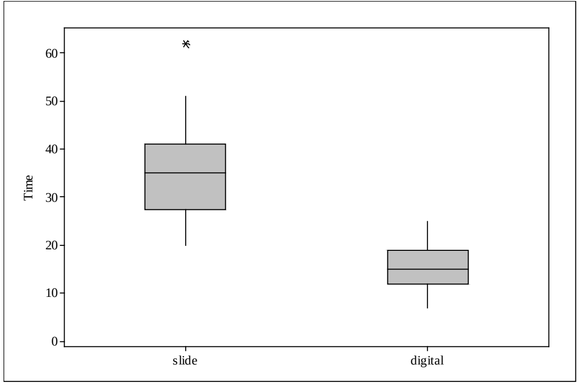
37.

a. This exercise calls for paired analysis. First, compute the difference between indoor and outdoor concentrations of hexavalent chromium for each of the 33 houses. These 33 differences are summarized as follows: $n = 33$, $\bar{d} = -.4239$, $s_D = .3868$, where d = (indoor value – outdoor value). Then $t_{.025,32} = 2.037$, and a 95% confidence interval for the population mean difference between indoor and outdoor concentration is $-.4239 \pm (2.037)\left(\dfrac{.3868}{\sqrt{33}}\right) = -.4239 \pm .13715 = (-.5611, -.2868)$. We can be highly confident, at the 95% confidence level, that the true average concentration of hexavalent chromium outdoors exceeds the true average concentration indoors by between .2868 and .5611 nanograms/m^3.

b. A 95% prediction interval for the difference in concentration for the 34th house is $\bar{d} \pm t_{.025,32}\left(s_D\sqrt{1+\tfrac{1}{n}}\right) = -.4239 \pm (2.037)\left(.3868\sqrt{1+\tfrac{1}{33}}\right) = (-1.224, .3758)$. This prediction interval means that the indoor concentration may exceed the outdoor concentration by as much as .3758 nanograms/m^3 and that the outdoor concentration may exceed the indoor concentration by a much as 1.224 nanograms/m^3, for the 34th house. Clearly, this is a wide prediction interval, largely because of the amount of variation in the differences.

38.

a. The boxplots indicate that retrieval time is much longer when the professional is accessing a library of slides than the digital resource.

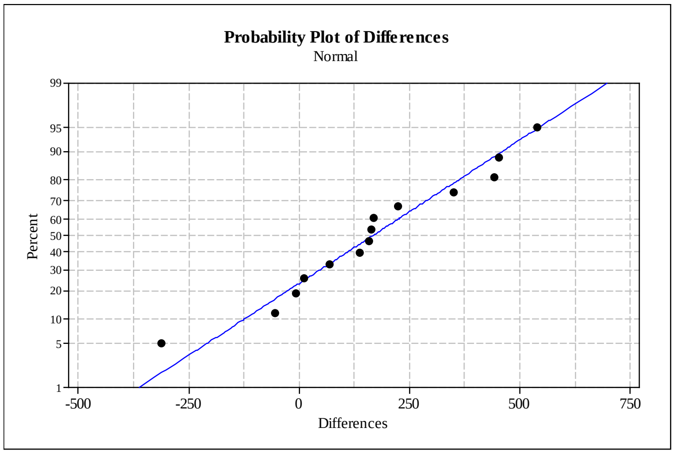


b. For the differences provided, $\bar{d}$ = 20.5 and s_D = 11.96. A normal probability plot of the differences is reasonably straight, so it's plausible that time differences follow a normal distribution and the paired t interval is valid.

With n = 13, the t critical value for the CI is $t_{.025,12}$ = 2.179. The resulting interval is $20.5 \pm 2.179\frac{11.96}{\sqrt{13}}$ = (13.3, 27.7). We are 95% confident that using the slide library takes, on average, between 13.3 and 27.7 seconds longer to retrieve a medical image. In particular, it is definitely not plausible that the true mean difference is zero.

39.

a. The accompanying normal probability plot shows that the differences are consistent with a normal population distribution.



b. We want to test H_0: $\mu_D = 0$ versus H_a: $\mu_D \neq 0$. The test statistic is $t = \frac{\bar{d}-0}{s_D/\sqrt{n}} = \frac{167.2-0}{228/\sqrt{14}} = 2.74$, and the two-tailed P-value is given by $2[P(T > 2.74)] \approx 2[P(T > 2.7)] = 2[.009] = .018$. Since .018 < .05, we reject H_0. There is evidence to support the claim that the true average difference between intake values measured by the two methods is not 0.

40. From the data, n = 10, $\bar{d}$ = 105.7, s_D = 103.845.

a. Let μ_D = true mean difference in TBBMC, postweaning minus lactation. We wish to test the hypotheses H_0: $\mu_D \leq 25$ v. H_a: $\mu_D > 25$. The test statistic is $t = \frac{105.7-25}{103.845/\sqrt{10}}$ = 2.46; at 9 df, the corresponding P-value is around .018. Hence, at the 5% significance level, we reject H_0 and conclude that true average TBBMC during postweaning does exceed the average during lactation by more than 25 grams.

b. A 95% upper confidence bound for $\mu_D = \bar{d} + t_{.05,9}s_D/\sqrt{n}$ = 105.7 + 1.833(103.845)/$\sqrt{10}$ = 165.89 g.

c. No. If we pretend the two samples are independent, the new standard error is is roughly 235, far greater than 103.845/$\sqrt{10}$. In turn, the resulting t statistic is just t = 0.45, with estimated df = 17 and P-value = .329 (all using a computer).

41.

a. Let μ_D denote the true mean change in total cholesterol under the aripiprazole regimen. A 95% CI for μ_D, using the "large-sample" method, is $\bar{d} \pm z_{\alpha/2}\frac{s_D}{\sqrt{n}} = 3.75 \pm 1.96(3.878) = (–3.85, 11.35)$.

b. Now let μ_D denote the true mean change in total cholesterol under the quetiapine regimen. The hypotheses are H_0: $\mu_D = 0$ versus H_a: $\mu_D > 0$. Assuming the distribution of cholesterol changes under this regimen is normal, we may apply a paired t test:

$$t = \frac{\bar{d} - \Delta_0}{s_D / \sqrt{n}} = \frac{9.05 - 0}{4.256} = 2.126 \Rightarrow P\text{-value} = P(T_{35} \geq 2.126) \approx P(T_{35} \geq 2.1) = .02.$$

Our conclusion depends on our significance level. At the $\alpha = .05$ level, there is evidence that the true mean change in total cholesterol under the quetiapine regimen is positive (i.e., there's been an increase); however, we do not have sufficient evidence to draw that conclusion at the $\alpha = .01$ level.

c. Using the "large-sample" procedure again, the 95% CI is $\bar{d} \pm 1.96\frac{s_D}{\sqrt{n}} = \bar{d} \pm 1.96SE(\bar{d})$. If this equals (7.38, 9.69), then midpoint = $\bar{d}$ = 8.535 and width = 2(1.96 $SE(\bar{d})$) = 9.69 – 7.38 = 2.31 ⇒ $SE(\bar{d}) = \frac{2.31}{2(1.96)} = .59$. Now, use these values to construct a 99% CI (again, using a "large-sample" z method): $\bar{d} \pm 2.576SE(\bar{d})$ = 8.535 ± 2.576(.59) = 8.535 ± 1.52 = (7.02, 10.06).

42. The $n = 6$ differences (Before – After) are –1, 2, 24, 35, –16, 1. A normal probability plot of these 6 differences does not suggest a significant violation of the normality assumption (although, with $n = 6$, we have essentially no power to detect such a difference). From the differences, $\bar{d} = 7.50$ and $s_D = 18.58$.

a. We wish to test H_0: $\mu_D = 0$ versus H_a: $\mu_D \neq 0$. The test statistic is $t = \frac{7.50 - 0}{18.58/\sqrt{6}} = 0.99$; at 5 df, the two-tailed P-value is roughly .368. Since .368 is larger than any reasonable α level, we fail to reject H_0. The data do not provide statistically significant evidence of a change in the average number of accidents after information was added to road signs.

b. A 95% prediction interval for a new difference value is $\bar{d} \pm t_{.025,5}s_D\sqrt{1+\frac{1}{6}}$ =

$7.50 \pm 2.571(18.58)\sqrt{1+\frac{1}{6}}$ = 7.50 ± 51.60 = (–44.1, 59.1). That is, we're 95% confident that the number of accidents at a 7th randomly-selected site after the signage changes will be anywhere from 44 fewer accidents to 59 more accidents than occurred before the information was added. (That's a pretty useless interval!)

43.

a. Although there is a "jump" in the middle of the Normal Probability plot, the data follow a reasonably straight path, so there is no strong reason for doubting the normality of the population of differences.

b. A 95% lower confidence bound for the population mean difference is:

$\bar{d} - t_{.05,14}\left(\frac{s_d}{\sqrt{n}}\right) = -38.60 - (1.761)\left(\frac{23.18}{\sqrt{15}}\right) = -38.60 - 10.54 = -49.14$. We are 95% confident that the true mean difference between age at onset of Cushing's disease symptoms and age at diagnosis is greater than -49.14.

c. A 95% upper confidence bound for the population mean difference is 38.60 + 10.54 = 49.14.

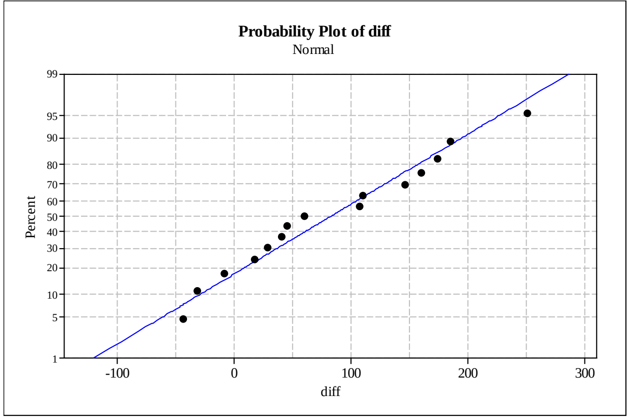
44.

a. No. The statement $\mu_D = 0$ implies that, on the average, the time difference between onset of symptoms and diagnosis of Cushing's disease is 0 months! That's impossible, since doctors wouldn'*t* run the tests to detect Cushing's disease until after a child has shown symptoms of the disease (Cushing screening is not a standard preventive procedure). For each child, the difference d = (age at onset) – (age at diagnosis) must be negative.

b. Using the subtraction order in (a), which matches the data in Exercise 43, we wish to test the hypotheses H_0: $\mu_D = –25$ versus H_a: $\mu_D < –25$ (this corresponds to age at diagnosis exceeding age at onset by more than 25 months, on average). The paired t statistic is $t = \dfrac{\bar{d} - \Delta_0}{s_D / \sqrt{n}} = \dfrac{-38.60 - (-25)}{23.18 / \sqrt{15}} = -2.27$, and the one-tailed P-value is $P(T_{14} \le -2.27) = P(T_{14} \ge 2.27) \approx P(T_{14} \ge 2.3) = .019$. This is a low P-value, so we have reasonably compelling evidence that, on the average, the first diagnosis of Cushing's disease happens more than 25 months after the first onset of symptoms.

45.

a. Yes, it's quite plausible that the population distribution of differences is normal, since the accompanying normal probability plot of the differences is quite linear.



b. No. Since the data is paired, the sample means and standard deviations are not useful summaries for inference. Those statistics would only be useful if we were analyzing two independent samples of data. (We could deduce $\bar{d}$ by subtracting the sample means, but there's no way we could deduce s_D from the separate sample standard deviations.)

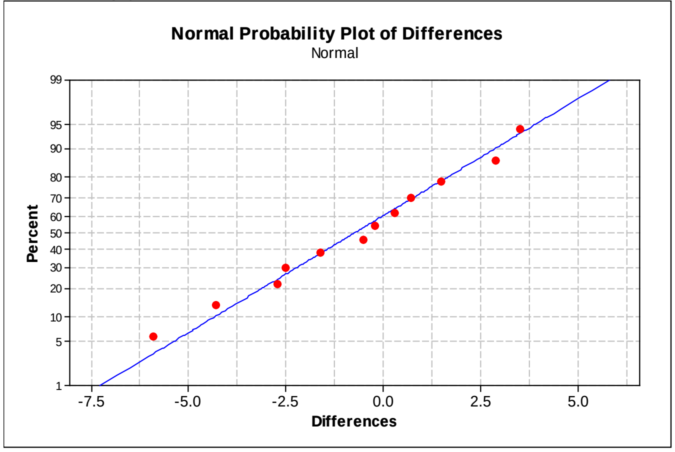
c. The hypotheses corresponding to an upper-tailed test are H_0: $\mu_D = 0$ versus H_a: $\mu_D > 0$. From the data provided, the paired t test statistic is $t = \dfrac{\bar{d} - \Delta_0}{s_D / \sqrt{n}} = \dfrac{82.5 - 0}{87.4 / \sqrt{15}}$ = 3.66. The corresponding P-value is $P(T_{14} \ge 3.66) \approx P(T_{14} \ge 3.7) = .001$. While the P-value stated in the article is inaccurate, the conclusion remains the same: we have strong evidence to suggest that the mean difference in ER velocity and IR velocity is positive. Since the measurements were negative (e.g. –130.6 deg/sec and –98.9 deg/sec), this actually means that the magnitude of IR velocity is significantly higher, on average, than the magnitude of ER velocity, as the authors of the article concluded.

46. We need to check the differences to see if the assumption of normality is plausible. A normal probability plot validates our use of the *t* distribution. A 95% upper confidence bound for μ_D is $\bar{d} + t_{.05,15}\left(\frac{s_d}{\sqrt{n}}\right) = 2635.63 + (1.753)\left(\frac{508.645}{\sqrt{16}}\right) = 2635.63 + 222.91$ = 2858.54. We are 95% confident that the true mean difference between modulus of elasticity after 1 minute and after 4 weeks is at most 2858.54.

47. From the data, $n = 12$, $\bar{d} = –0.73$, $s_D = 2.81$.

a. Let μ_D = the true mean difference in strength between curing under moist conditions and laboratory drying conditions. A 95% CI for μ_D is $\bar{d} \pm t_{.025,11}s_D/\sqrt{n}$ = –0.73 ± 2.201(2.81)/$\sqrt{10}$ = (–2.52 MPa, 1.05 MPa). In particular, this interval estimate includes the value zero, suggesting that true mean strength is not significantly different under these two conditions.

b. Since $n = 12$, we must check that the differences are plausibly from a normal population. The normal probability plot below strongly substantiates that condition.



48. With $(x_1, y_1) = (6, 5)$, $(x_2, y_2) = (15, 14)$, $(x_3, y_3) = (1, 0)$, and $(x_4, y_4) = (21, 20)$, $\bar{d} = 1$ and $s_D = 0$ (the d_i's are 1, 1, 1, and 1), while $s_1 = s_2 = 8.96$, so $s_p = 8.96$ and $t = .16$.

Section 9.4

49. Let p_1 denote the true proportion of correct responses to the first question; define p_2 similarly. The hypotheses of interest are H_0: $p_1 – p_2 = 0$ versus H_a: $p_1 – p_2 > 0$. Summary statistics are $n_1 = n_2 = 200$, $\hat{p}_1 = \frac{164}{200}$ = .82, $\hat{p}_2 = \frac{140}{200}$ = .70, and the pooled proportion is $\hat{p}$ = .76. Since the sample sizes are large, we may apply the two-proportion *z* test procedure.

The calculated test statistic is $z = \frac{(.82 - .70) - 0}{\sqrt{(.76)(.24)\left[\frac{1}{200} + \frac{1}{200}\right]}}$ = 2.81, and the *P*-value is $P(Z \geq 2.81) = .0025$.

Since .0025 ≤ .05, we reject H_0 at the $\alpha = .05$ level and conclude that, indeed, the true proportion of correct answers to the context-free question is higher than the proportion of right answers to the contextual one.

50.

a. With $\hat{p}_1 = \frac{63}{300} = .2100$, $\hat{p}_2 = \frac{75}{180} = .4167$, and $\hat{p} = \frac{63+75}{300+180} = .2875$, $z = \frac{.2100-.4167}{\sqrt{(.2875)(.7125)\left(\frac{1}{300}+\frac{1}{180}\right)}} = \frac{-.2067}{.0427} = -4.84$. The P-value is $2P(Z \le -4.84) \approx 2(0) = 0$. Thus, at the $\alpha = .01$ level (or any reasonable significance level) H_0 is rejected. The proportion of non-contaminated chickens differs for the two companies (Perdue and Tyson).

b. $\overline{p} = .275$ and $\sigma = .0432$, so power = $1-\left[\Phi\left(\frac{\left[(1.96)(.0421)+.2\right]}{.0432}\right)-\Phi\left(\frac{\left[-(1.96)(.0421)+.2\right]}{.0432}\right)\right] =$

$1-\left[\Phi(6.54)-\Phi(2.72)\right] = .9967$.

51. Let p_1 = the true proportion of patients that will experience erectile dysfunction when given no counseling, and define p_2 similarly for patients receiving counseling about this possible side effect. The hypotheses of interest are H_0: $p_1 - p_2 = 0$ versus H_a: $p_1 - p_2 < 0$.

The actual data are 8 out of 52 for the first group and 24 out of 55 for the second group, for a pooled proportion of $\hat{p} = \frac{8+24}{52+55}$ = .299. The two-proportion z test statistic is $\frac{(.153-.436)-0}{\sqrt{(.299)(.701)\left[\frac{1}{52}+\frac{1}{55}\right]}}$ = –3.20, and the P-value is $P(Z \le -3.20) = .0007$. Since .0007 < .05, we reject H_0 and conclude that a higher proportion of men will experience erectile dysfunction if told that it's a possible side effect of the BPH treatment, than if they weren't told of this potential side effect.

52. Let $\alpha = .05$. A 95% confidence interval is $\left(\hat{p}_1 - \hat{p}_2\right) \pm z_{\alpha/2}\sqrt{\left(\frac{\hat{p}_1\hat{q}_1}{m}+\frac{\hat{p}_2\hat{q}_2}{n}\right)}$

$$= \left(\tfrac{224}{395}-\tfrac{126}{266}\right) \pm 1.96\sqrt{\left(\frac{\left(\frac{224}{395}\right)\left(\frac{171}{395}\right)}{395}+\frac{\left(\frac{126}{266}\right)\left(\frac{140}{266}\right)}{266}\right)} = .0934 \pm .0774 = (.0160,.1708).$$

53.

a. Let p_1 and p_2 denote the true incidence rates of GI problems for the olestra and control groups, respectively. We wish to test H_0: $p_1 - \mu_2 = 0$ v. H_a: $p_1 - p_2 \ne 0$. The pooled proportion is $\hat{p} = \frac{529(.176)+563(.158)}{529+563}$ = .1667, from which the relevant test statistic is z = $\frac{.176-.158}{\sqrt{(.1667)(.8333)[529^{-1}+563^{-1}]}}$ = 0.78. The two-sided P-value is $2P(Z \ge 0.78) = .433 > \alpha = .05$, hence we fail to reject the null hypothesis. The data do not suggest a statistically significant difference between the incidence rates of GI problems between the two groups.

b. $n = \frac{\left(1.96\sqrt{(.35)(1.65)/2}+1.28\sqrt{(.15)(.85)+(.2)(.8)}\right)^2}{(.05)^2} = 1210.39$, so a common sample size of $m = n$ = 1211 would be required.

54. Let p_1 = true proportion of irradiated bulbs that are marketable; p_2 = true proportion of untreated bulbs that are marketable. The hypotheses are $H_0: p_1 - p_2 = 0$ v. $H_a: p_1 - p_2 > 0$. The test statistic is $z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{m}+\frac{1}{n}\right)}}$. With $\hat{p}_1 = \frac{153}{180} = .850$, and $\hat{p}_2 = \frac{119}{180} = .661$, $\hat{p} = \frac{272}{360} = .756$, $z = \frac{.850 - .661}{\sqrt{(.756)(.244)\left(\frac{1}{180}+\frac{1}{180}\right)}} = \frac{.189}{.045} = 4.2$. The *P*-value is 1 – Φ(4.2) ≈ 0, so reject H_0 at any reasonable level. Radiation appears to be beneficial.

55.

a. A 95% large sample confidence interval formula for ln(θ) is $\ln\left(\hat{\theta}\right) \pm z_{\alpha/2}\sqrt{\frac{m-x}{mx}+\frac{n-y}{ny}}$. Taking the antilogs of the upper and lower bounds gives the confidence interval for θ itself.

b. $\hat{\theta} = \frac{\frac{189}{11{,}034}}{\frac{104}{11{,}037}} = 1.818$, $\ln\left(\hat{\theta}\right) = .598$, and the standard deviation is $\sqrt{\frac{10{,}845}{(11{,}034)(189)}+\frac{10{,}933}{(11{,}037)(104)}} = .1213$, so the CI for ln(θ) is $.598 \pm 1.96(.1213) = (.360, .836)$.
Then taking the antilogs of the two bounds gives the CI for θ to be (1.43, 2.31). We are 95% confident that people who do not take the aspirin treatment are between 1.43 and 2.31 times more likely to suffer a heart attack than those who do. This suggests aspirin therapy may be effective in reducing the risk of a heart attack.

56.

a. The "after" success probability is $p_1 + p_3$ while the "before" probability is $p_1 + p_2$, so $p_1 + p_3 > p_1 + p_2$ becomes $p_3 > p_2$; thus, we wish to test $H_0: p_3 = p_2$ versus $H_a: p_3 > p_2$.

b. The estimator of $(p_1 + p_3) - (p_1 + p_2)$ is $\frac{(X_1 + X_3) - (X_1 + X_2)}{n} = \frac{X_3 - X_2}{n}$.

c. When H_0 is true, $p_2 = p_3$, so $V\left(\frac{X_3 - X_2}{n}\right) = \frac{p_2 + p_3}{n}$, which is estimated by $\frac{\hat{p}_2 + \hat{p}_3}{n}$. The *z* statistic is then $\frac{\frac{X_3 - X_2}{n}}{\sqrt{\frac{\hat{p}_2 + \hat{p}_3}{n}}} = \frac{X_3 - X_2}{\sqrt{X_2 + X_3}}$.

d. The computed value of *z* is $\frac{200 - 150}{\sqrt{200 + 150}} = 2.68$, so *P*-value = 1 – Φ(2.68) = .0037. At level .01, H_0 can be rejected, but at level .001, H_0 would not be rejected.

57. $\hat{p}_1 = \frac{15+7}{40} = .550$, $\hat{p}_2 = \frac{29}{42} = .690$, and the 95% CI is $(.550-.690) \pm 1.96(.106) = -.14 \pm .21 = (-.35, .07)$.

58. Using $p_1 = q_1 = p_2 = q_2 = .5$, $L = 2(1.96)\sqrt{\left(\frac{.25}{n} + \frac{.25}{n}\right)} = \frac{2.7719}{\sqrt{n}}$, so $L = .1$ requires $n = 769$.

Section 9.5

59.

a. From Table A.9, column 5, row 8, $F_{.01,5,8} = 3.69$.

b. From column 8, row 5, $F_{.01,8,5} = 4.82$.

c. $F_{.95,5,8} = \frac{1}{F_{.05,8,5}} = .207$.

d. $F_{.95,8,5} = \frac{1}{F_{.05,5,8}} = .271$

e. $F_{.01,10,12} = 4.30$

f. $F_{.99,10,12} = \frac{1}{F_{.01,12,10}} = \frac{1}{4.71} = .212$.

g. $F_{.05,6,4} = 6.16$, so $P(F \le 6.16) = .95$.

h. Since $F_{.99,10,5} = \frac{1}{5.64} = .177$, $P(.177 \le F \le 4.74) = P(F \le 4.74) - P(F \le .177) = .95 - .01 = .94$.

60.

a. Since the given *f* value of 4.75 falls between $F_{.05,5,10} = 3.33$ and $F_{.01,5,10} = 5.64$, we can say that the upper-tailed *P*-value is between .01 and .05.

b. Since the given *f* of 2.00 is less than $F_{.10,5,10} = 2.52$, the *P*-value > .10.

c. The two tailed *P*-value = $2P(F \ge 5.64) = 2(.01) = .02$.

d. For a lower tailed test, we must first use formula 9.9 to find the critical values:

$F_{.90,5,10} = \frac{1}{F_{.10,10,5}} = .3030$, $F_{.95,5,10} = \frac{1}{F_{.05,10,5}} = .2110$, $F_{.99,5,10} = \frac{1}{F_{.01,10,5}} = .0995$.

Since $.0995 < f = .200 < .2110$, $.01 < P\text{-value} < .05$ (but obviously closer to .05).

e. There is no column for numerator df of 35 in Table A.9, however looking at both df = 30 and df = 40 columns, we see that for denominator df = 20, our *f* value is between $F_{.01}$ and $F_{.001}$. So we can say $.001 < P\text{-value} < .01$.

61. We test $H_0 : \sigma_1^2 = \sigma_2^2$ v. $H_a : \sigma_1^2 \neq \sigma_2^2$. The calculated test statistic is $f = \frac{(2.75)^2}{(4.44)^2} = .384$. To use Table A.9, take the reciprocal: 1/f = 2.61. With numerator df = m – 1 = 5 – 1 = 4 and denominator df = n – 1 = 10 – 1 = 9 after taking the reciprocal, Table A.9 indicates the one-tailed probability is slightly more than .10, and so the two-sided P-value is slightly more than 2(.10) = .20.
Since .20 > .10, we do not reject H_0 at the α = .1 level and conclude that there is no significant difference between the two standard deviations.

62. With $\sigma_1 =$ true standard deviation for not-fused specimens and $\sigma_2 =$ true standard deviation for fused specimens, we test $H_0 : \sigma_1 = \sigma_2$ v. $H_a : \sigma_1 > \sigma_2$. The calculated test statistic is $f = \frac{(277.3)^2}{(205.9)^2} = 1.814$. With numerator df = m – 1 = 10 – 1 = 9, and denominator df = n – 1 = 8 – 1 = 7, $f = 1.814 < 2.72 = F_{.10,9,7}$. We can say that the P-value > .10, which is obviously > .01, so we cannot reject H_0. There is not sufficient evidence that the standard deviation of the strength distribution for fused specimens is smaller than that of not-fused specimens.

63. Let $\sigma_1^2 =$ variance in weight gain for low-dose treatment, and $\sigma_2^2 =$ variance in weight gain for control condition. We wish to test $H_0 : \sigma_1^2 = \sigma_2^2$ v. $H_a : \sigma_1^2 > \sigma_2^2$. The test statistic is $f = \frac{s_1^2}{s_2^2} = \frac{54^2}{32^2}$ = 2.85. From Table A.9 with df = (19, 22) ≈ (20, 22), the P-value is approximately .01, and we reject H_0 at level .05. The data do suggest that there is more variability in the low-dose weight gains.

64. The sample standard deviations are s_1 = 0.127 and s_2 = 0.060. For the hypotheses $H_0 : \sigma_1 = \sigma_2$ versus $H_a : \sigma_1 \neq \sigma_2$, we find a test statistic of $f = 0.127^2/0.060^2 = 4.548$. At df = (8, 8), Table A.9 indicates the <u>one</u>-tailed P-value is between .05 and .01 (since 3.44 < 4.548 < 6.03). This is a two-sided test, so the P-value is between 2(.05) and 2(.01), i.e., between .02 and .10. Hence, we reject H_0 at the α = .10 level. The data suggest a significant difference in the two population standard deviations.

65. $P\left(F_{1-\alpha/2,m-1,n-1} \leq \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \leq F_{\alpha/2,m-1,n-1} \right) = 1-\alpha$. The set of inequalities inside the parentheses is clearly equivalent to $\frac{S_2^2 F_{1-\alpha/2,m-1,n-1}}{S_1^2} \leq \frac{\sigma_2^2}{\sigma_1^2} \leq \frac{S_2^2 F_{\alpha/2,m-1,n-1}}{S_1^2}$. Substituting the sample values s_1^2 and s_2^2 yields the confidence interval for $\frac{\sigma_2^2}{\sigma_1^2}$, and taking the square root of each endpoint yields the confidence interval for $\frac{\sigma_2}{\sigma_1}$. With $m = n = 4$, we need $F_{.05,3,3} = 9.28$ and $F_{.95,3,3} = \frac{1}{9.28} = .108$. Then with s_1 = .160 and s_2 = .074, the CI for $\frac{\sigma_2^2}{\sigma_1^2}$ is (.023, 1.99), and for $\frac{\sigma_2}{\sigma_1}$ is (.15, 1.41).

66. A 95% upper bound for $\frac{\sigma_2}{\sigma_1}$ is $\sqrt{\frac{s_2^2 F_{.05,9,9}}{s_1^2}} = \sqrt{\frac{(3.59)^2(3.18)}{(.79)^2}} = 8.10$. We are confident that the ratio of the standard deviation of triacetate porosity distribution to that of the cotton porosity distribution is at most 8.10.

Supplementary Exercises

67. We test $H_0 : \mu_1 - \mu_2 = 0$ v. $H_a : \mu_1 - \mu_2 \neq 0$. The test statistic is

$$t = \frac{(\bar{x} - \bar{y}) - \Delta}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}} = \frac{807 - 757}{\sqrt{\frac{27^2}{10} + \frac{41^2}{10}}} = \frac{50}{\sqrt{241}} = \frac{50}{15.524} = 3.22.$$ The approximate df is

$$\nu = \frac{(241)^2}{\frac{(72.9)^2}{9} + \frac{(168.1)^2}{9}} = 15.6,$$ which we round down to 15. The *P*-value for a two-tailed test is approximately 2*P*(*T* > 3.22) = 2(.003) = .006. This small of a *P*-value gives strong support for the alternative hypothesis. The data indicates a significant difference. Due to the small sample sizes (10 each), we are assuming here that compression strengths for both fixed and floating test platens are normally distributed. And, as always, we are assuming the data were randomly sampled from their respective populations.

68.

a. From the first sample, a 99% lower prediction bound is $\bar{x} - t_{.01,16} s\sqrt{1 + \frac{1}{n}} = 6.2 - 2.583(4.5)\sqrt{1 + \frac{1}{17}}$ = 6.2 – 11.96 = –5.76. We can be 99% confident that the weight loss for a single individual under the supervised program will be more than –5.76 — that is, no worse than a gain of 5.76 kg. With the large standard deviation and small sample size, we cannot say with confidence that an individual will lose weight with this program.

b. The hypotheses of interest are $H_0 : \mu_1 - \mu_2 = 2$ versus $H_a : \mu_1 - \mu_2 > 2$. From the Minitab output, the test statistic and *P*-value are *t* = 1.89 and .035 at 28 df. Since .035 > .01, we cannot reject H_0 at the .01 significance level. (Notice, however, that we can conclude average weight loss is more than 2 kg better with the supervised program at the .05 level.)

69. Let p_1 = true proportion of returned questionnaires that included no incentive; p_2 = true proportion of returned questionnaires that included an incentive. The hypotheses are $H_0 : p_1 - p_2 = 0$ v. $H_a : p_1 - p_2 < 0$.

The test statistic is $z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{m} + \frac{1}{n}\right)}}$.

$\hat{p}_1 = \frac{75}{110} = .682$ and $\hat{p}_2 = \frac{66}{98} = .673$; at this point, you might notice that since $\hat{p}_1 > \hat{p}_2$, the numerator of the *z* statistic will be > 0, and since we have a lower tailed test, the *P*-value will be > .5. We fail to reject H_0. This data does not suggest that including an incentive increases the likelihood of a response.

70. Notice this study used a paired design, so we must apply the paired t test. To make the calculations easier, we have temporarily multiplied all values by 10,000. The 13 differences have a sample mean and sd of $\bar{d}$ = 2.462 and s_D = 3.307.

a. A 95% CI for μ_D is $2.462 \pm t_{.025,12}(3.307)/\sqrt{13}$ = (0.463, 4.460). Restoring the original units, we are 95% confident that the true mean difference is between .0000463 and .0004460 kcal/kg/lb. In particular, since this interval does not include 0, there is evidence of a difference in average energy expenditure with the two different types of shovel.

b. Now test H_0: $\mu_D = 0$ versus H_a: $\mu_D > 0$. The test statistic is $t = \dfrac{2.462-0}{3.307/\sqrt{13}}$ = 2.68, for a one-tailed P-value of roughly $P(T \geq 2.7$ when $T \sim t_{12}) = .010$. Since $.010 \leq .05$, we reject H_0 at the .05 level and conclude that average energy expenditure is greater with the conventional shovel than with the perforated shovel.

71. The center of any confidence interval for $\mu_1 - \mu_2$ is always $\bar{x}_1 - \bar{x}_2$, so $\bar{x}_1 - \bar{x}_2 = \dfrac{-473.3+1691.9}{2} = 609.3$. Furthermore, half of the width of this interval is $\dfrac{1691.9-(-473.3)}{2} = 1082.6$. Equating this value to the expression on the right of the 95% confidence interval formula, we find $\sqrt{\dfrac{s_1^2}{n_1}+\dfrac{s_2^2}{n_2}} = \dfrac{1082.6}{1.96} = 552.35$.

For a 90% interval, the associated z value is 1.645, so the 90% confidence interval is then $609.3 \pm (1.645)(552.35) = 609.3 \pm 908.6 \;= (-299.3, 1517.9)$.

72.

a. A 95% lower confidence bound for the true average strength of joints with a side coating is $\bar{x} - t_{.025,9}\left(\dfrac{s}{\sqrt{n}}\right) = 63.23 - (1.833)\left(\dfrac{5.96}{\sqrt{10}}\right) = 63.23 - 3.45 = 59.78$. That is, with a confidence level of 95%, the average strength of joints with a side coating is at least 59.78 (Note: this bound is valid only if the distribution of joint strength is normal.)

b. A 95% lower prediction bound for the strength of a single joint with a side coating is $\bar{x} - t_{.025,9}\left(s\sqrt{1+\frac{1}{n}}\right) = 63.23 - (1.833)\left(5.96\sqrt{1+\frac{1}{10}}\right) \;= 63.23 - 11.46 = 51.77$. That is, with a confidence level of 95%, the strength of a single joint with a side coating would be at least 51.77.

c. For a confidence level of 95%, a two-sided tolerance interval for capturing at least 95% of the strength values of joints with side coating is $\bar{x}$ ± (tolerance critical value) s. The tolerance critical value is obtained from Table A.6 with 95% confidence, k = 95%, and n = 10. Thus, the interval is $63.23 \pm (3.379)(5.96) = 63.23 \pm 20.14 = (43.09, 83.37)$. That is, we can be highly confident that at least 95% of all joints with side coatings have strength values between 43.09 and 83.37.

d. A 95% confidence interval for the difference between the true average strengths for the two types of joints is $\left(80.95-63.23\right)\pm t_{.025,\nu}\sqrt{\frac{\left(9.59\right)^2}{10}+\frac{\left(5.96\right)^2}{10}}$. The approximate degrees of freedom is $\nu=\frac{\left(\frac{91.9681}{10}+\frac{35.5216}{10}\right)^2}{\frac{\left(\frac{91.9681}{10}\right)^2}{9}+\frac{\left(\frac{35.5216}{10}\right)^2}{9}}=15.05\searrow 15$, and $t_{.025,15}=2.131$. The interval is, then, $17.72\pm\left(2.131\right)\left(3.57\right)=17.72\pm 7.61=\left(10.11,25.33\right)$. With 95% confidence, we can say that the true average strength for joints without side coating exceeds that of joints with side coating by between 10.11 and 25.33 lb-in./in.

73. Let μ_1 and μ_2 denote the true mean zinc mass for Duracell and Energizer batteries, respectively. We want to test the hypotheses H_0: $\mu_1 - \mu_2 = 0$ versus H_a: $\mu_1 - \mu_2 \neq 0$. Assuming that both zinc mass distributions are normal, we'll use a two-sample t test; the test statistic is $t=\frac{(\overline{x}-\overline{y})-\Delta_0}{\sqrt{\frac{s_1^2}{m}+\frac{s_2^2}{n}}}=\frac{(138.52-149.07)-0}{\sqrt{\frac{(7.76)^2}{15}+\frac{(1.52)^2}{20}}}=-5.19$.

The textbook's formula for df gives ν = 14. The P-value is $P(T_{14} \le -5.19) \approx 0$. Hence, we strongly reject H_0 and we conclude the mean zinc mass content for Duracell and Energizer batteries are <u>not</u> the same (they do differ).

74. This exercise calls for a paired analysis. First compute the difference between the amount of cone penetration for commutator and pinion bearings for each of the 17 motors. These 17 differences are summarized as follows: $n = 17$, $\overline{d}=-4.18$, $s_D = 35.85$, where d = (commutator value – pinion value). Then $t_{.025,16}=2.120$, and the 95% confidence interval for the population mean difference between penetration for the commutator armature bearing and penetration for the pinion bearing is: $-4.18\pm\left(2.120\right)\left(\frac{35.85}{\sqrt{17}}\right)=-4.18\pm 18.43=\left(-22.61,14.25\right)$. We would have to say that the population mean difference has not been precisely estimated. The bound on the error of estimation is quite large. Also, the confidence interval spans zero. Because of this, we have insufficient evidence to claim that the population mean penetration differs for the two types of bearings.

75. Since we can assume that the distributions from which the samples were taken are normal, we use the two-sample t test. Let μ_1 denote the true mean headability rating for aluminum killed steel specimens and μ_2 denote the true mean headability rating for silicon killed steel. Then the hypotheses are $H_0:\mu_1-\mu_2=0$ v. $H_a:\mu_1-\mu_2\neq 0$. The test statistic is $t=\frac{-.66}{\sqrt{.03888+.047203}}=\frac{-.66}{\sqrt{.086083}}=-2.25$. The approximate degrees of freedom are $\nu=\frac{\left(.086083\right)^2}{\frac{\left(.03888\right)^2}{29}+\frac{\left(.047203\right)^2}{29}}=57.5\searrow 57$. The two-tailed P-value ≈ 2(.014) = .028, which is less than the specified significance level, so we would reject H_0. The data supports the article's authors' claim.

76. Let μ_1 and μ_2 denote the true average number of cycles to break for polyisoprene and latex condoms, respectively. (Notice the order has been reversed.) We want to test the hypotheses H_0: $\mu_1 - \mu_2 = 1000$ versus H_a: $\mu_1 - \mu_2 > 1000$. Assuming that both cycle distributions are normal, we'll use a two-sample t test; the test statistic is $t = \dfrac{(\bar{x}-\bar{y})-\Delta_0}{\sqrt{\dfrac{s_1^2}{m}+\dfrac{s_2^2}{n}}} = \dfrac{(5805-4358)-0}{\sqrt{\dfrac{(3990)^2}{20}+\dfrac{(2218)^2}{20}}} = 2.40$.

The textbook's formula for df gives ν = 29. The P-value is $P(T_{29} \geq 2.40) \approx .012$. Hence, we reject H_0 and we conclude that the true average number of cycles to break for polyisoprene condoms exceeds the average for latex condoms by more than 1000 cycles.

77.

a. The relevant hypotheses are $H_0: \mu_1 - \mu_2 = 0$ v. $H_a: \mu_1 - \mu_2 \neq 0$. Assuming both populations have normal distributions, the two-sample t test is appropriate. $m = 11$, $\bar{x} = 98.1$, $s_1 = 14.2$, $n = 15$, $\bar{y} = 129.2$, $s_2 = 39.1$. The test statistic is $t = \dfrac{-31.1}{\sqrt{18.3309+101.9207}} = \dfrac{-31.1}{\sqrt{120.252}} = -2.84$. The approximate degrees of freedom $\nu = \dfrac{(120.252)^2}{\dfrac{(18.3309)^2}{10}+\dfrac{(101.9207)^2}{14}} = 18.64 \searrow 18$. From Table A.8, the two-tailed P-value ≈ 2(.006) = .012. No, obviously the results are different.

b. For the hypotheses $H_0: \mu_1 - \mu_2 = -25$ v. $H_a: \mu_1 - \mu_2 < -25$, the test statistic changes to $t = \dfrac{-31.1-(-25)}{\sqrt{120.252}} = -.556$. With df = 18, the P-value ≈ $P(T < -.6) = .278$. Since the P-value is greater than any sensible choice of α, we fail to reject H_0. There is insufficient evidence that the true average strength for males exceeds that for females by more than 25 N.

78.

a. The relevant hypotheses are $H_0: \mu_1^* - \mu_2^* = 0$ (which is equivalent to saying $\mu_1 - \mu_2 = 0$) versus $H_a: \mu_1^* - \mu_2^* \neq 0$ (aka $\mu_1 - \mu_2 \neq 0$). The pooled t test is based on df = $m + n - 2 = 8 + 9 - 2 = 15$. The pooled variance is $s_p^2 = \left(\dfrac{m-1}{m+n-2}\right)s_1^2 + \left(\dfrac{n-1}{m+n-2}\right)s_2^2$ $\left(\dfrac{8-1}{8+9-2}\right)(4.9)^2 + \left(\dfrac{9-1}{8+9-2}\right)(4.6)^2 = 22.49$, so $s_p = 4.742$. The test statistic is $t = \dfrac{\bar{x}^* - \bar{y}^*}{s_p\sqrt{\frac{1}{m}+\frac{1}{n}}} = \dfrac{18.0-11.0}{4.742\sqrt{\frac{1}{8}+\frac{1}{9}}} = 3.04 \approx 3.0$. From Table A.7, the P-value associated with $t = 3.0$ is $2P(T > 3.0) = 2(.004) = .008$. At significance level .05, H_0 is rejected and we conclude that there is a difference between μ_1^* and μ_2^*, which is equivalent to saying that there is a difference between μ_1 and μ_2.

b. No. The mean of a lognormal distribution is $\mu = e^{\mu^* + (\sigma^*)^2/2}$, where μ^* and σ^* are the parameters of the lognormal distribution (i.e., the mean and standard deviation of ln(x)). So when $\sigma_1^* = \sigma_2^*$, then $\mu_1^* = \mu_2^*$ would imply that $\mu_1 = \mu_2$. However, when $\sigma_1^* \neq \sigma_2^*$, then even if $\mu_1^* = \mu_2^*$, the two means μ_1 and μ_2 (given by the formula above) would not be equal.

79. To begin, we must find the % difference for each of the 10 meals! For the first meal, the % difference is $\frac{\text{measured} - \text{stated}}{\text{stated}} = \frac{212 - 180}{180}$ = .1778, or 17.78%. The other nine percentage differences are 45%, 21.58%, 33.04%, 5.5%, 16.49%, 15.2%, 10.42%, 81.25%, and 26.67%.

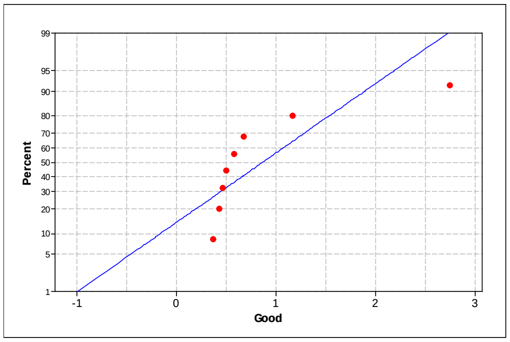
We wish to test the hypotheses H_0: $\mu = 0$ versus H_a: $\mu \neq 0$, where μ denotes the true average percent difference for all supermarket convenience meals. A normal probability plot of these 10 values shows some noticeable deviation from linearity, so a *t*-test is actually of questionable validity here, but we'll proceed just to illustrate the method.

For this sample, $n = 10$, $\bar{x}$ = 27.29%, and s = 22.12%, for a t statistic of $t = \frac{27.29 - 0}{22.12/\sqrt{10}} = 3.90$.

At df = $n - 1 = 9$, the *P*-value is $2P(T_9 \geq 3.90) \approx 2(.002) = .004$. Since this is smaller than any reasonable significance level, we reject H_0 and conclude that the true average percent difference between meals' stated energy values and their measured values is non-zero.

80. Since the *P*-value, .001, is very small, we would reject the null hypothesis of equal population means and conclude instead that the true mean arsenic concentration measurements are significantly different for the two methods. That is, the methods disagree. Assuming sample "1" corresponds to the lab method, the CI says we're 95% confident that the true mean arsenic concentration measurement using the lab method is between 6.498 μg/L and 11.102 μg/L higher than using the field method.

81. The normal probability plot below indicates the data for good visibility does <u>not</u> come from a normal distribution. Thus, a *t*-test is <u>not</u> appropriate for this small a sample size. (The plot for poor visibility isn'*t* as bad.) That is, a pooled *t* test should not be used here, nor should an "unpooled" two-sample *t* test be used (since it relies on the same normality assumption).



82.

a. A 95% CI for $\mu_{37,\text{dry}}$ = 325.73 ± $t_{.025,5}(34.97)/\sqrt{6}$ = 325.73 ± 2.571(14.276) = (289.03, 362.43). We are 95% confident that the true average breaking force in a dry medium at 37° is between 289.03 N and 362.43 N.

b. The relevant estimated df = 9. A 95% CI for $\mu_{37,\text{dry}} - \mu_{37,\text{wet}}$ = (325.73 – 306.09) ± $t_{.025,9}\sqrt{\dfrac{34.97^2}{6}+\dfrac{41.97^2}{6}}$ = (–30.81,70.09). We are 95% confident that the true average breaking force in a dry medium at 37° is between 30.81 N <u>less</u> and 70.09 N <u>more</u> than the true average breaking force in a wet medium at 37°.

c. We wish to test H_0: $\mu_{37,\text{dry}} - \mu_{22,\text{dry}} = 0$ v. Ha: $\mu_{37,\text{dry}} - \mu_{22,\text{dry}} > 0$. The relevant test statistic is t = $\dfrac{(325.73-170.60)-100}{\sqrt{\dfrac{34.97^2}{6}+\dfrac{39.08^2}{6}}}$ = 2.58. The estimated df = 9 again, and the approximate P-value is .015.

Hence, we reject H_0 and conclude that true average force in a dry medium at 37° is indeed more than 100 N greater than the average at 22°.

83. We wish to test H_0: $\mu_1 = \mu_2$ versus H_a: $\mu_1 \neq \mu_2$

Unpooled:

With H_0: $\mu_1 - \mu_2 = 0$ v. H_a: $\mu_1 - \mu_2 \neq 0$, we will reject H_0 if $p-value < \alpha$.

$\nu = \dfrac{\left(\frac{.79^2}{14}+\frac{1.52^2}{12}\right)^2}{\frac{\left(\frac{.79^2}{14}\right)^2}{13}+\frac{\left(\frac{1.52^2}{12}\right)^2}{11}} = 15.95 \downarrow 15$, and the test statistic $t = \dfrac{8.48-9.36}{\sqrt{\frac{.79^2}{14}+\frac{1.52^2}{12}}} = \dfrac{-.88}{.4869} = -1.81$ leads to a P-value of about 2P(T_{15} > 1.8) =2(.046) = .092.

Pooled:

The degrees of freedom are $\nu = m + n - 2 = 14 + 12 - 2 = 24$ and the pooled variance is $\left(\dfrac{13}{24}\right)(.79)^2 + \left(\dfrac{11}{24}\right)(1.52)^2 = 1.3970$, so $s_p = 1.181$. The test statistic is

$t = \dfrac{-.88}{1.181\sqrt{\frac{1}{14}+\frac{1}{12}}} = \dfrac{-.88}{.465} \approx -1.89$. The P-value = 2P(T_{24} > 1.9) = 2(.035) = .070.

With the pooled method, there are more degrees of freedom, and the P-value is smaller than with the unpooled method. That is, if we are willing to assume equal variances (which might or might not be valid here), the pooled test is more capable of detecting a significant difference between the sample means.

84. Because of the nature of the data, we will use a paired t test. We obtain the differences by subtracting intake value from expenditure value. We are testing the hypotheses H_0: $\mu_D = 0$ vs H_a: $\mu_D \neq 0$. The test statistic $t = \dfrac{1.757}{1.197/\sqrt{7}} = 3.88$ with df = n – 1 = 6 leads to a P-value of 2P(T > 3.88) ≈ .008. Using either significance level .05 or .01, we would reject the null hypothesis and conclude that there is a difference between average intake and expenditure. However, at significance level .001, we would not reject.

85.

a. With n denoting the second sample size, the first is $m = 3n$. We then wish $20 = 2\left(2.58\right)\sqrt{\frac{900}{3n}+\frac{400}{n}}$, which yields $n = 47$, $m = 141$.

b. We wish to find the n which minimizes $2z_{\alpha/2}\sqrt{\frac{900}{400-n}+\frac{400}{n}}$, or equivalently, the n which minimizes $\frac{900}{400-n}+\frac{400}{n}$. Taking the derivative with respect to n and equating to 0 yields $900\left(400-n\right)^{-2}-400n^{-2}=0$, whence $9n^2 = 4\left(400-n\right)^2$, or $5n^2+3200n-640,000=0$. The solution is $n = 160$, and thus $m = 400 - n = 240$.

86.

a. By appearing to administer the same pair of treatments to all children, any placebo or psychosomatic effects are reduced. For example, if parents are generally disposed to believe that injections are more effective, then those whose kids got only an injection (i.e., different from this study) might be more prone to ignoring flu symptoms, while parents of kids in the "nasal spray group" (again, in a different study design) might be more apt to notice flu symptoms.

b. Let p_1 = the true probability a child contracts the flu after vaccination by injection; define p_2 similarly for vaccination by nasal spray. Then $n_1 = n_2 = 4000$, $\hat{p}_1$ = 344/4000 = .086, $\hat{p}_2$ = 156/4000 = .039, and the pooled proportion is $\hat{p}$ = .0625. Consider the hypotheses H_0: $p_1 - \mu_2 = 0$ v. H_a: $p_1 - p_2 > 0$. The two-proportion z test statistic is $z = \frac{.086-.039}{\sqrt{.0625(.9375)\left[\frac{1}{4000}+\frac{1}{4000}\right]}}$ = 8.68, so the P-value is effectively 0 and we'd reject H_0 at any significance level. Hence, we conclude that kids are more likely to get the flue after vaccination by injection than after vaccination by nasal spray. Start using nasal spray vaccinations!

87. We want to test the hypothesis H_0: $\mu_1 \le 1.5\mu_2$ v. H_a: $\mu_1 > 1.5\mu_2$ — or, using the hint, H_0: $\theta \le 0$ v. H_a: $\theta > 0$. Our point estimate of θ is $\hat{\theta} = \overline{X}_1 - 1.5\overline{X}_2$, whose estimated standard error equals $s(\hat{\theta}) = \sqrt{\frac{s_1^2}{n_1}+(1.5)^2\frac{s_2^2}{n_2}}$, using the fact that $V(\hat{\theta}) = \frac{\sigma_1^2}{n_1}+(1.5)^2\frac{\sigma_2^2}{n_2}$. Plug in the values provided to get a test statistic t = $\frac{22.63-1.5(14.15)-0}{\sqrt{2.8975}}$ ≈ 0.83. A conservative df estimate here is ν = 50 – 1 = 49. Since $P(T \ge 0.83) \approx .20$ and .20 > .05, we fail to reject H_0 at the 5% significance level. The data does not suggest that the average tip after an introduction is more than 50% greater than the average tip without introduction.

88.

a. For the paired data on pitchers, $n = 17$, $\bar{d} = 4.066$, and $s_d = 3.955$. $t_{.025,16} = 2.120$, and the resulting 95% CI is (2.03, 6.10). We are 95% confident that the true mean difference between dominant and nondominant arm translation for pitchers is between 2.03 and 6.10.

b. For the paired data on position players, $n = 19$, $\bar{d} = 0.233$, and $s_d = 1.603$. $t_{.025,18} = 2.101$, and the resulting 95% CI is (–0.54, 1.01). We are 95% confident that the true mean difference between dominant and nondominant arm translation for position players is between 2.03 and 6.10.

c. Let μ_1 and μ_2 represent the true mean differences in side-to-side AP translation for pitchers and position players, respectively. We wish to test the hypotheses H_0: $\mu_1 - \mu_2 = 0$ v. H_a: $\mu_1 - \mu_2 > 0$. The data for this analysis are precisely the <u>differences</u> utilized in parts **a** and **b**. Hence, the test statistic is $t = \dfrac{4.066-0.233}{\sqrt{\dfrac{3.955^2}{17}+\dfrac{1.603^2}{19}}} = 3.73$. The estimated df = 20 (using software), and the corresponding P-value is $P(T > 3.73) = .001$. Hence, even at the 1% level, we concur with the authors' assessment that this difference is greater, on average, in pitchers than in position players.

89. $\Delta_0 = 0$, $\sigma_1 = \sigma_2 = 10$, $d = 1$, $\sigma = \sqrt{\dfrac{200}{n}} = \dfrac{14.142}{\sqrt{n}}$, so $\beta = \Phi\left(1.645 - \dfrac{\sqrt{n}}{14.142}\right)$, giving $\beta = .9015, .8264, .0294$, and $.0000$ for $n = 25, 100, 2500$, and 10,000 respectively. If the μ_is referred to true average IQs resulting from two different conditions, $\mu_1 - \mu_2 = 1$ would have little practical significance, yet very large sample sizes would yield statistical significance in this situation.

90. For the sandstone sample, $n = 20$, $\bar{x} = 2.36$, and $s_1 = 1.21$. For the shale sample, $m = 21$, $\bar{y} = 0.485$, and $s_2 = 0.157$. To construct a 95% confidence interval for the difference in population means, the df formula from the textbook gives $\nu = 19$. The resulting 95% CI is (1.306, 2.445). That is, we are 95% confident that the average roughness of sandstone (on this particular scale) is between 1.306 and 2.445 units higher than the average roughness of shale. In particular, since the interval does not include 0, the data suggest that average roughness is <u>not</u> the same for the two stone types.

91. $H_0: p_1 = p_2$ will be rejected at level α in favor of $H_a: p_1 > p_2$ if $z \ge z_\alpha$. With $\hat{p}_1 = \frac{250}{2500} = .10$ and $\hat{p}_2 = \frac{167}{2500} = .0668$, $\hat{p} = .0834$ and $z = \dfrac{.0332}{.0079} = 4.2$, so H_0 is rejected at any reasonable α level. It appears that a response is more likely for a white name than for a black name.

92. The computed value of z is $z = \dfrac{34-46}{\sqrt{34+46}} = -1.34$. A lower tailed test would be appropriate, so the P-value $= \Phi(-1.34) = .0901 > .05$, so we would not judge the drug to be effective.

93.

a. Let μ_1 and μ_2 denote the true average weights for operations 1 and 2, respectively. The relevant hypotheses are $H_0: \mu_1 - \mu_2 = 0$ v. $H_a: \mu_1 - \mu_2 \neq 0$. The value of the test statistic is

$$t = \frac{(1402.24 - 1419.63)}{\sqrt{\frac{(10.97)^2}{30} + \frac{(9.96)^2}{30}}} = \frac{-17.39}{\sqrt{4.011363 + 3.30672}} = \frac{-17.39}{\sqrt{7.318083}} = -6.43.$$

At df = $\nu = \dfrac{(7.318083)^2}{\frac{(4.011363)^2}{29} + \frac{(3.30672)^2}{29}} = 57.5 \searrow 57$, $2P(T \leq -6.43) \approx 0$, so we can reject H_0 at level .05. The data indicates that there is a significant difference between the true mean weights of the packages for the two operations.

b. H_0: $\mu_1 = 1400$ will be tested against H_a: $\mu_1 > 1400$ using a one-sample *t* test with test statistic $t = \dfrac{\bar{x} - 1400}{s_1 / \sqrt{m}}$. With degrees of freedom = 29, we reject H_0 if $t \geq t_{.05,29} = 1.699$. The test statistic value is $t = \dfrac{1402.24 - 1400}{10.97/\sqrt{30}} = \dfrac{2.24}{2.00} = 1.1$. Because 1.1 < 1.699, H_0 is not rejected. True average weight does not appear to exceed 1400.

94. First, $V(\bar{X} - \bar{Y}) = \dfrac{\lambda_1}{m} + \dfrac{\lambda_2}{n} = \lambda\left(\dfrac{1}{m} + \dfrac{1}{n}\right)$ under H_0, where λ can be estimated for the variance by the pooled estimate $\hat{\lambda}_{pooled} = \dfrac{m\bar{X} + n\bar{Y}}{m+n}$. With the obvious point estimates $\hat{\lambda}_1 = \bar{X}$ and $\hat{\lambda}_2 = \bar{Y}$, we have a large-sample test statistic of $Z = \dfrac{(\bar{X} - \bar{Y}) - 0}{\sqrt{\hat{\lambda}_{pooled}\left(\frac{1}{m} + \frac{1}{n}\right)}} = \dfrac{\bar{X} - \bar{Y}}{\sqrt{\frac{\bar{X}}{n} + \frac{\bar{Y}}{m}}}$.

With $\bar{x} = 1.616$ and $\bar{y} = 2.557$, $z = -5.3$ and *P*-value = $P(|Z| \geq |-5.3|) = 2\Phi(-5.3) \approx 0$, so we would certainly reject H_0: $\lambda_1 = \lambda_2$ in favor of H_a: $\lambda_1 \neq \lambda_2$.

95. A large-sample confidence interval for $\lambda_1 - \lambda_2$ is $(\hat{\lambda}_1 - \hat{\lambda}_2) \pm z_{\alpha/2}\sqrt{\dfrac{\hat{\lambda}_1}{m} + \dfrac{\hat{\lambda}_2}{n}}$, or $(\bar{x} - \bar{y}) \pm z_{\alpha/2}\sqrt{\dfrac{\bar{x}}{m} + \dfrac{\bar{y}}{n}}$.
With $\bar{x} = 1.616$ and $\bar{y} = 2.557$, the 95% confidence interval for $\lambda_1 - \lambda_2$ is –.94 ± 1.96(.177) = –.94 ± .35 = (–1.29, –.59).

CHAPTER 10

Section 10.1

1. The computed value of F is $f = \frac{\text{MSTr}}{\text{MSE}} = \frac{2673.3}{1094.2} = 2.44$. Degrees of freedom are $I - 1 = 4$ and $I(J - 1) = (5)(3) = 15$. From Table A.9, $F_{.05,4,15} = 3.06$ and $F_{.10,4,15} = 2.36$; since our computed value of 2.44 is between those values, it can be said that .05 < P-value < .10. Therefore, H_0 is <u>not</u> rejected at the $\alpha = .05$ level. The data do not provide statistically significant evidence of a difference in the mean tensile strengths of the different types of copper wires.

2.

Type of Box	$\overline{x}$	s
1	713.00	46.55
2	756.93	40.34
3	698.07	37.20
4	682.02	39.87

Grand mean = 712.51

$$\text{MSTr} = \frac{6}{4-1}\Big[(713.00-712.51)^2 + (756.93-712.51)^2 + (698.07-712.51)^2 + (682.02-712.51)^2\Big] = 6,223.0604$$

$$\text{MSE} = \frac{1}{4}\Big[(46.55)^2 + (40.34)^2 + (37.20)^2 + (39.87)^2\Big] = 1,691.9188$$

$$f = \frac{\text{MSTr}}{\text{MSE}} = \frac{6,223.0604}{1,691.9188} = 3.678$$

At df = (3, 20), 3.10 < 3.678 < 4.94 ⇒ .01 < P-value < .05. In particular, since P-value < α, we reject H_0. There is a difference in mean compression strengths among the four box types.

3. With μ_i = true average lumen output for brand i bulbs, we wish to test $H_0: \mu_1 = \mu_2 = \mu_3$ v. H_a: at least two μ_i's are different. $\text{MSTr} = \hat{\sigma}_B^2 = \frac{591.2}{2} = 295.60$, $\text{MSE} = \hat{\sigma}_W^2 = \frac{4773.3}{21} = 227.30$, so

$$f = \frac{\text{MSTr}}{\text{MSE}} = \frac{295.60}{227.30} = 1.30.$$

For finding the P-value, we need degrees of freedom $I - 1 = 2$ and $I(J - 1) = 21$. In the 2nd row and 21st column of Table A.9, we see that $1.30 < F_{.10,2,21} = 2.57$, so the P-value > .10. Since .10 is not < .05, we cannot reject H_0. There are no statistically significant differences in the average lumen outputs among the three brands of bulbs.

4. Let μ_i denote the true mean foam density from the ith manufacturer (i = 1, 2, 3, 4). The hypotheses are H_0: $\mu_1 = \mu_2 = \mu_3 = \mu_4$ versus H_a: at least two of the μ_i's are different. From the data provided, $\bar{x}_{1.} = 29.8, \bar{x}_{2.} = 27.4, \bar{x}_{3.} = 25.95, \bar{x}_{4.} = 27.15$; s_1 = .849, s_2 = .424, s_3 = 1.63, and s_4 = 2.33. From these we find SSTr = 15.60 and SSE = 8.99, so MSTr = $\frac{\text{SSTr}}{I-1} = \frac{15.60}{4-1}$ = 5.20, MSE = $\frac{\text{SSE}}{I(J-1)} = \frac{8.99}{4(2-1)}$ =2.25, and finally f = MSTr/MSE = 2.31.
Looking at the $F_{3,4}$ distribution, 2.31 is less than $F_{.10,3,4}$ = 4.19, so the P-value associated with this hypothesis test is <u>more</u> than .10. Thus, at any reasonable significance level, we would fail to reject H_0. The data provided do not provide statistically significant evidence of a difference in the true mean foam densities for these four manufacturers.

5. μ_i = true mean modulus of elasticity for grade i (i = 1, 2, 3). We test $H_0 : \mu_1 = \mu_2 = \mu_3$ vs. H_a: at least two μ_i's are different. Grand mean = 1.5367,

$$\text{MSTr} = \frac{10}{2}\left[(1.63-1.5367)^2 + (1.56-1.5367)^2 + (1.42-1.5367)^2\right] = .1143,$$

$\text{MSE} = \frac{1}{3}\left[(.27)^2 + (.24)^2 + (.26)^2\right] = .0660$, $f = \frac{\text{MSTr}}{\text{MSE}} = \frac{.1143}{.0660} = 1.73$. At df = (2,27), 1.73 < 2.51 $\Rightarrow$ the P-value is <u>more</u> than .10. Hence, we fail to reject H_0. The three grades do not appear to differ significantly.

6.

Source	df	SS	MS	F
Treatments	3	509.112	169.707	10.85
Error	36	563.134	15.643	
Total	39	1,072.256		

Use the df = (3,30) block of Table A.9, since df = 36 is not available. Since 10.85 > 7.05, P-value < .001. So, we strongly reject H_0 in favor of H_a: at least two of the four means differ.

7. Let μ_i denote the true mean electrical resistivity for the ith mixture (i = 1, …, 6).
The hypotheses are H_0: $\mu_1 = \ldots = \mu_6$ versus H_a: at least two of the μ_i's are different.
There are I = 6 different mixtures and J = 26 measurements for each mixture. That information provides the df values in the table. Working backwards, SSE = $I(J-1)$MSE = 2089.350; SSTr = SST – SSE = 3575.065; MSTr = SSTr/(I – 1) = 715.013; and, finally, f = MSTr/MSE = 51.3.

Source	df	SS	MS	f
Treatments	5	3575.065	715.013	51.3
Error	150	2089.350	13.929	
Total	155	5664.415		

The P-value is $P(F_{5,150} \geq 51.3) \approx 0$, and so H_0 will be rejected at any reasonable significance level. There is strong evidence that true mean electrical resistivity is <u>not</u> the same for all 6 mixtures.

8. The summary quantities are $x_{1.} = 2332.5$, $x_{2.} = 2576.4$, $x_{3.} = 2625.9$, $x_{4.} = 2851.5$, $x_{5.} = 3060.2$, $x_{..} = 13,446.5$, so CF = 5,165,953.21, SST = 75,467.58, SSTr = 43,992.55, SSE = 31,475.03, $\text{MSTr} = \frac{43,992.55}{4} = 10,998.14$, $\text{MSE} = \frac{31,475.03}{30} = 1049.17$ and $f = \frac{10,998.14}{1049.17} = 10.48$. (These values should be displayed in an ANOVA table as requested.)
At df = (4,30), 10.48 > 5.53 ⇒ P-value < .001. Hence, $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$ is rejected. There are differences in the true average axial stiffness for the different plate lengths.

9. The summary quantities are $x_{1.} = 34.3$, $x_{2.} = 39.6$, $x_{3.} = 33.0$, $x_{4.} = 41.9$, $x_{..} = 148.8$, $\Sigma\Sigma x_{ij}^2 = 946.68$, so $\text{CF} = \frac{(148.8)^2}{24} = 922.56$, $\text{SST} = 946.68 - 922.56 = 24.12$, $\text{SSTr} = \frac{(34.3)^2 + ... + (41.9)^2}{6} - 922.56 = 8.98$, $\text{SSE} = 24.12 - 8.98 = 15.14$.

Source	df	SS	MS	*F*
Treatments	3	8.98	2.99	3.95
Error	20	15.14	.757	
Total	23	24.12		

Since $3.10 = F_{.05,3,20} < 3.95 < 4.94 = F_{.01,3,20}$, .01 < *P*-value < .05, and H_0 is rejected at level .05.

10.

a. $E\left(\bar{X}_{..}\right) = \frac{\Sigma E\left(\bar{X}_{i.}\right)}{I} = \frac{\Sigma \mu_i}{I} = \mu$.

b. $E\left(\bar{X}_{i.}^2\right) = V\left(\bar{X}_{i.}\right) + \left[E\left(\bar{X}_{i.}\right)\right]^2 = \frac{\sigma^2}{J} + \mu_i^2$.

c. $E\left(\bar{X}_{..}^2\right) = V\left(\bar{X}_{..}\right) + \left[E\left(\bar{X}_{..}\right)\right]^2 = \frac{\sigma^2}{IJ} + \mu^2$.

d. $E(\text{SSTr}) = E\left[J\Sigma\bar{X}_{i.}^2 - IJ\bar{X}_{..}^2\right] = J\sum\left(\frac{\sigma^2}{J} + \mu_i^2\right) - IJ\left(\frac{\sigma^2}{IJ} + \mu^2\right)$

$= I\sigma^2 + J\Sigma\mu_i^2 - \sigma^2 - IJ\mu^2 = (I-1)\sigma^2 + J\Sigma(\mu_i - \mu)^2$, so

$E(\text{MSTr}) = \frac{E(\text{SSTr})}{I-1} = \sigma^2 + J\sum\frac{(\mu_i - \mu)^2}{I-1}$.

e. When H_0 is true, $\mu_1 = \cdots = \mu_i = \mu$, so $\Sigma(\mu_i - \mu)^2 = 0$ and $E(\text{MSTr}) = \sigma^2$. When H_0 is false, $\Sigma(\mu_i - \mu)^2 > 0$, so $E(\text{MSTr}) > \sigma^2$ (on average, MSTr overestimates σ^2).

Section 10.2

11. $Q_{.05,5,15} = 4.37$, $w = 4.37\sqrt{\frac{272.8}{4}} = 36.09$. The brands seem to divide into two groups: 1, 3, and 4; and 2 and 5; with no significant differences within each group but all between group differences are significant.

3	1	4	2	5
437.5	462.0	469.3	512.8	532.1

12. Brands 2 and 5 do not differ significantly from one another, but both differ significantly from brands 1, 3, and 4. While brands 3 and 4 do differ significantly, there is not enough evidence to indicate a significant difference between 1 and 3 or 1 and 4.

3	1	4	2	5
427.5	462.0	469.3	512.8	532.1

13. Brand 1 does not differ significantly from 3 or 4, 2 does not differ significantly from 4 or 5, 3 does not differ significantly from1, 4 does not differ significantly from 1 or 2, 5 does not differ significantly from 2, but all other differences (e.g., 1 with 2 and 5, 2 with 3, etc.) do appear to be significant.

3	1	4	2	5
427.5	462.0	469.3	502.8	532.1

14. We'll use $\alpha = .05$. In Example 10.3, $I = 5$ and $J = 10$, so the critical value is $Q_{.05,5,45} \approx Q_{.05,5,40} = 4.04$, and MSE = 37.926. So, $w \approx 4.04\sqrt{\frac{37.926}{10}} = 7.87$. Conveniently, the sample means are already in numerical order. Starting with 10.5, the lowest sample mean is only significantly different (by Tukey's method) from the highest mean, 21.6. No other differences are significant. The mean shear bond strengths for treatments 1 and 5 are significantly different, but no others are.

$\bar{x}_{1\cdot}$	$\bar{x}_{2\cdot}$	$\bar{x}_{3\cdot}$	$\bar{x}_{4\cdot}$	$\bar{x}_{5\cdot}$
10.5	14.8	15.7	16.0	21.6

15. In Exercise 10.7, $I = 6$ and $J = 26$, so the critical value is $Q_{.05,6,150} \approx Q_{.05,6,120} = 4.10$, and MSE = 13.929. So, $w \approx 4.10\sqrt{\frac{13.929}{26}} = 3.00$. So, sample means less than 3.00 apart will belong to the same underscored set. Three distinct groups emerge: the first mixture (in the above order), then mixtures 2-4, and finally mixtures 5-6.

14.18	17.94	18.00	18.00	25.74	27.67

16.

a. Since the largest standard deviation (s_4 = 44.51) is only slightly more than twice the smallest (s_3 = 20.83) it is plausible that the population variances are equal (see text, top of p. 395).

b. The relevant hypotheses are $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$ vs. H_a: at least two of the μ_i's are different. With the given f of 10.48 and associated P-value of 0.000, we can reject H_0 and conclude that there is a difference in axial stiffness for the different plate lengths.

c. There is no significant difference in the axial stiffness for lengths 4, 6, and 8, and for lengths 6, 8, and 10, yet 4 and 10 differ significantly. Length 12 differs from 4, 6, and 8, but does not differ from 10.

4	6	8	10	12
333.21	368.06	375.13	407.36	437.17
———	———	———		
	———	———	———	
			———	———

17. $\theta = \Sigma c_i \mu_i$ where $c_1 = c_2 = .5$ and $c_3 = -1$, so $\hat{\theta} = .5\bar{x}_{1.} + .5\bar{x}_{2.} - \bar{x}_{3.} = -.527$ and $\Sigma c_i^2 = 1.50$. With $t_{.025,27} = 2.052$ and MSE = .0660, the desired CI is (from (10.5))

$$-.527 \pm (2.052)\sqrt{\frac{(.0660)(1.50)}{10}} = -.527 \pm .204 = (-.731, -.323).$$

18.

a. Let μ_i = true average growth when hormone the ith is applied. The hypotheses are $H_0 : \mu_1 = ... = \mu_5$ versus H_a: at least two of the μ_i's are different. With $\frac{x_{..}^2}{IJ} = \frac{(278)^2}{20} = 3864.20$ and $\Sigma\Sigma x_{ij}^2 = 4280$, SST = 415.80. $\frac{\Sigma x_{i.}^2}{J} = \frac{(51)^2 + (71)^2 + (70)^2 + (46)^2 + (40)^2}{4} = 4064.50$, so SSTr = 4064.50 – 3864.20 = 200.3, and SSE = 415.80 – 200.30 = 215.50. Thus $\text{MSTr} = \frac{200.3}{4} = 50.075$, $\text{MSE} = \frac{215.5}{15} = 14.3667$, and

$$f = \frac{50.075}{14.3667} = 3.49.$$

At df = (4, 15), 3.05 < 3.49 < 4.89 ⇒ .01 < P-value < .05. In particular, we reject H_0 at the α = .05 level. There appears to be a difference in the average growth with the application of the different growth hormones.

b. $Q_{.05,5,15} = 4.37$, $w = 4.37\sqrt{\frac{14.3667}{4}} = 8.28$. The sample means are, in increasing order, 10.00, 11.50, 12.75, 17.50, and 17.75. The most extreme difference is 17.75 – 10.00 = 7.75 which doesn't exceed 8.28, so no differences are judged significant. Tukey's method and the F test are at odds.

19. MSTr = 140, error df = 12, so $f = \frac{140}{\text{SSE}/12} = \frac{1680}{\text{SSE}}$ and $F_{.05,2,12} = 3.89$.

$w = Q_{.05,3,12}\sqrt{\frac{\text{MSE}}{J}} = 3.77\sqrt{\frac{\text{SSE}}{60}} = .4867\sqrt{\text{SSE}}$. Thus we wish $\frac{1680}{SSE} > 3.89$ (significant f) and $.4867\sqrt{\text{SSE}} > 10$ (= 20 – 10, the difference between the extreme $\bar{x}_{i.}$'s, so no significant differences are identified). These become 431.88 > SSE and SSE > 422.16, so SSE = 425 will work.

20. Now MSTr = 125, so $f = \dfrac{1500}{\text{SSE}}$, $w = .4867\sqrt{\text{SSE}}$ as before, and the inequalities become 385.60 > SSE and SSE > 422.16. Clearly no value of SSE can satisfy both inequalities.

21.

a. The hypotheses are H_0: $\mu_1 = \ldots = \mu_6$ v. H_a: at least two of the μ_i's are different. Grand mean = 222.167, MSTr = 38,015.1333, MSE = 1,681.8333, and f = 22.6.
At df = (5, 78) ≈ (5, 60), 22.6 ≥ 4.76 ⇒ P-value < .001. Hence, we reject H_0. The data indicate there is a dependence on injection regimen.

b. Assume $t_{.005,78} \approx 2.645$.

i) Confidence interval for $\mu_1 - \frac{1}{5}\left(\mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6\right)$: $\Sigma c_i \overline{x}_i \pm t_{\alpha/2, I(J-1)} \sqrt{\dfrac{\text{MSE}\left(\Sigma c_i^2\right)}{J}}$

$$= -67.4 \pm (2.645)\sqrt{\frac{1,681.8333(1.2)}{14}} = (-99.16, -35.64).$$

ii) Confidence interval for $\frac{1}{4}\left(\mu_2 + \mu_3 + \mu_4 + \mu_5\right) - \mu_6$:

$$= 61.75 \pm (2.645)\sqrt{\frac{1,681.8333(1.25)}{14}} = (29.34, 94.16)$$

Section 10.3

22. Summary quantities are $x_{1.} = 291.4$, $x_{2.} = 221.6$, $x_{3.} = 203.4$, $x_{4.} = 227.5$, $x_{..} = 943.9$, $CF = 49,497.07$, $\Sigma\Sigma x_{ij}^2 = 50,078.07$, from which SST = 581,

$\text{SSTr} = \dfrac{(291.4)^2}{5} + \dfrac{(221.6)^2}{4} + \dfrac{(203.4)^2}{4} + \dfrac{(227.5)^2}{5} - 49,497.07 = 49,953.57 - 49,497.07 = 456.50$, and

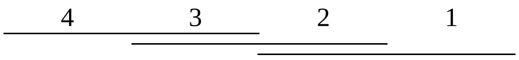
SSE = 124.50. Thus $\text{MSTr} = \dfrac{456.50}{3} = 152.17$, $\text{MSE} = \dfrac{124.50}{18-4} = 8.89$, and f = 17.12. Because $17.12 > F_{.001,3,14}$ = 9.73, P-value < .001 and $H_0 : \mu_1 = \ldots = \mu_4$ is rejected at level .05. There is a difference in true average yield of tomatoes for the four different levels of salinity.

23. $J_1 = 5$, $J_2 = 4$, $J_3 = 4$, $J_4 = 5$, $\overline{x}_{1.} = 58.28$, $\overline{x}_{2.} = 55.40$, $\overline{x}_{3.} = 50.85$, $\overline{x}_{4.} = 45.50$, MSE = 8.89.

With $W_{ij} = Q_{.05,4,14} \cdot \sqrt{\dfrac{\text{MSE}}{2}\left(\dfrac{1}{J_i} + \dfrac{1}{J_j}\right)} = 4.11\sqrt{\dfrac{8.89}{2}\left(\dfrac{1}{J_i} + \dfrac{1}{J_j}\right)}$,

$\overline{x}_{1.} - \overline{x}_{2.} \pm W_{12} = (2.88) \pm (5.81)$; $\overline{x}_{1.} - \overline{x}_{3.} \pm W_{13} = (7.43) \pm (5.81)$ *; $\overline{x}_{1.} - \overline{x}_{4.} \pm W_{14} = (12.78) \pm (5.48)$ *;
$\overline{x}_{2.} - \overline{x}_{3.} \pm W_{23} = (4.55) \pm (6.13)$; $\overline{x}_{2.} - \overline{x}_{4.} \pm W_{24} = (9.90) \pm (5.81)$ *; $\overline{x}_{3.} - \overline{x}_{4.} \pm W_{34} = (5.35) \pm (5.81)$.

A * indicates an interval that doesn't include zero, corresponding to μ's that are judged significantly different. This underscoring pattern does not have a very straightforward interpretation.



24. Let μ_i denote the true average skeletal-muscle activity the *i*th group (i = 1, 2, 3). The hypotheses are H_0: $\mu_1 = \mu_2 = \mu_3$ versus H_a: at least two of the μ_i's are different.

From the summary information provided, $\bar{x}_{..} = 51.10$, from which

$$\text{SSTr} = \sum_{i=1}^{3}\sum_{j=1}^{J_i}(\bar{x}_{i\cdot} - \bar{x}_{..})^2 = \sum_{i=1}^{3} J_i(\bar{x}_{i\cdot} - \bar{x}_{..})^2 = 797.1$$

Also,

$$\text{SSE} = \sum_{i=1}^{3}\sum_{j=1}^{J_i}(x_{ij} - \bar{x}_{i\cdot})^2 = \sum_{i=1}^{3}(J_i - 1)s_i^2 = 1319.7$$

The numerator and denominator df are $I - 1 = 2$ and $n - I = 28 - 3 = 25$, from which the F statistic is

$$f = \frac{\text{MSTr}}{\text{MSE}} = \frac{791.1/2}{1319.7/25} = 7.55.$$

Since $F_{.01,2,25} = 5.57$ and $F_{.001,2,25} = 9.22$, the P-value for this hypothesis test is between .01 and .001. There is strong evidence to suggest the population mean skeletal-muscle activity for these three groups is not the same.

To compare a group of size 10 to a group of size 8, Tukey's "honestly significant difference" at the .05 level is

$$w = Q_{.05,3,25}\sqrt{\frac{\text{MSE}}{2}\left(\frac{1}{10}+\frac{1}{8}\right)} \approx 3.53\sqrt{\frac{52.8}{2}\left(\frac{1}{10}+\frac{1}{8}\right)} = 8.60.$$

So, the "old, active" group has a significantly higher mean s-m activity than the other two groups, but young and old, sedentary populations are not significantly different in this regard.

Young	Old sedentary	Old active
46.68	47.71	58.24

25.

a. The distributions of the polyunsaturated fat percentages for each of the four regimens must be normal with equal variances.

b. We have all the $\bar{x}_{i.}$s, and we need the grand mean:

$$\bar{x}_{..} = \frac{8(43.0)+13(42.4)+17(43.1)+14(43.5)}{52} = \frac{2236.9}{52} = 43.017;$$

$$\text{SSTr} = \sum J_i\left(\bar{x}_{i.} - \bar{x}_{..}\right)^2 = 8(43.0-43.017)^2 + 13(42.4-43.017)^2$$

$$+17(43.1-43.017)^2 + 13(43.5-43.017)^2 = 8.334 \text{ and } \text{MSTr} = \frac{8.334}{3} = 2.778$$

$$\text{SSE} = \sum(J_i - 1)s^2 = 7(1.5)^2 + 12(1.3)^2 + 16(1.2)^2 + 13(1.2)^2 = 77.79 \text{ and } \text{MSE} = \frac{77.79}{48} = 1.621.$$

Then

$$f = \frac{\text{MSTr}}{\text{MSE}} = \frac{2.778}{1.621} = 1.714$$

Since $1.714 < F_{.10,3,50} = 2.20$, we can say that the P-value is > .10. We do not reject the null hypothesis at significance level .10 (or any smaller), so we conclude that the data suggests no difference in the percentages for the different regimens.

26.

a.

i:	1	2	3	4	5	6	
J_i:	4	5	4	4	5	4	
$x_{i.}$:	56.4	64.0	55.3	52.4	85.7	72.4	$x_{..} = 386.2$
$\overline{x}_{i.}$:	14.10	12.80	13.83	13.10	17.14	18.10	$\Sigma\Sigma x_{ij}^2 = 5850.20$

Thus SST = 113.64, SSTr = 108.19, SSE = 5.45, MSTr = 21.64, MSE = .273, f = 79.3. Since $79.3 \geq F_{.01,5,20} = 4.10$, P-value < .01 and $H_0 : \mu_1 = \cdots = \mu_6$ is rejected.

b. The modified Tukey intervals are as follows; the first number is $\overline{x}_{i.} - \overline{x}_{j.}$ and the second number is $W_{ij} = Q_{.01} \cdot \sqrt{\frac{\text{MSE}}{2}\left(\frac{1}{J_i} + \frac{1}{J_j}\right)}$:

Pair	Interval	Pair	Interval	Pair	Interval
1,2	1.30 ± 1.37	2,3	-1.03 ± 1.37	3,5	$-3.31 \pm 1.37*$
1,3	$.27 \pm 1.44$	2,4	$-.30 \pm 1.37$	3,6	$-4.27 \pm 1.44*$
1,4	1.00 ± 1.44	2,5	$-4.34 \pm 1.29*$	4,5	$-4.04 \pm 1.37*$
1,5	$-3.04 \pm 1.37*$	2,6	$-5.30 \pm 1.37*$	4,6	$-5.00 \pm 1.44*$
1,6	$-4.00 \pm 1.44*$	3,4	$.37 \pm 1.44$	5,6	$-.96 \pm 1.37$

Asterisks identify pairs of means that are judged significantly different from one another.

c. The confidence interval is $\Sigma c_i \overline{x}_{i.} \pm t_{\alpha/2,n-I}\sqrt{\text{MSE}\sum \frac{c_i^2}{J_i}}$.

$\Sigma c_i \overline{x}_{i.} = \frac{1}{4}\overline{x}_{1.} + \frac{1}{4}\overline{x}_{2.} + \frac{1}{4}\overline{x}_{3.} + \frac{1}{4}\overline{x}_{4.} - \frac{1}{2}\overline{x}_{5.} - \frac{1}{2}\overline{x}_{6.} = -4.16$, $\sum \frac{c_i^2}{J_i} = .1719$, MSE = .273, $t_{.025,20} = 2.086$.

The resulting 95% confidence interval is
$-4.16 \pm (2.845)\sqrt{(.273)(.1719)} = -4.16 \pm .45 = (-4.61, -3.71)$.

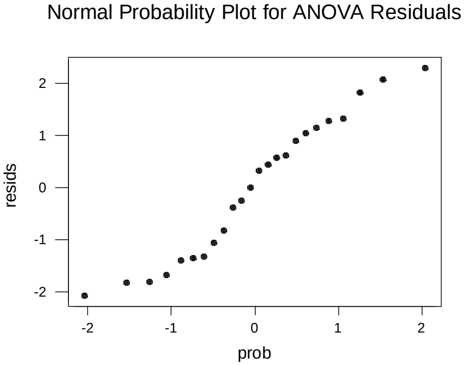
27.

a. Let μ_i = true average folacin content for specimens of brand i. The hypotheses to be tested are $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$ vs. H_a: at least two of the μ_i's are different . $\Sigma\Sigma x_{ij}^2 = 1246.88$ and $\frac{x_{..}^2}{n} = \frac{(168.4)^2}{24} = 1181.61$, so SST = 65.27; $\frac{\Sigma x_{i.}^2}{J_i} = \frac{(57.9)^2}{7} + \frac{(37.5)^2}{5} + \frac{(38.1)^2}{6} + \frac{(34.9)^2}{6} = 1205.10$, so $\text{SSTr} = 1205.10 - 1181.61 = 23.49$.

Source	df	SS	MS	F
Treatments	3	23.49	7.83	3.75
Error	20	41.78	2.09	
Total	23	65.27		

With numerator df = 3 and denominator df = 20, $F_{.05,3,20} = 3.10 < 3.75 < F_{.01,3,20} = 4.94$, so the *P*-value is between .01 and .05. We reject H_0 at the .05 level: at least one of the pairs of brands of green tea has different average folacin content.

b. With $\bar{x}_{i\cdot} = 8.27$, 7.50, 6.35, and 5.82 for *i* = 1, 2, 3, 4, we calculate the residuals $x_{ij} - \bar{x}_{i\cdot}$ for all observations. A normal probability plot appears below and indicates that the distribution of residuals could be normal, so the normality assumption is plausible. The sample standard deviations are 1.463, 1.681, 1.060, and 1.551, so the equal variance assumption is plausible (since the largest sd is less than twice the smallest sd).



c. $Q_{.05,4,20} = 3.96$ and $W_{ij} = 3.96 \cdot \sqrt{\dfrac{2.09}{2}\left(\dfrac{1}{J_i} + \dfrac{1}{J_j}\right)}$, so the Modified Tukey intervals are:

Pair	Interval	Pair	Interval
1,2	$.77 \pm 2.37$	2,3	$1.15 \pm 2.45a$
1,3	1.92 ± 2.25	2,4	1.68 ± 2.45
1,4	2.45 ± 2.25 *	3,4	$.53 \pm 2.34$

$$\begin{array}{cccc} \underline{4 \quad\; 3 \quad\; 2} & & & 1 \\ & \underline{3 \quad\; 2 \quad\; 1} & & \end{array}$$

Only Brands 1 and 4 are significantly different from each other.

28. $\text{SSTr} = \sum_i \left\{ \sum_j \left(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot} \right)^2 \right\} = \sum_i J_i \left(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot} \right)^2 = \sum_i J_i \bar{X}_{i\cdot}^2 - 2\bar{X}_{\cdot\cdot} \sum_i J_i \bar{X}_{i\cdot} + \bar{X}_{\cdot\cdot}^2 \sum_i J_i$

$= \sum_i J_i \bar{X}_{i\cdot}^2 - 2\bar{X}_{\cdot\cdot} X_{\cdot\cdot} + n\bar{X}_{\cdot\cdot}^2 = \sum_i J_i \bar{X}_{i\cdot}^2 - 2n\bar{X}_{\cdot\cdot}^2 + n\bar{X}_{\cdot\cdot}^2 = \sum_i J_i \bar{X}_{i\cdot}^2 - n\bar{X}_{\cdot\cdot}^2$.

29. $E\left(\text{SSTr}\right) = E\left(\sum_i J_i \bar{X}_{i\cdot}^2 - n\bar{X}_{\cdot\cdot}^2\right) = \Sigma J_i E\left(\bar{X}_{i\cdot}^2\right) - nE\left(\bar{X}_{\cdot\cdot}^2\right)$

$= \Sigma J_i \left[V\left(\bar{X}_{i\cdot}\right) + \left(E\left(\bar{X}_{i\cdot}\right)\right)^2\right] - n\left[V\left(\bar{X}_{\cdot\cdot}\right) + \left(E\left(\bar{X}_{\cdot\cdot}\right)\right)^2\right] = \sum J_i \left[\frac{\sigma^2}{J_i} + \mu_i^2\right] - n\left[\frac{\sigma^2}{n} + \left(\frac{\Sigma J_i \mu_i}{n}\right)^2\right]$

$= I\sigma^2 + \Sigma J_i \left(\mu + \alpha_i\right)^2 - \sigma^2 - \frac{1}{n}\left[\Sigma J_i \left(\mu + \alpha_i\right)\right]^2 = \left(I - 1\right)\sigma^2 + \Sigma J_i \mu^2 + 2\mu \Sigma J_i \alpha_i + \Sigma J_i \alpha_i^2 - \frac{1}{n}\left[n\mu + 0\right]^2$

$= \left(I - 1\right)\sigma^2 + \mu^2 n + 2\mu 0 + \Sigma J_i \alpha_i^2 - n\mu^2 = (I - 1)\sigma^2 + \Sigma J_i \alpha_i^2$, from which E(MSTr) is obtained through division by $(I - 1)$.

30.

a. $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = -1$, $\alpha_4 = 1$, so $\phi^2 = \frac{8\left(0^2 + 0^2 + \left(-1\right)^2 + 1^2\right)}{4} = 4$, $\phi = 2$, and from figure (10.5), power ≈ .92, so $\beta \approx .08$.

b. $\phi^2 = .5J$, so $\phi = .707\sqrt{J}$ and $\nu_2 = 4\left(J - 1\right)$. By inspection of figure (10.5), $J = 9$ looks to be sufficient.

c. $\mu_1 = \mu_2 = \mu_3 = \mu_4, \mu_5 = \mu_1 + 1$, so $\mu = \mu_1 + \frac{1}{5}$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\frac{1}{5}$, $\alpha_4 = \frac{4}{5}$,

$\phi^2 = \frac{10\left({}^{20}\!/_{25}\right)}{5} = 1.60$ $\phi = 1.26$, $\nu_1 = 4$, $\nu_2 = 45$.

By inspection of figure (10.6), power ≈ .5, so $\beta \approx .5$.

31. With $\sigma = 1$ (any other σ would yield the same ϕ), $\alpha_1 = -1$, $\alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1$,

$\phi^2 = \frac{1\left(5\left(-1\right)^2 + 4\left(0\right)^2 + 4\left(0\right)^2 + 5\left(1\right)^2\right)}{4} = 2.5$, $\phi = 1.58$, $\nu_1 = 3$, $\nu_2 = 14$, and power ≈ .65.

32. With Poisson data, the ANOVA should be performed using $y_{ij} = \sqrt{x_{ij}}$. This gives $y_{1\cdot} = 15.43$, $y_{2\cdot} = 17.15$, $y_{3\cdot} = 19.12$, $y_{4\cdot} = 20.01$, $y_{\cdot\cdot} = 71.71$, $\Sigma\Sigma y_{ij}^2 = 263.79$, CF = 257.12, SST = 6.67, SSTr = 2.52, SSE = 4.15, MSTr = .84, MSE = .26, $f = 3.23$. Since $3.23 \approx F_{.05,3,16}$, P-value ≈ .05 > .01 and H_0 cannot be rejected at the .01 level. The expected number of flaws per reel does not seem to depend upon the brand of tape.

33. $g(x) = x\left(1 - \frac{x}{n}\right) = nu\left(1 - u\right)$ where $u = \frac{x}{n}$, so $h(x) = \int\left[u\left(1 - u\right)\right]^{-1/2} du$. From a table of integrals, this gives $h(x) = \arcsin\left(\sqrt{u}\right) = \arcsin\left(\sqrt{\frac{x}{n}}\right)$ as the appropriate transformation.

34. $E\left(\text{MSTr}\right) = \sigma^2 + \frac{1}{I - 1}\left(n - \frac{IJ^2}{n}\right)\sigma_A^2 = \sigma^2 + \frac{n - J}{I - 1}\sigma_A^2 = \sigma^2 + J\sigma_A^2.$

Supplementary Exercises

35.

a. The hypotheses are $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$ v. H_a: at least two of the μ_i's are different. The calculated test statistic is $f = 3.68$. Since $F_{.05,3,20} = 3.10 < 3.68 < F_{.01,3,20} = 4.94$, the P-value is between .01 and .05. Thus, we fail to reject H_0 at $\alpha = .01$. At the 1% level, the means do not appear to differ significantly.

b. We reject H_0 when the P-value $\le \alpha$. Since .029 is not < .01, we still fail to reject H_0.

36. Let μ_1, μ_2, μ_3 denote the population mean cortisol concentrations for these three groups. The hypotheses are H_0: $\mu_1 = \mu_2 = \mu_3$ versus H_a: at least two of the μ_i's are different.

The grand mean is $\bar{x}_{..} = \dfrac{\sum x_{ij}}{n} = \dfrac{47(174.7)+36(160.2)+50(153.5)}{47+36+50} = \dfrac{21653.1}{133}$ = 162.8. From this,

SSTr = $\sum_{i=1}^{I}\sum_{j=1}^{J_i}(\bar{x}_{i.}-\bar{x}_{..})^2 = \sum_{i=1}^{I} J_i(\bar{x}_{i.}-\bar{x}_{..})^2$ = $47(174.7 – 162.8)^2 + 36(160.2 – 162.8)^2 + 50(153.5 – 162.8)^2$ = 11223.53, while

SSE = $\sum_{i=1}^{I}\sum_{j=1}^{J_i}(x_{ij}-\bar{x}_{i.})^2 = \sum_{i=1}^{I}(J_i-1)s_i^2$ = $(47 – 1)(50.9)^2 + (36 – 1)(37.2)^2 + (50 – 1)(45.9)^2$ = 270845.35.

The calculated value of the test statistic is $f = \dfrac{\text{MSTr}}{\text{MSE}} = \dfrac{\text{SSTr}/(I-1)}{\text{SSE}/(n-I)} = \dfrac{11223.53/(3-1)}{270845.35/(133-3)}$ = 2.69. Using approximate df (2, 130) ≈ (2, 100), 2.36 < 2.69 < 3.09 ⇒ .05 < P-value < .10. In particular, we cannot reject H_0 at the .05 level: there is insufficient evidence to conclude that the population mean cortisol levels differ for these three groups. [From software, the P-value is .071.]

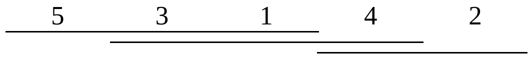
37. Let μ_i = true average amount of motor vibration for each of five bearing brands. Then the hypotheses are $H_0 : \mu_1 = ... = \mu_5$ vs. H_a: at least two of the μ_i's are different. The ANOVA table follows:

Source	df	SS	MS	F
Treatments	4	30.855	7.714	8.44
Error	25	22.838	0.914	
Total	29	53.694		

8.44 > $F_{.001,4,25}$ = 6.49, so P-value < .001 < .05, so we reject H_0. At least two of the means differ from one another. The Tukey multiple comparisons are appropriate. $Q_{.05,5,25} = 4.15$ from Minitab output; or, using Table A.10, we can approximate with $Q_{.05,5,24} = 4.17$. $W_{ij} = 4.15\sqrt{.914/6} = 1.620$.

Pair	$\bar{x}_{i.} - \bar{x}_{j.}$	Pair	$\bar{x}_{i.} - \bar{x}_{j.}$
1,2	-2.267*	2,4	1.217
1,3	0.016	2,5	2.867*
1,4	-1.050	3,4	-1.066
1,5	0.600	3,5	0.584
2,3	2.283*	4,5	1.650*

*Indicates significant pairs.



38. $x_{1.} = 15.48$, $x_{2.} = 15.78$, $x_{3.} = 12.78$, $x_{4.} = 14.46$, $x_{5.} = 14.94$ $x_{..} = 73.44$, so CF = 179.78, SST = 3.62, SSTr = 180.71 – 179.78 = .93, SSE = 3.62 – .93 = 2.69.

Source	df	SS	MS	F
Treatments	4	.93	.233	2.16
Error	25	2.69	.108	
Total	29	3.62		

Since 2.16 < $F_{.10,4,25}$ = 2.18, P-value > .10 > .05. Do not reject H_0 at level .05.

39. $\hat{\theta} = 2.58 - \frac{2.63+2.13+2.41+2.49}{4} = .165$, $t_{.025,25} = 2.060$, MSE = .108, and $\Sigma c_i^2 = (1)^2 + (-.25)^2 + (-.25)^2 + (-.25)^2 + (-.25)^2 = 1.25$, so a 95% confidence interval for θ is $.165 \pm 2.060\sqrt{\frac{(.108)(1.25)}{6}} = .165 \pm .309 = (-.144, .474)$. This interval does include zero, so 0 is a plausible value for θ.

40. $\mu_1 = \mu_2 = \mu_3, \mu_4 = \mu_5 = \mu_1 - \sigma$, so $\mu = \mu_1 - \frac{2}{5}\sigma$, $\alpha_1 = \alpha_2 = \alpha_3 = \frac{2}{5}\sigma$, $\alpha_4 = \alpha_5 = -\frac{3}{5}\sigma$. Then $\phi^2 = \frac{J}{I}\sum\frac{\alpha_i^2}{\sigma^2} = \frac{6}{5}\left[\frac{3\left(\frac{2}{5}\sigma\right)^2}{\sigma^2} + \frac{2\left(-\frac{3}{5}\sigma\right)^2}{\sigma^2}\right] = 1.632$ and $\phi = 1.28$, $\nu_1 = 4$, $\nu_2 = 25$. By inspection of figure (10.6), power ≈ .48, so $\beta \approx .52$.

41. This is a random effects situation. $H_0 : \sigma_A^2 = 0$ states that variation in laboratories doesn't contribute to variation in percentage. SST = 86,078.9897 – 86,077.2224 = 1.7673, SSTr = 1.0559, and SSE = .7114. At df = (3, 8), 2.92 < 3.96 < 4.07 ⇒ .05 < P-value < .10, so H_0 cannot be rejected at level .05. Variation in laboratories does not appear to be present.

42.

a. μ_i = true average CFF for the three iris colors. Then the hypotheses are $H_0 : \mu_1 = \mu_2 = \mu_3$ vs. H_a: at least two of the μ_i's are different. SST = 13,659.67 – 13,598.36 = 61.31, $\text{SSTr} = \left(\frac{(204.7)^2}{8} + \frac{(134.6)^2}{5} + \frac{(169.0)^2}{6}\right) - 13{,}598.36 = 23.00$ The ANOVA table follows:

Source	df	SS	MS	F
Treatments	2	23.00	11.50	4.803
Error	16	38.31	2.39	
Total	18	61.31		

Because $F_{.05,2,16} = 3.63 < 4.803 < F_{.01,2,16} = 6.23$, .01 < P-value < .05, so we reject H_0. There are differences in CFF based on iris color.

b. $Q_{.05,3,16} = 3.65$ and $W_{ij} = 3.65 \cdot \sqrt{\dfrac{2.39}{2}\left(\dfrac{1}{J_i} + \dfrac{1}{J_j}\right)}$, so the modified Tukey intervals are:

Pair	$\left(\bar{x}_{i\cdot} - \bar{x}_{j\cdot}\right) \pm W_{ij}$
1,2	-1.33 ± 2.27
1,3	-2.58 ± 2.15 *
2,3	-1.25 ± 2.42

Brown	Green	Blue
25.59	26.92	28.17

The CFF is only significantly different for brown and blue iris colors.

43. $\sqrt{(I-1)(\text{MSE})\left(F_{.05,I-1,n-I}\right)} = \sqrt{(2)(2.39)(3.63)} = 4.166$. For $\mu_1 - \mu_2$, $c_1 = 1$, $c_2 = -1$, and $c_3 = 0$, so $\sqrt{\sum \dfrac{c_i^2}{J_i}} = \sqrt{\dfrac{1}{8} + \dfrac{1}{5}} = .570$. Similarly, for $\mu_1 - \mu_3$, $\sqrt{\sum \dfrac{c_i^2}{J_i}} = \sqrt{\dfrac{1}{8} + \dfrac{1}{6}} = .540$; for $\mu_2 - \mu_3$, $\sqrt{\sum \dfrac{c_i^2}{J_i}} = \sqrt{\dfrac{1}{5} + \dfrac{1}{6}} = .606$, and for $.5\mu_2 + .5\mu_2 - \mu_3$, $\sqrt{\sum \dfrac{c_i^2}{J_i}} = \sqrt{\dfrac{.5^2}{8} + \dfrac{.5^2}{5} + \dfrac{(-1)^2}{6}} = .498$.

Contrast	Estimate	Interval
$\mu_1 - \mu_2$	25.59 – 26.92 = –1.33	$(-1.33) \pm (.570)(4.166) = (-3.70, 1.04)$
$\mu_1 - \mu_3$	25.59 – 28.17 = –2.58	$(-2.58) \pm (.540)(4.166) = (-4.83, -.33)$
$\mu_2 - \mu_3$	26.92 – 28.17 = –1.25	$(-1.25) \pm (.606)(4.166) = (-3.77, 1.27)$
$.5\mu_2 + .5\mu_2 - \mu_3$	–1.92	$(-1.92) \pm (.498)(4.166) = (-3.99, 0.15)$

The contrast between μ_1 and μ_3, since the calculated interval is the only one that does not contain 0.

44.

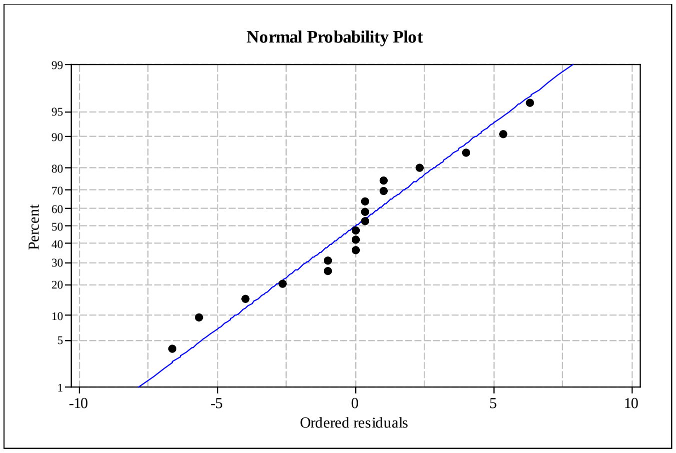
Source	df	SS	MS	*F*
Treatments	3	24,937.63	8312.54	1117.8
Error	8	59.49	7.44	
Total	11	24,997.12		

Because 1117.8 > $F_{.001,3,8}$ = 15.83, *P*-value < .001 and $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$ is rejected.

$Q_{.05,4,8} = 4.53$, so $w = 4.53\sqrt{\dfrac{7.44}{3}} = 7.13$. The four sample means are $\bar{x}_{4\cdot} = 29.92$, $\bar{x}_{1\cdot} = 33.96$, $\bar{x}_{3\cdot} = 115.84$, and $\bar{x}_{2\cdot} = 129.30$. Only $\bar{x}_{1\cdot} - \bar{x}_{4\cdot} < 7.13$, so all means are judged significantly different from one another except for μ_4 and μ_1 (corresponding to PCM and OCM).

45. $Y_{ij} - \overline{Y}_{..} = c\left(X_{ij} - \overline{X}_{..}\right)$ and $\overline{Y}_{i.} - \overline{Y}_{..} = c\left(\overline{X}_{i.} - \overline{X}_{..}\right)$, so each sum of squares involving *Y* will be the corresponding sum of squares involving *X* multiplied by c^2. Since *F* is a ratio of two sums of squares, c^2 appears in both the numerator and denominator. So c^2 cancels, and *F* computed from Y_{ij}'s = *F* computed from X_{ij}'s.

46. The ordered residuals are –6.67, –5.67, –4, –2.67, –1, –1, 0, 0, 0, .33, .33, .33, 1, 1, 2.33, 4, 5.33, 6.33. The corresponding *z* percentiles are –1.91, –1.38, –1.09, –.86, –.67, –.51, –.36, –.21, –.07, .07, .21, .36, .51, .67, .86, 1.09, 1.38, and 1.91. The resulting plot of the respective pairs (the Normal Probability Plot) has some curvature to it, but not enough to invalidate the normality assumption. (A formal test of normality has an approximate *P*-value of .2.) Minitab's version of the residual plot appears below.



CHAPTER 11

Section 11.1

1.

a. The test statistic is $f_A = \frac{\text{MSA}}{\text{MSE}} = \frac{\text{SSA}/(I-1)}{\text{SSE}/(I-1)(J-1)} = \frac{442.0/(4-1)}{123.4/(4-1)(3-1)}$ = 7.16. Compare this to the F distribution with df = (4 –1, (4 – 1)(3 – 1)) = (3, 6): 4.76 < 7.16 < 9.78 ⇒ .01 < P-value < .05. In particular, we reject $H_{0\text{A}}$ at the .05 level and conclude that at least one of the factor A means is different (equivalently, at least one of the α_i's is not zero).

b. Similarly, $f_B = \frac{\text{SSB}/(J-1)}{\text{SSE}/(I-1)(J-1)} = \frac{428.6/(3-1)}{123.4/(4-1)(3-1)}$ = 10.42. At df = (2, 6), 5.14 < 10.42 < 10.92 ⇒ .01 < P-value < .05. In particular, we reject $H_{0\text{B}}$ at the .05 level and conclude that at least one of the factor B means is different (equivalently, at least one of the β_j's is not zero).

2.

a. $x_{1.}$ = 163, $x_{2.}$ = 152, $x_{3.}$ = 142, $x_{4.}$ = 146, $x_{.1}$ = 215, $x_{.2}$ = 188, $x_{.3}$ = 200, $x_{..}$ = 603, $\Sigma\Sigma x_{ij}^2$ = 30599, CF = $(603)^2/12$ = 30300.75 ⇒ SST = 298.25, SSA = $[163^2 + 152^2 + 142^2 + 146^2]/3$ – 30300.75 = 83.58, SSB = 30392.25 – 30300.75 = 91.50, SSE = 298.25 – 83.58 – 91.50 = 123.17.

Source	df	SS	MS	*F*
A	3	83.58	27.86	1.36
B	2	91.50	45.75	2.23
Error	6	123.17	20.53	
Total	11	298.25		

Since 1.35 < $F_{.05,3,6}$ = 4.76, the P-value for the factor A test is > .05. Since 2.23 < $F_{.02,3,6}$ = 5.14, the P-value for the factor B test is also > .05.Therefore, neither $H_{0\text{A}}$ nor $H_{0\text{B}}$ is rejected.

b. $\hat{\mu} = \overline{x}_{..} = 50.25$, $\hat{\alpha}_1 = \overline{x}_{1.} - \overline{x}_{..} = 4.08$, $\hat{\alpha}_2 = .42$, $\hat{\alpha}_3 = -2.92$, $\hat{\alpha}_4 = -1.58$, $\hat{\beta}_1 = \overline{x}_{.1} - \overline{x}_{..} = 3.50$, $\hat{\beta}_2 = -3.25$, $\hat{\beta}_3 = -.25$.

3.

a. The entries of this ANOVA table were produced with software.

Source	**df**	**SS**	**MS**	***F***
Medium	1	0.053220	0.0532195	18.77
Current	3	0.179441	0.0598135	21.10
Error	3	0.008505	0.0028350	
Total	7	0.241165		

To test H_{0A}: $\alpha_1 = \alpha_2 = 0$ (no liquid medium effect), the test statistic is $f_A = 18.77$; at df = (1, 3), the *P*-value is .023 from software (or between .01 and .05 from Table A.9). Hence, we reject H_{0A} and conclude that medium (oil or water) affects mean material removal rate.

To test H_{0B}: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ (no current effect), the test statistic is $f_B = 21.10$; at df = (3, 3), the *P*-value is .016 from software (or between .01 and .05 from Table A.9). Hence, we reject H_{0B} and conclude that working current affects mean material removal rate as well.

b. Using a .05 significance level, with $J = 4$ and error df = 3 we require $Q_{.05,4,3} = 6.825$. Then, the metric for significant differences is $w = 6.825\sqrt{0.0028530/2} = 0.257$. The means happen to increase with current; sample means and the underscore scheme appear below.

Current:	10	15	20	25
$\overline{x}_{.j}$:	0.201	0.324	0.462	0.602
	———	———		
		———	———	
			———	———

4.

a. The entries of this ANOVA table were produced with software.

Source	**df**	**SS**	**MS**	***F***
Paint Brand	3	159.58	53.19	7.85
Roller Brand	2	38.00	19.00	2.80
Error	6	40.67	6.78	
Total	11	238.25		

b. At df = (3, 6), .01 < *P*-value < .05. At the .05 level, we reject H_{0A}: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. The mean amount of coverage depends on the paint brand.

c. At df = (2, 6), *P*-value > .1. At the .05 level, do <u>not</u> reject H_{0B}: $\beta_1 = \beta_2 = \beta_3 = 0$. The mean amount of coverage does not depend significantly on the roller brand.

d. Because H_{0B} was not rejected. Tukey's method is used only to identify differences in levels of factor A (brands of paint). $Q_{.05,4,6} = 4.90$, from which $w = 7.37$.

i:	4	3	2	1
$\overline{x}_{i.}$:	41.3	41.7	45.7	50.3
	———	———	———	

Brand 1 differs significantly from all other brands.

5.

Source	**df**	**SS**	**MS**	***F***
Angle	3	58.16	19.3867	2.5565
Connector	4	246.97	61.7425	8.1419
Error	12	91.00	7.5833	
Total	19	396.13		

We're interested in $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus H_a: at least one $\alpha_i \neq 0$. $f_A = 2.5565 < F_{.01,3,12} = 5.95 \Rightarrow$ *P*-value > .01, so we fail to reject H_0. The data fails to indicate any effect due to the angle of pull, at the .01 significance level.

6.

a. $MSA = \frac{11.7}{2} = 5.85$, $MSE = \frac{25.6}{8} = 3.20$, $f_A = \frac{5.85}{3.20} = 1.83$, which is not significant at level .05.

b. Otherwise extraneous variation associated with houses would tend to interfere with our ability to assess assessor effects. If there really was a difference between assessors, house variation might have hidden such a difference. Alternatively, an observed difference between assessors might have been due just to variation among houses and the manner in which assessors were allocated to homes.

7.

a. The entries of this ANOVA table were produced with software.

Source	**df**	**SS**	**MS**	***F***
Brand	2	22.8889	11.4444	8.96
Operator	2	27.5556	13.7778	10.78
Error	4	5.1111	1.2778	
Total	8	55.5556		

The calculated test statistic for the *F*-test on brand is $f_A = 8.96$. At df = (2, 4), the *P*-value is .033 from software (or between .01 and .05 from Table A.9). Hence, we reject H_0 at the .05 level and conclude that lathe brand has a statistically significant effect on the percent of acceptable product.

b. The block-effect test statistic is $f = 10.78$, which is quite large (a *P*-value of .024 at df = (2, 4)). So, yes, including this operator blocking variable was a good idea, because there is significant variation due to different operators. If we had not controlled for such variation, it might have affected the analysis and conclusions.

8.

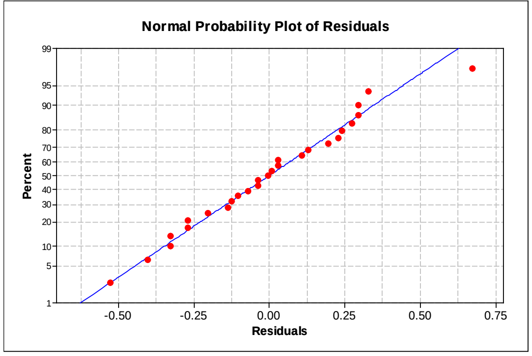
a. Software gives the output below. In particular, against the null hypothesis H_{0A}: $\mu_{\text{cold}} = \mu_{\text{neutral}} = \mu_{\text{hot}}$, $f_A = 1.03370/0.11787 = 8.77$ with associated *P*-value = .003 at df = (2, 16). Hence, we reject H_{0A} at the 1% significance level and conclude, after blocking by subject, that temperature does affect true average body mass loss.

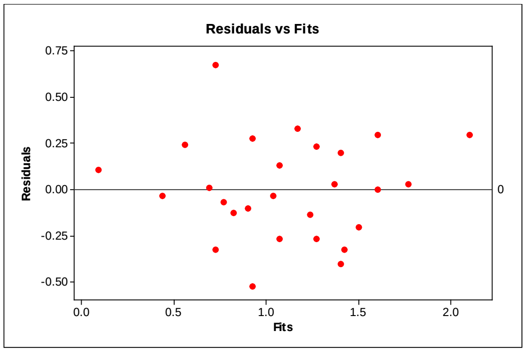
Two–way ANOVA: BMLoss versus Temp, Subject

```
Source   DF       SS       MS     F      P
Temp      2  2.06741  1.03370  8.77  0.003
Subject   8  2.98519  0.37315  3.17  0.024
Error    16  1.88593  0.11787
Total    26  6.93852
```

b. $Q_{.05,3,16} = 3.65$, so $w = 3.65\sqrt{0.11787/9} = .418$. $\bar{x}_{1.} = .767$, $\bar{x}_{2.} = 1.111$, and $\bar{x}_{3.} = 1.444$. So the only significant difference is between average body mass loss at 6°C and at 30°C.

c. A normal probability plot of the residuals shows that, with the exception of one large residual, normality is plausible. A residual versus fit plot substantiates the assumption of constant variance.





9. The entries of this ANOVA table were produced with software.

Source	**df**	**SS**	**MS**	***F***
Treatment	3	81.1944	27.0648	22.36
Block	8	66.5000	8.3125	6.87
Error	24	29.0556	1.2106	
Total	35	176.7500		

At df = (3, 24), $f = 22.36 > 7.55 \Rightarrow$ *P*-value < .001. Therefore, we strongly reject H_{0A} and conclude that there is an effect due to treatments. We follow up with Tukey's procedure:

$Q_{.05,4,24} = 3.90$; $w = 3.90\sqrt{1.2106/9} = 1.43$

1	4	3	2
8.56	9.22	10.78	12.44

10. The entries of this ANOVA table were produced with software.

Source	**df**	**SS**	**MS**	***F***
Method	2	23.23	11.61	8.69
Batch	9	86.79	9.64	7.22
Error	18	24.04	1.34	
Total	29	134.07		

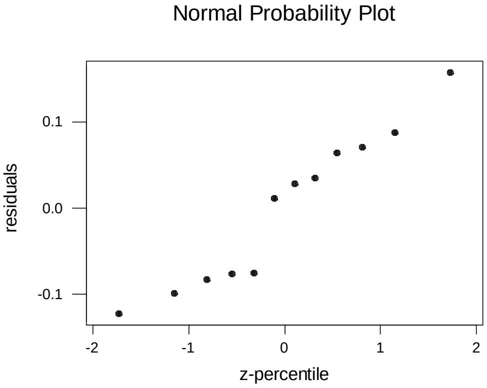
$F_{.01,2,18} = 6.01 < 8.69 < F_{.001,2,18} = 10.33 \Rightarrow .001 <$ *P*-value < .01. We reject H_{0A} and conclude that at least two of the curing methods produce differing average compressive strengths. (Note: With *P*-value < .001, there are differences between batches as well — that is, blocking by batch effective reduced variation.)

$Q_{.05,3,18} = 3.61$; $w = 3.61\sqrt{1.34/10} = 1.32$

Method A	Method B	Method C
29.49	31.31	31.40

Methods B and C produce strengths that are not significantly different, but Method A produces strengths that are different (less) than those of both B and C.

11. The residual, percentile pairs are (–0.1225, –1.73), (–0.0992, –1.15), (–0.0825, –0.81), (–0.0758, –0.55), (–0.0750, –0.32), (0.0117, –0.10), (0.0283, 0.10), (0.0350, 0.32), (0.0642, 0.55), (0.0708, 0.81), (0.0875, 1.15), (0.1575, 1.73).



The pattern is sufficiently linear, so normality is plausible.

12. $MSB = \frac{113.5}{4} = 28.38$ and $MSE = \frac{25.6}{8} = 3.20$ $\Rightarrow f_B = 8.87$. At df = (4, 8), 8.87 > 7.01 $\Rightarrow$ *P*-value < .01. Thus, we reject H_{0B} and conclude that $\sigma_B^2 > 0$.

13.

a. With $Y_{ij} = X_{ij} + d$, $\overline{Y}_{i.} = \overline{X}_{i.} + d$ and $\overline{Y}_{.j} = \overline{X}_{.j} + d$ and $\overline{Y}_{..} = \overline{X}_{..} + d$, so all quantities inside the parentheses in (11.5) remain unchanged when the *Y* quantities are substituted for the corresponding *X*'s (e.g., $\overline{Y}_{i.} - \overline{Y}_{..} = \overline{X}_{i.} - \overline{X}_{..}$, etc.).

b. With $Y_{ij} = cX_{ij}$, each sum of squares for *Y* is the corresponding SS for *X* multiplied by c^2. However, when *F* ratios are formed the c^2 factors cancel, so all *F* ratios computed from *Y* are identical to those computed from *X*. If $Y_{ij} = cX_{ij} + d$, the conclusions reached from using the *Y*'s will be identical to those reached using the *X*'s.

14.

$$E\left(\overline{X}_{i.} - \overline{X}_{..}\right) = E\left(\overline{X}_{i.}\right) - E\left(\overline{X}_{..}\right) = \frac{1}{J}E\left(\sum_j X_{ij}\right) - \frac{1}{IJ}E\left(\sum_i\sum_j X_{ij}\right)$$
$$= \frac{1}{J}\sum_j\left(\mu + \alpha_i + \beta_j\right) - \frac{1}{IJ}\sum_i\sum_j\left(\mu + \alpha_i + \beta_j\right)$$
$$= \mu + \alpha_i + \frac{1}{J}\sum_j \beta_j - \mu - \frac{1}{I}\sum_i \alpha_i - \frac{1}{J}\sum_j \beta_j = \alpha_i$$

as desired.

15.

a. $\Sigma\alpha_i^2 = 24$, so $\phi^2 = \left(\frac{3}{4}\right)\left(\frac{24}{16}\right) = 1.125$, $\phi = 1.06$, $\nu_1 = 3$, $\nu_2 = 6$, and from Figure 10.5, power $\approx .2$.

For the second alternative, $\phi = 1.59$, and power $\approx .43$.

b. $\phi^2 = \left(\frac{I}{J}\right)\sum\frac{\beta_j^2}{\sigma^2} = \left(\frac{4}{5}\right)\left(\frac{20}{16}\right) = 1.00$, so $\phi = 1.00$, $\nu_1 = 4$, $\nu_2 = 12$, and power $\approx .3$.

Section 11.2

16.

a.

Source	**df**	**SS**	**MS**	***F***
A	2	30,763.0	15,381.50	3.79
B	3	34,185.6	11,395.20	2.81
AB	6	43,581.2	7263.53	1.79
Error	24	97,436.8	4059.87	
Total	35	205,966.6		

b. The test statistic is $f_{AB} = 1.79$. $1.79 < F_{.05,6,24} = 2.51 \Rightarrow$ *P*-value > .05, so H_{0AB} cannot be rejected, and we conclude that no statistically significant interaction is present.

c. The test statistic is $f_A = 3.79$. $3.79 > F_{.05,2,24} = 3.40 \Rightarrow$ *P*-value < .05, so H_{0A} is rejected at level .05. There is a statistically significant factor A main effect.

d. The test statistic is $f_B = 2.81$. $2.81 < F_{.05,3,24} = 3.01 \Rightarrow$ *P*-value > .05, so H_{0B} is not rejected. There is not a statistically significant factor B main effect.

e. $Q_{.05,3,24} = 3.53$, $w = 3.53\sqrt{4059.87/12} = 64.93$.

3	1	2
3960.02	4010.88	4029.10
———————	———————	
	———————	———————

Only times 2 and 3 yield significantly different strengths.

17.

a.

Source	**df**	**SS**	**MS**	***F***	***P*-value**
Sand	2	705	352.5	3.76	.065
Fiber	2	1,278	639.0	6.82	.016
Sand × Fiber	4	279	69.75	0.74	.585
Error	9	843	93.67		
Total	17	3,105			

P-values were obtained from software; approximations can also be acquired using Table A.9. There appears to be an effect due to carbon fiber addition, but not due to any other effect (interaction effect or sand addition main effect).

b.

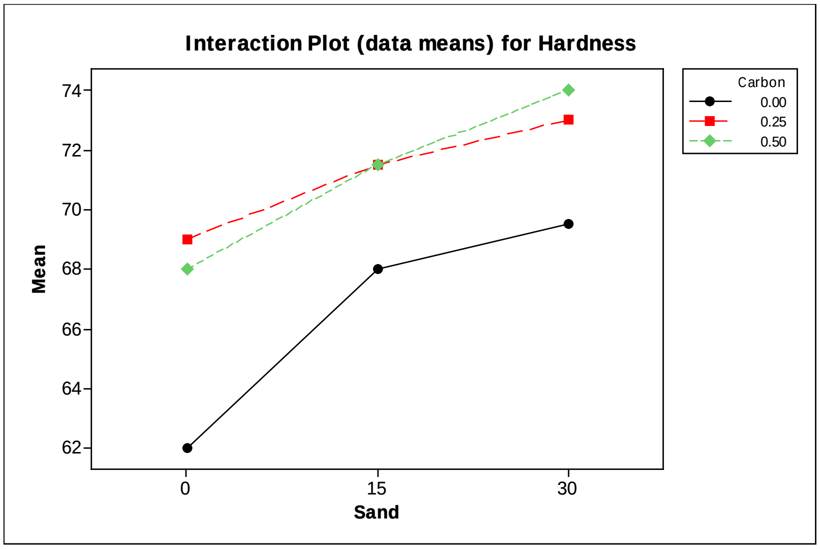
Source	**df**	**SS**	**MS**	***F***	***P*-value**
Sand	2	106.78	53.39	6.54	.018
Fiber	2	87.11	43.56	5.33	.030
Sand × Fiber	4	8.89	2.22	0.27	.889
Error	9	73.50	8.17		
Total	17	276.28			

There appears to be an effect due to both sand and carbon fiber addition to casting hardness, but no interaction effect.

c.

Sand%	0	15	30	0	15	30	0	15	30
Fiber%	0	0	0	0.25	0.25	0.25	0.5	0.5	0.5
$\bar{x}$	62	68	69.5	69	71.5	73	68	71.5	74

The plot below indicates some effect due to sand and fiber addition with no significant interaction. This agrees with the statistical analysis in part **b**.



18.

Source	**df**	**SS**	**MS**	***F***
Formulation	1	2,253.44	2,253.44	376.20
Speed	2	230.81	115.41	19.27
Formulation × Speed	2	18.58	9.29	1.55
Error	12	71.87	5.99	
Total	17	2,574.70		

a. There appears to be no interaction between the two factors: $f_{AB} = 1.55$, df = (2, 12), *P*-value = .25.

b. Both formulation and speed appear to have a highly statistically significant effect on yield (the calculated *F* statistics are enormous).

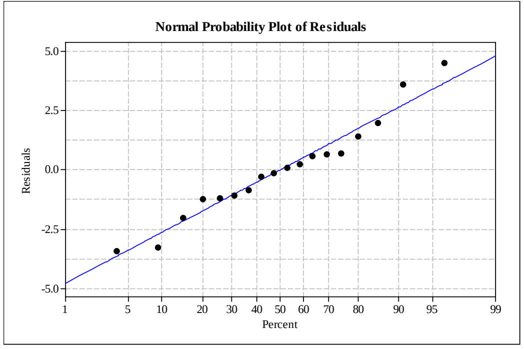
c. Let Factor A = formulation and Factor B = speed. Begin by estimating the μ's:
$\hat{\mu} = \bar{x}_{\ldots} = 175.84$; $\hat{\mu}_{1\cdot} = \frac{1}{3}\sum_j \bar{x}_{1j\cdot} = 187.03$ and $\hat{\mu}_{2\cdot} = 164.66$; $\hat{\mu}_{\cdot 1} = \frac{1}{2}\sum_i \bar{x}_{i1\cdot} = 177.83$, $\hat{\mu}_{\cdot 2} = 170.82$, and $\hat{\mu}_{\cdot 3} = 178.88$.

Since $\alpha_i = \mu_{i\cdot} - \mu$, $\hat{\alpha}_1 = 187.03 - 175.84 = 11.19$ and $\hat{\alpha}_2 = 164.66 - 175.84 = -11.18$; these sum to 0 except for rounding error. Similarly, $\hat{\beta}_1 = \hat{\mu}_{\cdot 1} - \hat{\mu} = 177.83 - 175.84 = 1.99$, $\hat{\beta}_2 = -5.02$, and $\hat{\beta}_3 = 3.04$; these sum to 0 except for rounding error.

d. Using $\gamma_{ij} = \mu_{ij} - (\mu + \alpha_i + \beta_j)$ and techniques similar to above, we find the following estimates of the interaction effects: $\hat{\gamma}_{11} = .45$, $\hat{\gamma}_{12} = -1.41$, $\hat{\gamma}_{13} = .96$, $\hat{\gamma}_{21} = -.45$, $\hat{\gamma}_{22} = 1.39$, and $\hat{\gamma}_{23} = -.97$. Again, there are some minor rounding errors.

Observed	**Fitted**	**Residual**	**Observed**	**Fitted**	**Residual**
189.7	189.47	0.23	161.7	161.03	0.67
188.6	189.47	–0.87	159.8	161.03	–1.23
190.1	189.47	0.63	161.6	161.03	0.57
165.1	166.20	–1.1	189.0	191.03	–2.03
165.9	166.20	–0.3	193.0	191.03	1.97
167.6	166.20	1.4	191.1	191.03	0.07
185.1	180.60	4.5	163.3	166.73	–3.43
179.4	180.60	–1.2	166.6	166.73	–0.13
177.3	180.60	–3.3	170.3	166.73	3.57

e. The residual plot is reasonably linear, suggesting the true errors could be assumed normal.



19.

Source	**df**	**SS**	**MS**	***F***
Farm Type	2	35.75	17.875	0.94
Tractor Maint. Method	5	861.20	172.240	9.07
Type × Method	10	603.51	60.351	3.18
Error	18	341.82	18.990	
Total	35	1842.28		

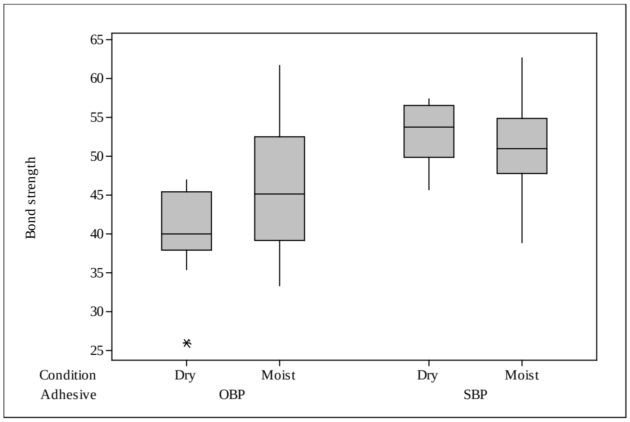
For the interaction effect, $f_{AB} = 3.18$ at df = (10, 18) gives P-value = .016 from software. Hence, we do not reject H_{0AB} at the .01 level (although just barely). This allows us to proceed to the main effects.

For the factor A main effect, $f_A = 0.94$ at df = (2, 18) gives P-value = .41 from software. Hence, we clearly fail to reject H_{0A} at the .01 level — there is not statistically significant effect due to type of farm.

Finally, $f_B = 9.07$ at df = (5, 18) gives P-value < .0002 from software. Hence, we strongly reject H_{0B} at the .01 level — there is a statistically significant effect due to tractor maintenance method.

20.

a. The accompanying comparative box plot indicates that bond strength is typical greater with SBP adhesive than with OBP adhesive, but that the "condition" of the adhesive (dry or moist) doesn't have much of an impact.

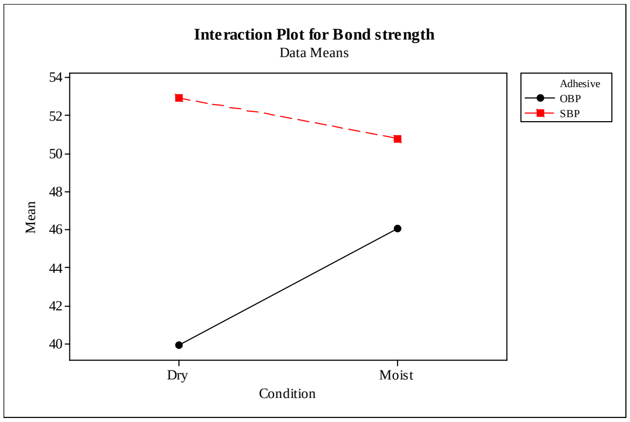


b. Minitab provides the accompanying two-way ANOVA table. At the .01 level, there is not a statistically significant interaction between adhesive and condition's effects on shear bond strength (although it's worth noting that the P-value of .031 is not large). Ignoring the interaction effect, condition (dry/moist) is clearly not statistically significant, while adhesive (OBP/SBP) is highly statistically significant.

Two-way ANOVA: Bond strength versus Adhesive, Condition

```
Source       DF       SS       MS      F      P
Adhesive      1   951.41  951.410  22.85  0.000
Condition     1    48.20   48.200   1.16  0.288
Interaction   1   207.92  207.917   4.99  0.031
Error        44  1832.32   41.644
Total        47  3039.85
```

An interaction plot reinforces that SBP adhesive is superior to OBP adhesive. The slopes of the two line segments aren't very steep, suggesting a negligible "condition effect"; however, the fact that the slopes have opposite signs (and, in particular, are clearly different) speaks to the interaction effect: OBP works better moist, while SBP works better dry.



c. Minitab performs a one-way ANOVA as summarized below. Consistent with the two-way ANOVA, we reject the hypothesis that “type” (i.e., adhesive-condition pairing) has no effect on bond strength. Using Tukey’s procedure results in the underscoring scheme seen below. The OBP-D setting is statistically significantly different at the .05 level from either of the SBP settings, but no other differences are statistically significant.

One-way ANOVA: Bond strength versus Type

```
Source  DF      SS     MS     F      P
Type     3  1207.5  402.5  9.67  0.000
Error   44  1832.3   41.6
Total   47  3039.8
```

OBP-D	OBP-M	SBP-M	SBP-D
39.9	46.1	50.8	53.0

21. From the provided SS, $SSAB = 64{,}954.70 - \left[22{,}941.80 + 22{,}765.53 + 15{,}253.50\right] = 3993.87$. This allows us to complete the ANOVA table below.

Source	**df**	**SS**	**MS**	***F***
A	2	22,941.80	11,470.90	22.98
B	4	22,765.53	5691.38	11.40
AB	8	3993.87	499.23	.49
Error	15	15,253.50	1016.90	
Total	29	64,954.70		

f_{AB} = .49 is clearly not significant. Since $22.98 \geq F_{.05,2,8} = 4.46$, the *P*-value for factor A is < .05 and H_{0A} is rejected. Since $11.40 \geq F_{.05,4,8} = 3.84$, the *P*-value for factor B is < .05 and H_{0B} is also rejected. We conclude that the different cement factors affect flexural strength differently and that batch variability contributes to variation in flexural strength.

22. This is a mixed effects model. In particular, the relevant null hypotheses are $H_{0A}: \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$; $H_{0B}: \sigma_B^2 = 0$; $H_{0AB}: \sigma_G^2 = 0$. Software gives the ANOVA table below. Interaction between brand and writing surface has no significant effect on the lifetime of the pen (*P*-value = .150). Since neither of the main effect *P*-values are significant (.798 and .400), we can conclude that neither the surface nor the brand of pen has a significant effect on the writing lifetime.

Source	**df**	**SS**	**MS**	***F***	***P*-value**
A	3	1387.50	462.05	$\frac{MSA}{MSAB} = .34$	.798
B	2	2888.08	1444.04	$\frac{MSB}{MSAB} = 1.07$	.400
AB	6	8100.25	,350.04	$\frac{MSAB}{MSE} = 1.97$	.150
Error	12	8216.00	684.67		
Total	23	20591.83			

23. Summary quantities include $x_{1..} = 9410$, $x_{2..} = 8835$, $x_{3..} = 9234$, $x_{.1.} = 5432$, $x_{.2.} = 5684$, $x_{.3.} = 5619$, $x_{.4.} = 5567$, $x_{.5.} = 5177$, $x_{...} = 27{,}479$, $CF = 16{,}779{,}898.69$, $\Sigma x_{i..}^2 = 251{,}872{,}081$, $\Sigma x_{.j.}^2 = 151{,}180{,}459$, resulting in the accompanying ANOVA table.

Source	**df**	**SS**	**MS**	***F***
A	2	11,573.38	5786.69	$\frac{MSA}{MSAB} = 26.70$
B	4	17,930.09	4482.52	$\frac{MSB}{MSAB} = 20.68$
AB	8	1734.17	216.77	$\frac{MSAB}{MSE} = 1.38$
Error	30	4716.67	157.22	
Total	44	35,954.31		

Since $1.38 < F_{.01,8,30} = 3.17$, the interaction *P*-value is > .01 and H_{0G} cannot be rejected. We continue: $26.70 \geq F_{.01,2,8} = 8.65 \Rightarrow$ factor A *P*-value < .01 and $20.68 \geq F_{.01,4,8} = 7.01 \Rightarrow$ factor B *P*-value < .01, so both H_{0A} and H_{0B} are rejected. Both capping material and the different batches affect compressive strength of concrete cylinders.

24.

a.

$$\begin{aligned} E\left(\bar{X}_{i..} - \bar{X}_{...}\right) &= \frac{1}{JK}\sum_j\sum_k E\left(X_{ijk}\right) - \frac{1}{IJK}\sum_i\sum_j\sum_k E\left(X_{ijk}\right) \\ &= \frac{1}{JK}\sum_j\sum_k\left(\mu + \alpha_i + \beta_j + \gamma_{ij}\right) - \frac{1}{IJK}\sum_i\sum_j\sum_k\left(\mu + \alpha_i + \beta_j + \gamma_{ij}\right) \\ &= \mu + \alpha_i - \mu = \alpha_i \end{aligned}$$

b.

$$\begin{aligned} E\left(\hat{\gamma}_{ij}\right) &= \frac{1}{K}\sum_k E\left(X_{ijk}\right) - \frac{1}{JK}\sum_j\sum_k E\left(X_{ijk}\right) - \frac{1}{IK}\sum_i\sum_k E\left(X_{ijk}\right) + \frac{1}{IJK}\sum_i\sum_j\sum_k E\left(X_{ijk}\right) \\ &= \mu + \alpha_i + \beta_j + \gamma_{ij} - \left(\mu + \alpha_i\right) - \left(\mu + \beta_j\right) + \mu = \gamma_{ij} \end{aligned}$$

25. With $\theta = \alpha_i - \alpha_i'$, $\hat{\theta} = \overline{X}_{i..} - \overline{X}_{i'..} = \frac{1}{JK}\sum_j \sum_k \left(X_{ijk} - X_{i'jk}\right)$, and since $i \neq i'$, X_{ijk} and $X_{i'jk}$ are independent for every *j*, *k*. Thus, $V\left(\hat{\theta}\right) = V\left(\overline{X}_{i..}\right) + V\left(\overline{X}_{i'..}\right) = \frac{\sigma^2}{JK} + \frac{\sigma^2}{JK} = \frac{2\sigma^2}{JK}$ (because $V\left(\overline{X}_{i..}\right) = V\left(\overline{\varepsilon}_{i..}\right)$ and $V\left(\varepsilon_{ijk}\right) = \sigma^2$) so $\hat{\sigma}_{\hat{\theta}} = \sqrt{\frac{2MSE}{JK}}$. The appropriate number of df is *IJ*(*K* – 1), so the CI is $\left(\overline{x}_{i..} - \overline{x}_{i'..}\right) \pm t_{\alpha/2, IJ(K-1)} \sqrt{\frac{2MSE}{JK}}$. For the data of exercise 19, $\overline{x}_{2..} = 8.192$, $\overline{x}_{3..} = 8.395$, *MSE* = .0170, $t_{.025,9} = 2.262$, *J* = 3, *K* = 2, so the 95% C.I. for $\alpha_2 - \alpha_3$ is (8.182 – 8.395) ± 2.262 $\sqrt{\frac{.0340}{6}}$ = –0.203 ± 0.170 = (–0.373, –0.033).

26.

a. $\frac{E(MSAB)}{E(MSE)} = 1 + \frac{K\sigma_G^2}{\sigma^2} = 1$ if $\sigma_G^2 = 0$ and > 1 if $\sigma_G^2 > 0$, so $\frac{MSAB}{MSE}$ is the appropriate *F* ratio.

b. $\frac{E(MSA)}{E(MSAB)} = \frac{\sigma^2 + K\sigma_G^2 + JK\sigma_A^2}{\sigma^2 + K\sigma_G^2} = 1 + \frac{JK\sigma_A^2}{\sigma^2 + K\sigma_G^2} = 1$ if $\sigma_A^2 = 0$ and > 1 if $\sigma_A^2 > 0$, so $\frac{MSA}{MSAB}$ is the appropriate *F* ratio for H_{0A} versus H_{aA}. Similarly, *MSB/MSAB* is the appropriate *F* ratio for H_{0B} versus H_{aB}.

Section 11.3

27.

a. The last column will be used in part **b.**

Source	df	SS	MS	F	$F_{.05,\text{num df, den df}}$
A	2	14,144.44	7072.22	61.06	3.35
B	2	5,511.27	2755.64	23.79	3.35
C	2	244,696.39	122.348.20	1056.24	3.35
AB	4	1,069.62	267.41	2.31	2.73
AC	4	62.67	15.67	.14	2.73
BC	4	331.67	82.92	.72	2.73
ABC	8	1,080.77	135.10	1.17	2.31
Error	27	3,127.50	115.83		
Total	53	270,024.33			

b. The computed F-statistics for all four interaction terms (2.31, .14, .72, 1.17) are less than the tabled values for statistical significance at the level .05 (2.73 for AB/AC/BC, 2.31 for ABC). Hence, all four P-values exceed .05. This indicates that none of the interactions are statistically significant.

c. The computed F-statistics for all three main effects (61.06, 23.79, 1056.24) exceed the tabled value for significance at level .05 ($3.35 = F_{.05,2,27}$). Hence, all three P-values are less than .05 (in fact, all three P-values are less than .001), which indicates that all three main effects are statistically significant.

d. Since $Q_{.05,3,27}$ is not tabled, use $Q_{.05,3,24} = 3.53$, $w = 3.53\sqrt{\frac{115.83}{(3)(3)(2)}} = 8.95$. All three levels differ significantly from each other.

28.

Source	df	SS	MS	F	$F_{.01,\text{ num df, den df}}$
A	3	19,149.73	6,383.24	2.70	4.72
B	2	2,589,047.62	1,294,523.81	546.79	5.61
C	1	157,437.52	157,437.52	66.50	7.82
AB	6	53,238.21	8,873.04	3.75	3.67
AC	3	9,033.73	3,011.24	1.27	4.72
BC	2	91,880.04	45,940.02	19.40	5.61
ABC	6	6,558.46	1,093.08	.46	3.67
Error	24	56,819.50	2,367.48		
Total	47	2,983,164.81			

For each effect, if the calculated F statistic is more than the .01 critical value (far right column), then the P-value is less than .01 and, thus, the effect is deemed significant.

The statistically significant interactions are AB and BC. Factor A appears to be the least significant of all the factors. It does not have a significant main effect and the significant interaction (AB) is only slightly greater than the tabled value at significance level .01 (hence, P-value barely below .01).

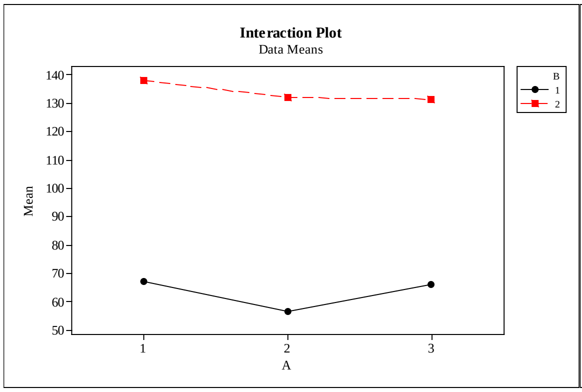
29.

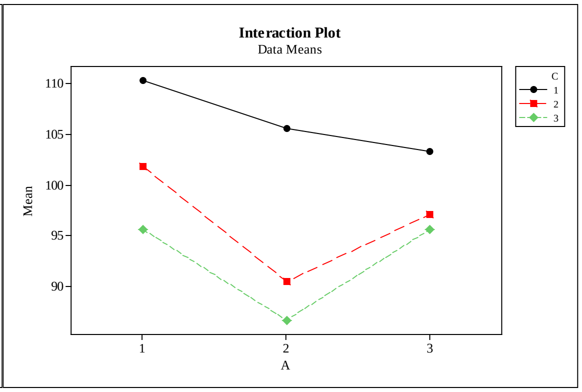
a.

Source	df	SS	MS	*F*	***P*-value**
A	2	1043.27	521.64	110.69	<.001
B	1	112148.10	112148.10	23798.01	<.001
C	2	3020.97	1510.49	320.53	<.001
AB	2	373.52	186.76	39.63	<.001
AC	4	392.71	98.18	20.83	<.001
BC	2	145.95	72.98	15.49	<.001
ABC	4	54.13	13.53	2.87	.029
Error	72	339.30	4.71		
Total	89	117517.95			

P-values were obtained using software. At the .01 significance level, all main and two-way interaction effects are statistically significant (in fact, extremely so), but the three-way interaction is not statistically significant (.029 > .01).

b. The means provided allow us to construct an AB interaction plot and an AC interaction plot. Based on the first plot, it's actually surprising that the AB interaction effect is significant: the "bends" of the two paths (B = 1, B = 2) are different but not that different. The AC interaction effect is more clear: the effect of C = 1 on mean response decreases with A (= 1, 2, 3), while the pattern for C = 2 and C = 3 is very different (a sharp up-down-up trend).





30.

a. See ANOVA table below. Notice that with no replication, there is no error term from which to estimate the error variance. In part **b**, the three-way interaction term will be absorbed into the error (i.e., become the error "effect"); the *F*-statistics in the table reflect that denominator.

b.

Source	**df**	**SS**	**MS**	***F***
A	3	.226250	.0754170	77.35
B	1	.000025	.0000250	.03
C	1	.003600	.0036000	3.69
AB	3	.004325	.0014417	1.48
AC	3	.000650	.0002170	.22
BC	1	.000625	.0006250	.64
ABC	3	.002925	.0009750	
Error	—	—	—	
Total	15	.238400		

The only statistically significant effect at the level .05 is the factor A main effect: levels of nitrogen ($f_A = 77.35$, *P*-value < .001). The next most significant effect is factor C main effect, but $f_C = 3.69$ at df = (1, 3) yields a *P*-value of about .15.

c. $Q_{.05,4,3} = 6.82$; $w = 6.82\sqrt{\frac{.002925}{(2)(2)}} = .1844$.

1	2	3	4
1.1200	1.3025	1.3875	1.4300

31.

a. The following ANOVA table was created with software.

Source	**df**	**SS**	**MS**	***F***	***P*-value**
A	2	124.60	62.30	4.85	.042
B	2	20.61	10.30	0.80	.481
C	2	356.95	178.47	13.89	.002
AB	4	57.49	14.37	1.12	.412
AC	4	61.39	15.35	1.19	.383
BC	4	11.06	2.76	0.22	.923
Error	8	102.78	12.85		
Total	26	734.87			

b. The *P*-values for the *AB*, *AC*, and *BC* interaction effects are provided in the table. All of them are much greater than .1, so none of the interaction terms are statistically significant.

c. According to the *P*-values, the factor *A* and *C* main effects are statistically significant at the .05 level. The factor *B* main effect is not statistically significant.

d. The paste thickness (factor *C*) means are 38.356, 35.183, and 29.560 for thickness .2, .3, and .4, respectively. Applying Tukey's method, $Q_{.05,3,8} = 4.04 \Rightarrow w = 4.04\sqrt{12.85/9} = 4.83$.

Thickness:	.4	.3	.2
Mean:	29.560	35.183	38.356

32.

a. Since $\frac{E(MSABC)}{E(MSE)} = \frac{\sigma^2 + L\sigma^2_{ABC}}{\sigma^2} = 1$ if $\sigma^2_{ABC} = 0$ and > 1 if $\sigma^2_{ABC} > 0$, $\frac{MSABC}{MSE}$ is the appropriate *F* ratio for testing $H_0 : \sigma^2_{ABC} = 0$. Similarly, $\frac{MSC}{MSE}$ is the *F* ratio for testing $H_0 : \sigma^2_C = 0$; $\frac{MSAB}{MSABC}$ is the *F* ratio for testing $H_0 :$ all $\gamma^{AB}_{ij} = 0$; and $\frac{MSA}{MSAC}$ is the *F* ratio for testing $H_0 :$ all $\alpha_i = 0$.

b.

Source	**df**	**SS**	**MS**	***F***	$F_{.01,\text{ num df, den df}}$
A	1	14,318.24	14,318.24	$\frac{MSA}{MSAC} = 19.85$	98.50
B	3	9656.4	3218.80	$\frac{MSB}{MSBC} = 6.24$	9.78
C	2	2270.22	1135.11	$\frac{MSC}{MSE} = 3.15$	5.61
AB	3	3408.93	1136.31	$\frac{MSAB}{MSABC} = 2.41$	9.78
AC	2	1442.58	721.29	$\frac{MSAC}{MSABC} = 1.53$	5.61
BC	6	3096.21	516.04	$\frac{MSBC}{MSE} = 1.43$	3.67
ABC	6	2832.72	472.12	$\frac{MSABC}{MSE} = 1.31$	3.67
Error	24	8655.60	360.65		
Total	47				

To have a *P*-value less than .01, the calculated *F* statistic must be greater than the value in the far-right column. At level .01, no H_0's can be rejected, so there appear to be no interaction or main effects present.

33. The various sums of squares yield the accompanying ANOVA table.

Source	**df**	**SS**	**MS**	***F***
A	6	67.32	11.02	
B	6	51.06	8.51	
C	6	5.43	.91	.61
Error	30	44.26	1.48	
Total	48	168.07		

We're interested in factor C. At df = (6, 30), .61 < $F_{.05,6,30} = 2.42 \Rightarrow$ *P*-value > .05. Thus, we fail to reject H_{0C} and conclude that heat treatment had no effect on aging.

34. Rewrite the data to get sums within levels:

	1	**2**	**3**	**4**	**5**	**6**
$x_{i..}$	144	205	272	293	85	98
$x_{.j.}$	171	199	147	221	177	182
$x_{..k}$	180	161	186	171	169	230

Thus $x... = 1097$, $CF = 1097^2/36 = 33428.03$, $\Sigma\Sigma x^2_{ij(k)} = 42,219$, $\Sigma x^2_{i..} = 239,423$, $\Sigma x^2_{.j.} = 203,745$, $\Sigma x^2_{..k} = 203.619$. From these, we can construct the ANOVA table.

Source	**df**	**SS**	**MS**	***F***
A	5	6475.80	1295.16	
B	5	529.47	105.89	
C	5	508.47	101.69	1.59
Error	20	1277.23	63.89	
Total	35	8790.97		

Since $1.59 < F_{.01,5,20} = 4.10$, the *P*-value for the factor C effect is > .01. Hence, H_{0C} is not rejected; shelf space does not appear to affect sales at the .01 level (even adjusting for variation in store and week).

35.

	1	**2**	**3**	**4**	**5**	
$x_{i..}$	40.68	30.04	44.02	32.14	33.21	$\Sigma x^2_{i..} = 6630.91$
$x_{.j.}$	29.19	31.61	37.31	40.16	41.82	$\Sigma x^2_{.j.} = 6605.02$
$x_{..k}$	36.59	36.67	36.03	34.50	36.30	$\Sigma x^2_{..k} = 6489.92$

$x_{...} = 180.09$, CF = 1297.30, $\Sigma\Sigma x^2_{ij(k)} = 1358.60$

Source	**df**	**SS**	**MS**	***F***
A	4	28.89	7.22	10.71
B	4	23.71	5.93	8.79
C	4	0.63	0.16	0.23
Error	12	8.09	0.67	
Total	24	61.30		

$F_{.05,4,12} = 3.26$, so the *P*-values for factor A and B effects are < .05 (10.71 > 3.26, 8.79 > 3.26), but the *P*-value for the factor C effect is > .05 (0.23 < 3.26). Both factor A (plant) and B(leaf size) appear to affect moisture content, but factor C (time of weighing) does not.

36.

Source	**df**	**SS**	**MS**	***F***	$F_{.01,\text{num df, den df}}$*
A (laundry treatment)	3	39.171	13.057	16.23	3.95
B (pen type)	2	.665	.3325	.41	4.79
C (fabric type)	5	21.508	4.3016	5.35	3.17
AB	6	1.432	.2387	.30	2.96
AC	15	15.953	1.0635	1.32	2.19
BC	10	1.382	.1382	.17	2.47
ABC	30	9.016	.3005	.37	1.86
Error	144	115.820	.8043		
Total	215	204.947			

*Because denominator df = 144 is not tabled, we have used 120.

To be significant at the .01 level (*P*-value < .01), the calculated *F* statistic must be greater than the .01 critical value in the far right column. At the level .01, there are two statistically significant main effects: laundry treatment and fabric type. There are no statistically significant interactions.

37. SST = (71)(93.621) = 6,647.091. Computing all other sums of squares and adding them up = 6,645.702. Thus SSABCD = 6,647.091 – 6,645.702 = 1.389 and MSABCD = 1.389/4 = .347.

Source	**df**	**MS**	***F***	$F_{.01,\text{num df, den df}}$*
A	2	2207.329	2259.29	5.39
B	1	47.255	48.37	7.56
C	2	491.783	503.36	5.39
D	1	.044	.05	7.56
AB	2	15.303	15.66	5.39
AC	4	275.446	281.93	4.02
AD	2	.470	.48	5.39
BC	2	2.141	2.19	5.39
BD	1	.273	.28	7.56
CD	2	.247	.25	5.39
ABC	4	3.714	3.80	4.02
ABD	2	4.072	4.17	5.39
ACD	4	.767	.79	4.02
BCD	2	.280	.29	5.39
ABCD	4	.347	.355	4.02
Error	36	.977		
Total	71			

*Because denominator df for 36 is not tabled, use df = 30.

To be significant at the .01 level (*P*-value < .01), the calculated *F* statistic must be greater than the .01 critical value in the far right column. At level .01 the statistically significant main effects are A, B, C. The interaction AB and AC are also statistically significant. No other interactions are statistically significant.

Section 11.4

38.

a. Apply Yates' method:

Condition	$x_{ijk.}$	1	2	Effect Contrast	$SS = \frac{(contrast)^2}{16}$
$(1) = x_{111.}$	404.2	839.2	1991.0	3697.0	
$a = x_{211.}$	435.0	1151.8	1706.0	164.2	1685.1
$b = x_{121.}$	549.6	717.6	83.4	583.4	21,272.2
$ab = x_{221.}$	602.2	988.4	80.8	24.2	36.6
$c = x_{112.}$	339.2	30.8	312.6	–285.0	5076.6
$ac = x_{212.}$	378.4	52.6	270.8	–2.6	.4
$bc = x_{122.}$	473.4	39.2	21.8	–41.8	109.2
$abc = x_{222.}$	515.0	41.6	2.4	–19.4	23.5

That verifies all the treatment SS. From the original data, $\Sigma\Sigma\Sigma\Sigma x_{ijkl}^2 = 882{,}573.38$ and $x_{....} = 3697$, so SST = 882,573.38 – 3697^2/16 = 28,335.3. This verifies the last SS.

b. The important effects are those with small associated *P*-values, indicating statistical significance. Those effects significant at level .05 (i.e., *P*-value < .05) are the three main effects and the speed by distance interaction.

39. Start by applying Yates' method. Each sum of squares is given by SS = (effect contrast)^2/24.

Condition	Total x_{ijk}	1	2	Effect Contrast	SS
(1)	315	927	2478	5485	
a	612	1551	3007	1307	SSA = 71,177.04
b	584	1163	680	1305	SSB = 70,959.38
ab	967	1844	627	199	SSAB = 1650.04
c	453	297	624	529	SSC = 11,660.04
ac	710	383	681	–53	SSAC = 117.04
bc	737	257	86	57	SSBC = 135.38
abc	1107	370	113	27	SSABC = 30.38

a. Totals appear above. From these,

$$\hat{\beta}_1 = \overline{x}_{.2..} - \overline{x}_{....} = \frac{584 + 967 + 737 + 1107 - 315 - 612 - 453 - 710}{24} = 54.38\,;$$

$$\hat{\gamma}_{11}^{AC} = \frac{315 - 612 + 584 - 967 - 453 + 710 - 737 + 1107}{24} = 2.21\,;\ \hat{\gamma}_{21}^{AC} = -\hat{\gamma}_{11}^{AC} = 2.21\,.$$

b. Factor sums of squares appear in the preceding table. From the original data, $\sum\sum\sum\sum x_{ijkl}^2$ = 1,411,889 and $x_{\ldots\ldots}$ = 5485, so SST = 1,411,889 – $5485^2/24$ = 158,337.96, from which SSE = 2608.7 (the remainder).

Source	df	SS	MS	*F*	*P*-value
A	1	71,177.04	71,177.04	435.65	<.001
B	1	70,959.38	70,959.38	435.22	<.001
AB	1	1650.04	1650.04	10.12	.006
C	1	11,660.04	11,660.04	71.52	<.001
AC	1	117.04	117.04	0.72	.409
BC	1	135.38	135.38	0.83	.376
ABC	1	30.38	30.38	0.19	.672
Error	16	2608.7	163.04		
Total	23	158,337.96			

P-values were obtained from software. Alternatively, a *P*-value less than .05 requires an *F* statistic greater than $F_{.05,1,16}$ = 4.49. We see that the AB interaction and all the main effects are significant.

c. Yates' algorithm generates the 15 effect SS's in the ANOVA table; each SS is $(\text{effect contrast})^2/48$. From the original data, $\sum\sum\sum\sum\sum x_{ijklm}^2$ = 3,308,143 and $x_{\ldots\ldots\ldots}$ = 11,956 $\Rightarrow$ SST = 3,308,143 – $11{,}956^2/48$ 328,607.98. SSE is the remainder: SSE = SST – [sum of effect SS's] = … = 4,339.33.

Source	df	SS	MS	*F*
A	1	136,640.02	136,640.02	1007.6
B	1	139,644.19	139,644.19	1029.8
C	1	24,616.02	24,616.02	181.5
D	1	20,377.52	20,377.52	150.3
AB	1	2,173.52	2,173.52	16.0
AC	1	2.52	2.52	0.0
AD	1	58.52	58.52	0.4
BC	1	165.02	165.02	1.2
BD	1	9.19	9.19	0.1
CD	1	17.52	17.52	0.1
ABC	1	42.19	42.19	0.3
ABD	1	117.19	117.19	0.9
ACD	1	188.02	188.02	1.4
BCD	1	13.02	13.02	0.1
ABCD	1	204.19	204.19	1.5
Error	32	4,339.33	135.60	
Total	47	328,607.98		

In this case, a *P*-value less than .05 requires an *F* statistic greater than $F_{.05,1,32} \approx 4.15$. Thus, all four main effects and the AB interaction effect are statistically significant at the .05 level (and no other effects are).

40.

a. In the accompanying ANOVA table, effects are listed in the order implied by Yates' algorithm. $\Sigma\Sigma\Sigma\Sigma\Sigma x_{ijklm}^2 = 4783.16$, $x_{.....} = 388.14$, so $SST = 4783.16 - \frac{388.14^2}{32} = 72.56$ and SSE = 72.56 – (sum of all other SS's) = 35.85.

Source	**df**	**SS**	**MS**	***F***
A	1	.17	.17	< 1
B	1	1.94	1.94	< 1
AB	1	3.42	3.42	1.53
C	1	8.16	8.16	3.64
AC	1	.26	.26	< 1
BC	1	.74	.74	< 1
ABC	1	.02	.02	< 1
D	1	13.08	13.08	5.84
AD	1	.91	.91	< 1
BD	1	.78	.78	< 1
ABD	1	.78	.78	< 1
CD	1	6.77	6.77	3.02
ACD	1	.62	.62	< 1
BCD	1	1.76	1.76	< 1
ABCD	1	.00	.00	< 1
Error	16	35.85	2.24	
Total	31	72.56		

b. To be significant at the .05 level (i.e., *P*-value < .05) requires the calculated *F* statistic to be greater than $F_{.05,1,16} = 4.49$. So, none of the interaction effects is judged significant, and only the D main effect is significant.

41. The accompanying ANOVA table was created using software. All *F* statistics are quite large (some extremely so) and all *P*-values are very small. So, in fact, all seven effects are statistically significant for predicting quality.

Source	**df**	**SS**	**MS**	***F***	***P*-value**
A	1	.003906	.003906	25.00	.001
B	1	.242556	.242556	1552.36	<.001
C	1	.003906	.003906	25.00	.001
AB	1	.178506	.178506	1142.44	<.001
AC	1	.002256	.002256	14.44	.005
BC	1	.178506	.178506	1142.44	<.001
ABC	1	.002256	.002256	14.44	.005
Error	8	.000156	.000156		
Total	15	.613144			

42. Use Yates' algorithm to find the contrasts, then find MS by $SS = \frac{(contrast)^2}{48}$ and MS = SS/1. Then, $\Sigma\Sigma\Sigma\Sigma\Sigma x_{ijklm}^2 = 32{,}917{,}817$ and $x_{.....} = 39{,}371 \Rightarrow$ SST = 624,574, whereby SSE = SST – [sum of all other SS] = 164528, and error df = 32.

Effect	MS	*F*	Effect	MS	*F*
A	3120	< 1	AD	16170	3.15
B	332168	64.61	BD	3519	< 1
AB	14876	2.89	ABD	2120	< 1
C	43140	8.39	CD	4701	< 1
AC	776	< 1	ACD	1964	< 1
BC	3554	< 1	BCD	10355	2.01
ABC	1813	< 1	ABCD	1313	< 1
D	20460	3.98	Error	5142	

$F_{.01,1,32} \approx 7.5$, so to get a *P*-value < .01 requires an *F* statistic greater than 7.5. Only the B and C main effects are judged significant at the 1% level.

43.

Condition/ Effect	$SS = \frac{(contrast)^2}{16}$	*F*	Condition/ Effect	$SS = \frac{(contrast)^2}{16}$	*F*
(1)	—		D	414.123	850.77
A	.436	< 1	AD	.017	< 1
B	.099	< 1	BD	.456	< 1
AB	.003	< 1	ABD	.990	—
C	.109	< 1	CD	2.190	4.50
AC	.078	< 1	ACD	1.020	—
BC	1.404	3.62	BCD	.133	—
ABC	.286	—	ABCD	.004	—

SSE = .286 + .990 + 1.020 + .133 + .004 =2.433, df = 5, so MSE = .487, which forms the denominators of the *F* values above. A *P*-value less than .05 requires an *F* statistic greater than $F_{.05,1,5} = 6.61$, so only the D main effect is significant.

44.

a. The eight treatment conditions which have even number of letters in common with *abcd* and thus go in the first (principal) block are (1), *ab, ac, bc, ad, bd, cd, abcd*; the other eight conditions are placed in the second block.

b. and **c.**

$x_{....} = 1290$, $\Sigma\Sigma\Sigma\Sigma x_{ijkl}^2 = 105{,}160$, so SST = 1153.75. The two block totals are 639 and 651, so $SSBl = \frac{639^2}{8} + \frac{651^2}{8} - \frac{1290^2}{16} = 9.00$, which is identical (as it must be here) to SSABCD computed from Yates algorithm.

Condition/Effect	**Block**	$SS = \frac{(contrast)^2}{16}$	***F***
(1)	1	—	
A	2	25.00	1.93
B	2	9.00	< 1
AB	1	12.25	< 1
C	2	49.00	3.79
AC	1	2.25	< 1
BC	1	.25	< 1
ABC	2	9.00	—
D	2	930.25	71.90
AD	1	36.00	2.78
BD	1	25.00	1.93
ABD	2	20.25	—
CD	1	4.00	< 1
ACD	2	20.25	—
BCD	2	2.25	—
ABCD=Blocks	1	9.00	< 1
Total		1153.75	

$SSE = 9.0 + 20.25 + 20.25 + 2.25 = 51.75$, df = 4, so MSE = 12.9375, $F_{.05,1,4} = 7.71$, so only the D main effect is significant.

45.

a. The allocation of treatments to blocks is as given in the answer section (see back of book), with block #1 containing all treatments having an even number of letters in common with both *ab* and *cd*, block #2 those having an odd number in common with *ab* and an even number with *cd*, etc.

b. $\Sigma\Sigma\Sigma\Sigma x_{ijklm}^2 = 9{,}035{,}054$ and $x_{.....} = 16{,}898$, so $SST = 9{,}035{,}054 - \frac{16{,}898^2}{32} = 111{,}853.875$. The eight block-replication totals are 2091 (= 618 + 421 + 603 + 449, the sum of the four observations in block #1 on replication #1), 2092, 2133, 2145, 2113, 2080, 2122, and 2122, so $SSBl = \frac{2091^2}{4} + ... + \frac{2122^2}{4} - \frac{16{,}898^2}{32} = 898.875$. The effect SS's can be computed via Yates' algorithm; those we keep appear below. SSE is computed by SST – [sum of all other SS]. MSE = 5475.75/12 = 456.3125, which forms the denominator of the F ratios below. With $F_{.01,1,12} = 9.33$, only the A and B main effects are significant.

Source	**df**	**SS**	***F***
A	1	12403.125	27.18
B	1	92235.125	202.13
C	1	3.125	0.01
D	1	60.500	0.13
AC	1	10.125	0.02
BC	1	91.125	0.20
AD	1	50.000	0.11
BC	1	420.500	0.92
ABC	1	3.125	0.01
ABD	1	0.500	0.00
ACD	1	200.000	0.44
BCD	1	2.000	0.00
Block	7	898.875	0.28
Error	12	5475.750	
Total	31	111853.875	

46. The result is clearly true if either defining effect is represented by either a single letter (e.g., A) or a pair of letters (e.g. AB). The only other possibilities are for both to be "triples" (e.g. ABC or ABD, all of which must have two letters in common.) or one a triple and the other ABCD. But the generalized interaction of ABC and ABD is CD, so a two-factor interaction is confounded, and the generalized interaction of ABC and ABCD is D, so a main effect is confounded. Similar comments apply to any other triple(s).

47.

a. The third nonestimable effect is (*ABCDE*)(*CDEFG*) = *ABFG*. The treatments in the group containing (1) are (1), *ab, cd, ce, de, fg, acf, adf, adg, aef, acg, aeg, bcg, bcf, bdf, bdg, bef, beg, abcd, abce, abde, abfg, cdfg, cefg, defg, acdef, acdeg, bcdef, bcdeg, abcdfg, abcefg, abdefg.* The alias groups of the seven main effects are{*A, BCDE, ACDEFG, BFG*}, {*B, ACDE, BCDEFG, AFG*}, {*C, ABDE, DEFG, ABCFG*}, {*D, ABCE, CEFG, ABDFG*}, {*E, ABCD, CDFG, ABEFG*}, {*F, ABCDEF, CDEG, ABG*}, and {*G, ABCDEG, CDEF, ABF*}.

b. 1: (1), *aef, beg, abcd, abfg, cdfg, acdeg, bcdef;* 2: *ab, cd, fg, aeg, bef, acdef, bcdeg, abcdfg;* 3: *de, acg, adf, bcf, bdg, abce, cefg, abdefg;* 4: *ce, acf, adg, bcg, bdf, abde, defg, abcefg.*

48.

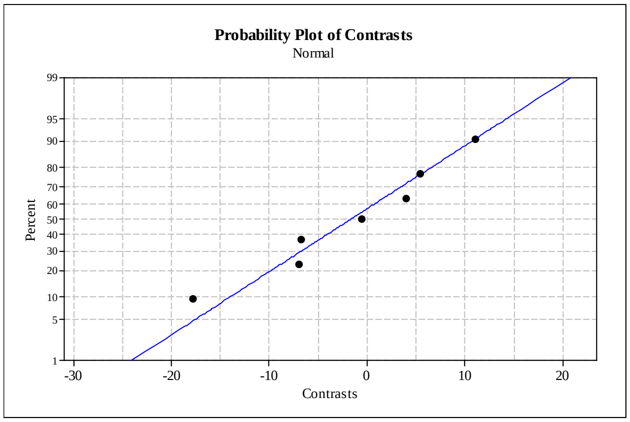
a. The treatment conditions in the observed group are, in standard order, (1), *ab*, *ac*, *bc*, *ad*, *bd*, *cd*, and *abcd*. The alias pairs are {*A*, *BCD*}, {*B*, *ACD*}, {*C*, *ABD*}, {*D*, *ABC*}, {*AB*, *CD*}, {*AC*, *BD*}, and {*AD*, *BC*}.

b. The accompanying sign chart allows us to compute the contrasts.

	A	***B***	***C***	***D***	***AB***	***AC***	***AD***
(1) = 8.936	–	–	–	–	+	+	+
ab = 9.130	+	+	–	–	+	–	–
ac = 4.314	+	–	+	–	–	+	–
bc = 7.692	–	+	+	–	–	–	+
ad = 0.415	+	–	–	+	–	–	+
bd = 6.061	–	+	–	+	–	+	–
cd = 1.984	–	–	+	+	+	–	–
abcd = 3.830	+	+	+	+	+	+	+
Contrast	–6.984	11.064	–6.722	–17.782	5.398	3.92	–0.616
SS	6.10	15.30	5.65	39.52	3.64	1.92	0.05
F	3.26	8.18	3.02	21.14			

To test main effects, we use SSE = SSAB + SSAC + SSAD = 5.61, so MSE = 5.61/3 = 1.87. The test statistics above are calculated by f = [SSTr/1]/[SSE/3] = SSTr/MSE. A *P*-value less than .05 requires a test statistic greater than $F_{.05,1,3}$ = 10.13, so only the D main effect is judged to be statistically significant.

The accompanying normal probability plot is quite linear, with the point corresponding to the D main effect (contrast = –17.782) the only noteworthy value.



49.

		A	B	C	D	E	AB	AC	AD	AE	BC	BD	BE	CD	CE	DE
a	70.4	+	–	–	–	–	–	–	–	–	+	+	+	+	+	+
b	72.1	–	+	–	–	–	–	+	+	+	–	–	–	+	+	+
c	70.4	–	–	+	–	–	+	–	+	+	–	+	+	–	–	+
abc	73.8	+	+	+	–	–	+	+	–	–	+	–	–	–	–	+
d	67.4	–	–	–	+	–	+	+	–	+	+	–	+	–	+	–
abd	67.0	+	+	–	+	–	+	–	+	–	–	+	–	–	+	–
acd	66.6	+	–	+	+	–	–	+	+	–	–	–	+	+	–	–
bcd	66.8	–	+	+	+	–	–	–	–	+	+	+	–	+	–	–
e	68.0	–	–	–	–	+	+	+	+	–	+	+	–	+	–	–
abe	67.8	+	+	–	–	+	+	–	–	+	–	–	+	+	–	–
ace	67.5	+	–	+	–	+	–	+	–	+	–	+	–	–	+	–
bce	70.3	–	+	+	–	+	–	–	+	–	+	–	+	–	+	–
ade	64.0	+	–	–	+	+	–	–	+	+	+	–	–	–	–	+
bde	67.9	–	+	–	+	+	–	+	–	–	–	+	+	–	–	+
cde	65.9	–	–	+	+	+	+	–	–	–	–	–	–	+	+	+
abcde	68.0	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

Thus $SSA = \dfrac{(70.4 - 72.1 - 70.4 + ... + 68.0)^2}{16} = 2.250$, SSB = 7.840, SSC = .360, SSD = 52.563, SSE = 10.240, SSAB = 1.563, SSAC = 7.563, SSAD = .090, SSAE = 4.203, SSBC = 2.103, SSBD = .010, SSBE = .123, SSCD = .010, SSCE = .063, SSDE = 4.840, Error SS = sum of two factor SS's = 20.568, Error MS = 2.057, $F_{.01,1,10} = 10.04$, so only the D main effect is significant.

Supplementary Exercises

50.

Source	**df**	**SS**	**MS**	***f***
Treatment	4	14.962	3.741	36.7
Block	8	9.696		
Error	32	3.262	.102	
Total	44	27.920		

The *P*-value for testing $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$ is based on the $F_{4,32}$ distribution. Since the calculated value of 36.7 is off the chart for df = (4, 30), we infer the *P*-value is < .001, and H_0 is rejected. We conclude that expected smoothness score does depend somehow on the drying method used.

51.

Source	**df**	**SS**	**MS**	***F***
A	1	322.667	322.667	980.38
B	3	35.623	11.874	36.08
AB	3	8.557	2.852	8.67
Error	16	5.266	.329	
Total	23	372.113		

We first test the null hypothesis of no interactions ($H_{0AB}:\gamma_{ij}=0$ for all *i, j*). At df = (3, 16), 5.29 < 8.67 < 9.01 $\Rightarrow$ .01 < *P*-value < .001. Therefore, H_0 is rejected. Because we have concluded that interaction is present, tests for main effects are not appropriate.

52. Let X_{ij} = the amount of clover accumulation when the i^{th} sowing rate is used in the j^{th} plot = $\mu+\alpha_i+\beta_j+e_{ij}$.

Source	**df**	**SS**	**MS**	***F***
Treatment	3	3,141,153.5	1,040,751.17	2.28
Block	3	19,470,550.0		
Error	9	4,141,165.5	460,129.50	
Total	15	26,752,869.0		

At df = (3, 9), 2.28 < 2.81 $\Rightarrow$ *P*-value > .10. Hence, H_0 is not rejected. Expected accumulation does not appear to depend on sowing rate.

53. Let A = spray volume, B = belt speed, C = brand. The Yates table and ANOVA table are below. At degrees of freedom = (1, 8), a *P*-value less than .05 requires $F > F_{.05,1,8}$ = 5.32. So all of the main effects are significant at level .05, but none of the interactions are significant.

Condition	Total	1	2	Contrast	$SS=\frac{(contrast)^2}{16}$
(1)	76	129	289	592	21,904.00
A	53	160	303	22	30.25
B	62	143	13	48	144.00
AB	98	160	9	134	1122.25
C	88	–23	31	14	12.25
AC	55	36	17	–4	1.00
BC	59	–33	59	–14	12.25
ABC	101	42	75	16	16.00

Effect	**df**	**MS**	***F***
A	1	30.25	6.72
B	1	144.00	32.00
AB	1	1122.25	249.39
C	1	12.25	2.72
AC	1	1.00	.22
BC	1	12.25	2.72
ABC	1	16.00	3.56
Error	8	4.50	
Total	15		

54. We use Yates' method for calculating the sums of squares, and for ease of calculation, we divide each observation by 1000.

Condition	Total	1	2	Contrast	$SS = \frac{(contrast)^2}{8}$
(1)	23.1	66.1	213.5	317.2	–
A	43.0	147.4	103.7	20.2	51.005
B	71.4	70.2	24.5	44.6	248.645
AB	76.0	33.5	–4.3	–12.0	18.000
C	37.0	19.9	81.3	–109.8	1507.005
AC	33.2	4.6	–36.7	–28.8	103.68
BC	17.0	–3.8	–15.3	–118.0	1740.5
ABC	16.5	–.5	3.3	18.6	43.245

We assume that there is no three-way interaction, so MSABC becomes the MSE for ANOVA:

Source	**df**	**MS**	***F***
A	1	51.005	1.179
B	1	248.645	5.750
AB	1	18.000	< 1
C	1	1507.005	34.848
AC	1	103.68	2.398
BC	1	1740.5	40.247
Error	1	43.245	
Total	7		

A *P*-value less than .05 requires an *F* statistic greater than $F_{.05,1,1}$. Unfortunately, $F_{.05,1,1} = 161.45$, so that we cannot conclude any of these terms is significant at the 5% level. Even if we assume two-way interactions are absent, none of the main effects is statistically significant (again due to extremely low power via the error df).

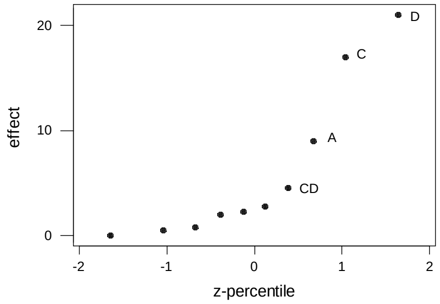
55.

a.

Effect	**%Iron**	**1**	**2**	**3**	**Effect Contrast**	**SS**
	7	18	37	174	684	
A	11	19	137	510	144	1296
B	7	62	169	50	36	81
AB	12	75	341	94	0	0
C	21	79	9	14	272	4624
AC	41	90	41	22	32	64
BC	27	165	47	2	12	9
ABC	48	176	47	–2	–4	1
D	28	4	1	100	336	7056
AD	51	5	13	172	44	121
BD	33	20	11	32	8	4
ABD	57	21	11	0	0	0
CD	70	23	1	12	72	324
ACD	95	24	1	0	–32	64
BCD	77	25	1	0	–12	9
ABCD	99	22	–3	–4	–4	1

We use *estimate* = *contrast*/2^p when $n = 1$ to get $\hat{\alpha}_1 = \frac{144}{2^4} = \frac{144}{16} = 9.00$, $\hat{\beta}_1 = \frac{36}{16} = 2.25$, $\hat{\delta}_1 = \frac{272}{16} = 17.00$, $\hat{\gamma}_1 = \frac{336}{16} = 21.00$. Similarly, $\left(\widehat{\alpha\beta}\right)_{11} = 0$, $\left(\widehat{\alpha\delta}\right)_{11} = 2.00$, $\left(\widehat{\alpha\gamma}\right)_{11} = 2.75$, $\left(\widehat{\beta\delta}\right)_{11} = .75$, $\left(\widehat{\beta\gamma}\right)_{11} = .50$, and $\left(\widehat{\delta\gamma}\right)_{11} = 4.50$.

b. The plot suggests main effects *A*, *C,* and *D* are quite important, and perhaps the interaction *CD* as well. In fact, pooling the 4 three-factor interaction SS's and the four-factor interaction SS to obtain an SSE based on 5 df and then constructing an ANOVA table suggests that these are the most important effects.



56. The entries of this ANOVA table were produced with software.

Source	**df**	**SS**	**MS**	***F***
Health	1	0.5880	0.5880	27.56
pH	2	0.6507	0.3253	15.25
Interaction	2	0.1280	0.0640	3.00
Error	24	0.5120	0.0213	
Total	29	1.8787		

First we test the interaction effect: at df = (2, 24), 2.54 < 3.00 < 3.40 ⇒ .05 < *P*-value < .10. Hence, we can fail to reject the no-interaction hypothesis at the .05 significance level and proceed to consider the main effects.

Both *F* statistics are highly statistically significant at the relevant df (*P*-values < .001), so we conclude that both the health of the seedling and its pH level have an effect on the average rating.

57. The ANOVA table is:

Source	df	SS	MS	F	$F_{.01,\text{ num df, den df}}$
A	2	67553	33777	11.37	5.49
B	2	72361	36181	12.18	5.49
C	2	442111	221056	74.43	5.49
AB	4	9696	2424	0.82	4.11
AC	4	6213	1553	0.52	4.11
BC	4	34928	8732	2.94	4.11
ABC	8	33487	4186	1.41	3.26
Error	27	80192	2970		
Total	53	746542			

A *P*-value less than .01 requires an *F* statistic greater than the $F_{.01}$ value at the appropriate df (see the far right column). All three main effects are statistically significant at the 1% level, but no interaction terms are statistically significant at that level.

58.

Source	**df**	**SS**	**MS**	$\boldsymbol{F}$	$\boldsymbol{F_{.05,\text{ num df, den df}}}$
A(pressure)	1	6.94	6.940	11.57	4.26
B(time)	3	5.61	1.870	3.12	3.01
C(concen.)	2	12.33	6.165	10.28	3.40
AB	3	4.05	1.350	2.25	3.01
AC	2	7.32	3.660	6.10	3.40
BC	6	15.80	2.633	4.39	2.51
ABC	6	4.37	.728	1.21	2.51
Error	24	14.40	.600		
Total	47	70.82			

A *P*-value less than .05 requires an *F* statistic greater than the $F_{.05}$ value at the appropriate df (see the far right column). The three-factor interaction (ABC) is not statistically significant. However, both the AC and BC two-factor interactions appear to be present.

59. Based on the *P*-values in the ANOVA table, statistically significant factors at the level .01 are adhesive type and cure time. The conductor material does not have a statistically significant effect on bond strength. There are no significant interactions.

60. Minitab provides the accompanying 3-way ANOVA, assuming no interaction effects. We see that both cutting speed and feed have statistically significant effects on tool life (both *P*-values are less than .01), but cutting depth has a marginally significant effect (*P*-value = .081).

```
Analysis of Variance for Tool Life

Source     DF      SS      MS        F      P
Cut speed   1  850.78  850.78  1815.00  0.000
Feed        1   11.28   11.28    24.07  0.008
Cut depth   1    2.53    2.53     5.40  0.081
Error       4    1.88    0.47
Total       7  866.47
```

61. $SSA = \sum_i \sum_j \left(\bar{X}_{i...} - \bar{X}_{....}\right)^2 = \frac{1}{N}\Sigma X_{i...}^2 - \frac{X_{....}^2}{N}$, with similar expressions for SSB, SSC, and SSD, each having $N - 1$ df.

$SST = \sum_i \sum_j \left(X_{ij(kl)} - \bar{X}_{....}\right)^2 = \sum_i \sum_j X_{ij(kl)}^2 - \frac{X_{....}^2}{N}$ with $N^2 - 1$ df, leaving $N^2 - 1 - 4(N - 1)$ df for error.

	1	2	3	4	5	Σx^2
$x_{i...}$:	482	446	464	468	434	1,053,916
$x_{.j..}$:	470	451	440	482	451	1,053,626
$x_{..k.}$:	372	429	484	528	481	1,066,826
$x_{...l}$:	340	417	466	537	534	1,080,170

Also, $\Sigma\Sigma x_{ij(kl)}^2 = 220{,}378$, $x_{....} = 2294$, and CF = 210,497.44.

Source	**df**	**SS**	**MS**	***F***
A	4	285.76	71.44	.594
B	4	227.76	56.94	.473
C	4	2867.76	716.94	5.958
D	4	5536.56	1384.14	11.502
Error	8	962.72	120.34	
Total	24			

At df = (4, 8), a *P*-value less than .05 requires an *F*-statistic greater than $F_{.05,4,8} = 3.84$. H_{0A} and H_{0B} cannot be rejected, while H_{0C} and H_{0D} are rejected.

CHAPTER 12

Section 12.1

1.

a. Stem and Leaf display of temp:

17	0	
17	23	stem = tens
17	445	leaf = ones
17	67	
17		
18	0000011	
18	2222	
18	445	
18	6	
18	8	

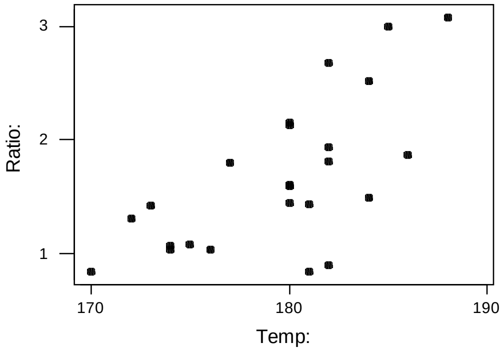
180 appears to be a typical value for this data. The distribution is reasonably symmetric in appearance and somewhat bell-shaped. The variation in the data is fairly small since the range of values (188 – 170 = 18) is fairly small compared to the typical value of 180.

0	889	
1	0000	stem = ones
1	3	leaf = tenths
1	4444	
1	66	
1	8889	
2	11	
2		
2	5	
2	6	
2		
3	00	

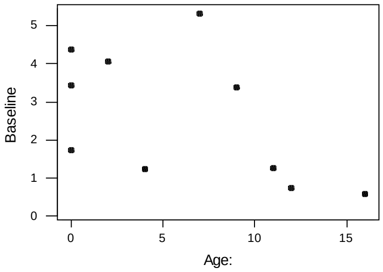
For the ratio data, a typical value is around 1.6 and the distribution appears to be positively skewed. The variation in the data is large since the range of the data (3.08 - .84 = 2.24) is very large compared to the typical value of 1.6. The two largest values could be outliers.

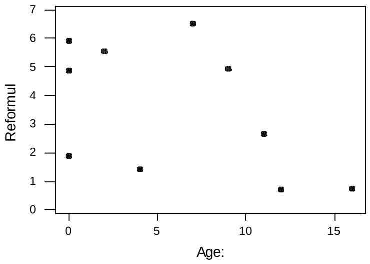
b. The efficiency ratio is not uniquely determined by temperature since there are several instances in the data of equal temperatures associated with different efficiency ratios. For example, the five observations with temperatures of 180 each have different efficiency ratios.

c. A scatter plot of the data appears below. The points exhibit quite a bit of variation and do not appear to fall close to any straight line or simple curve.



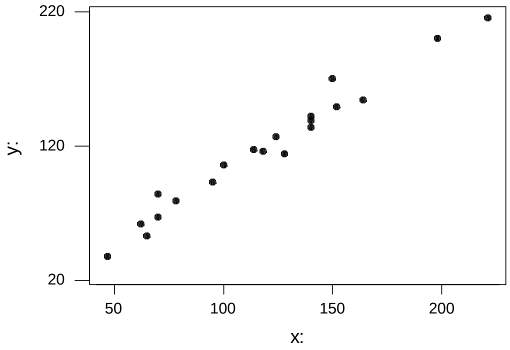
2. Scatter plots for the emissions vs age:



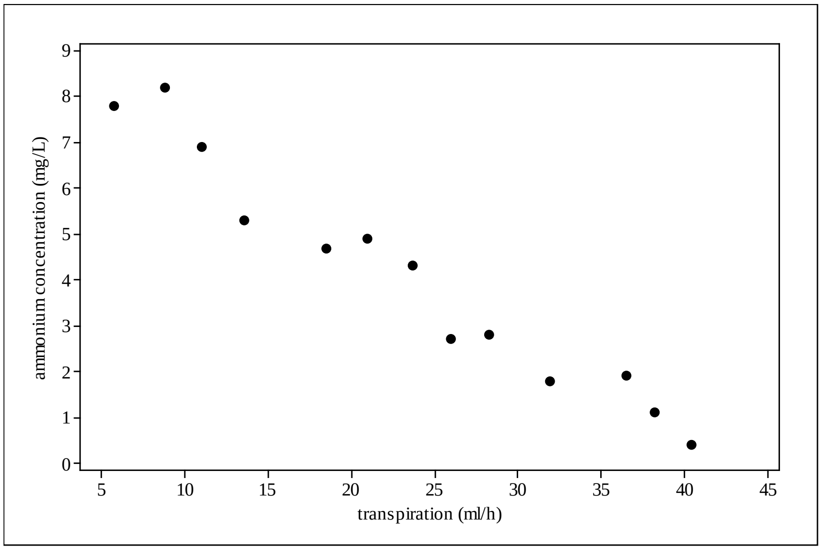


With this data the relationship between the age of the lawn mower and its NO_x emissions seems somewhat dubious. One might have expected to see that as the age of the lawn mower increased the emissions would also increase. We certainly do not see such a pattern. Age does not seem to be a particularly useful predictor of NO_x emission.

3. A scatter plot of the data appears below. The points fall very close to a straight line with an intercept of approximately 0 and a slope of about 1. This suggests that the two methods are producing substantially the same concentration measurements.

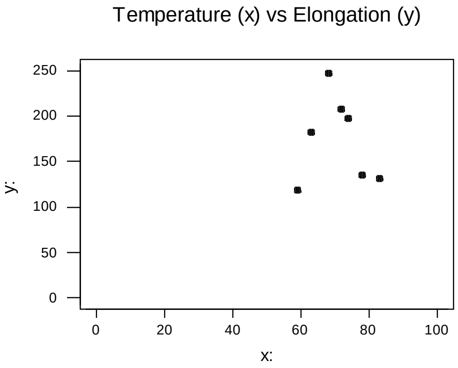


4. The accompanying scatterplot shows a strong, negative, linear association between transpiration and ammonium concentration. Based on the strong linearity of the scatterplot, it does seem reasonable to use simple linear regression modeling for these two variables.

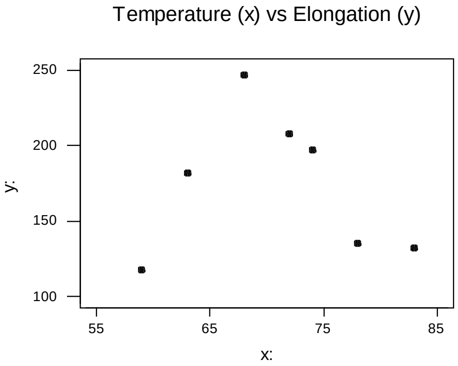


5.

a. The scatter plot with axes intersecting at (0,0) is shown below.



b. The scatter plot with axes intersecting at (55, 100) is shown below.



c. A parabola appears to provide a good fit to both graphs.

6. There appears to be a linear relationship between racket resonance frequency and sum of peak-to-peak acceleration. As the resonance frequency increases the sum of peak-to-peak acceleration tends to decrease. However, there is not a perfect relationship. Variation does exist. One should also notice that there are two tennis rackets that appear to differ from the other 21 rackets. Both have very high resonance frequency values. One might investigate if these rackets differ in other ways as well.

7.

a. $\mu_{Y \cdot 2500} = 1800 + 1.3(2500) = 5050$

b. expected change = slope = β_1 = 1.3

c. expected change = $100\beta_1$ = 130

d. expected change = $-100\beta_1$ = –130

8.

a. $\mu_{Y \cdot 2000} = 1800 + 1.3(2000) = 4400$, and σ = 350, so $P(Y > 5000) = P\left(Z > \frac{5000-4400}{350}\right) = P(Z > 1.71)$ = .0436.

b. Now $E(Y)$ = 5050, so $P(Y > 5000) = P(Z > -.14) = .5557$.

c. $E(Y_2 - Y_1) = E(Y_2) - E(Y_1) = 5050 - 4400 = 650$, and $V(Y_2 - Y_1) = V(Y_2) + V(Y_1) = (350)^2 + (350)^2 = 245{,}000$, so the sd of $Y_2 - Y_1$ is 494.97. Thus $P(Y_2 - Y_1 > 0) = P\left(Z > \frac{1000-650}{494.97}\right) = P(Z > .71) = .2389$.

d. The standard deviation of $Y_2 - Y_1$ is 494.97 (from **c**), and $E(Y_2 - Y_1) = 1800 + 1.3x_2 - (1800 + 1.3x_1) = 1.3(x_2 - x_1)$. Thus $P(Y_2 > Y_1) = P(Y_2 - Y_1 > 0) = P\left(z > \frac{-1.3(x_2 - x_1)}{494.97}\right) = .95$ implies that $-1.645 = \frac{-1.3(x_2 - x_1)}{494.97}$, so $x_2 - x_1$ = 626.33.

9.

a. β_1 = expected change in flow rate (y) associated with a one inch increase in pressure drop (x) = .095.

b. We expect flow rate to decrease by $5\beta_1$ = .475.

c. $\mu_{Y \cdot 10} = -.12 + .095(10) = .83$, and $\mu_{Y \cdot 15} = -.12 + .095(15) = 1.305$.

d. $P(Y > .835) = P\left(Z > \frac{.835 - .830}{.025}\right) = P(Z > .20) = .4207$.

$P(Y > .840) = P\left(Z > \frac{.840 - .830}{.025}\right) = P(Z > .40) = .3446$.

e. Let Y_1 and Y_2 denote pressure drops for flow rates of 10 and 11, respectively. Then $\mu_{Y \cdot 11} = .925$, so $Y_1 - Y_2$ has expected value .830 – .925 = –.095, and sd $\sqrt{(.025)^2 + (.025)^2} = .035355$. Thus $P(Y_1 > Y_2) = P(Y_1 - Y_2 > 0) = P\left(z > \frac{0 - (-.095)}{.035355}\right) = P(Z > 2.69) = .0036$.

10. No. Y has expected value 5000 when x = 100 and 6000 when x = 200, so the two probabilities become $P\left(z > \frac{500}{\sigma}\right) = .05$ and $P\left(z > \frac{500}{\sigma}\right) = .10$. These two equations are contradictory.

11.

a. β_1 = expected change for a one degree increase = –.01, and $10\beta_1$ = –.1 is the expected change for a 10 degree increase.

b. $\mu_{Y \cdot 200} = 5.00 - .01(200) = 3$, and $\mu_{Y \cdot 250} = 2.5$.

c. The probability that the first observation is between 2.4 and 2.6 is $P(2.4 \le Y \le 2.6) = P\left(\frac{2.4-2.5}{.075} \le Z \le \frac{2.6-2.5}{.075}\right) = P(-1.33 \le Z \le 1.33) = .8164$. The probability that any particular one of the other four observations is between 2.4 and 2.6 is also .8164, so the probability that all five are between 2.4 and 2.6 is $(.8164)^5 = .3627$.

d. Let Y_1 and Y_2 denote the times at the higher and lower temperatures, respectively. Then $Y_1 - Y_2$ has expected value $5.00 - .01(x+1) - (5.00 - .01x) = -.01$. The standard deviation of $Y_1 - Y_2$ is $\sqrt{(.075)^2 + (.075)^2} = .10607$. Thus $P(Y_1 - Y_2 > 0) = P\left(Z > \frac{-(-.01)}{.10607}\right) = P(Z > .09) = .4641$.

Section 12.2

12.

a. From the summary provided, $\hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}} = \dfrac{-341.959231}{1585.230769}$ = –0.21572 and $\hat{\beta}_0 = \dfrac{\Sigma y - \hat{\beta}_1 \Sigma x}{n} = \dfrac{52.8 - (-.21572)(303.7)}{13}$ = 9.1010. So the equation of the least squares regression line is $\hat{y}$ = 9.1010 – .21572x.
Based on this equation, the predicted ammonium concentration (y) when transpiration (x) is 25 ml/h is $\hat{y}(25)$ = 9.1010 – .21572(25) = 3.708 mg/L.

b. If you plug x = 45 into the least squares regression line, you get $\hat{y}(45)$ = 9.1010 – .21572(45) = –0.606. That's an impossible ammonium concentration level, since concentration can't be negative. But it doesn't make sense to predict y at x = 45 from this data set, because x = 45 is well outside of the scope of the data (this is an example of extrapolation and its potential adverse consequences).

c. With the aid of software, SSE = $\sum_{i=1}^{13} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{13} (y_i - [9.1010 - .21572 x_i])^2$ = … = 3.505. Or, using the available sum of squares and a derivation similar to the one described in the section, SSE = $S_{yy} - \hat{\beta}_1 S_{xy}$ = 77.270769 – (–.21572)(–341.959231) = 3.505. Either way, the residual standard deviation is $s = \sqrt{\dfrac{\text{SSE}}{n-2}} = \sqrt{\dfrac{3.505}{13-2}}$ = 0.564.
The typical difference between a sample's actual ammonium concentration and the concentration predicted by the least squares regression line is about ±0.564 mg/L.

d. With SST = S_{yy} = 77.270769, $r^2 = 1 - \dfrac{\text{SSE}}{\text{SST}} = 1 - \dfrac{3.505}{77.270769}$ = .955. So, the least squares regression line helps to explain 95.5% of the total variation in ammonium concentration. Given that high percentage and the linear relationship visible in the scatterplot (see Exercise 4), yes, the model does a good job of explaining observed variation in ammonium concentration.

13. For this data, $n = 4$, $\Sigma x_i = 200$, $\Sigma y_i = 5.37$, $\Sigma x_i^2 = 12.000$, $\Sigma y_i^2 = 9.3501$, $\Sigma x_i y_i = 333 \Rightarrow$

$$S_{xx} = 12,000 - \frac{(200)^2}{4} = 2000, \text{ SST = } S_{yy} = 9.3501 - \frac{(5.37)^2}{4} = 2.140875, \; S_{xy} = 333 - \frac{(200)(5.37)}{4} = 64.5$$

$$\Rightarrow \hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{64.5}{2000} = .03225 \Rightarrow SSE = S_{yy} - \hat{\beta}_1 S_{xy} = 2.14085 - (.03225)(64.5) = .060750.$$

From these calculations, $r^2 = 1 - \frac{SSE}{SST} = 1 - \frac{.060750}{2.14085} = .972$. This is a very high value of r^2, which confirms the authors' claim that there is a strong linear relationship between the two variables. (A scatter plot also shows a strong, linear relationship.)

14.

a. $n = 24$, $\Sigma x_i = 4308$, $\Sigma y_i = 40.09$, $\Sigma x_i^2 = 773,790$, $\Sigma y_i^2 = 76.8823$, $\Sigma x_i y_i = 7,243.65 \Rightarrow$

$$S_{xx} = 773,790 - \frac{(4308)^2}{24} = 504.0, \; S_{yy} = 76.8823 - \frac{(40.09)^2}{24} = 9.9153,$$

$$S_{xy} = 7,243.65 - \frac{(4308)(40.09)}{24} = 45.8246 \Rightarrow \hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{45.8246}{504} = .09092,$$

$\hat{\beta}_0 = \frac{40.09}{24} - (.09092)\frac{4308}{24} = -14.6497$. Therefore, the equation of the estimated regression line is $\hat{y} = -14.6497 + .09092x$.

b. When $x = 182$, $\hat{y} = -14.6497 + .09092(182) = 1.8997$. So when the tank temperature is 182, we would predict an efficiency ratio of 1.8997.

c. The four observations for which temperature is 182 are: (182, .90), (182, 1.81), (182, 1.94), and (182, 2.68). Their corresponding residuals are: $.90 - 1.8997 = -0.9977$, $1.81 - 1.8997 = -0.0877$, $1.94 - 1.8997 = 0.0423$, $2.68 - 1.8997 = 0.7823$. These residuals do not all have the same sign because in the cases of the first two pairs of observations, the observed efficiency ratios were smaller than the predicted value of 1.8997. Whereas, in the cases of the last two pairs of observations, the observed efficiency ratios were larger than the predicted value.

d. $\text{SSE} = S_{yy} - \hat{\beta}_1 S_{xy} = 9.9153 - (.09092)(45.8246) = 5.7489 \Rightarrow \quad r^2 = 1 - \frac{\text{SSE}}{\text{SST}} = 1 - \frac{5.7489}{9.9153} = .4202$.

42.02% of the observed variation in efficiency ratio can be attributed to the approximate linear relationship between the efficiency ratio and the tank temperature.

15.

a. The following stem and leaf display shows that: a typical value for this data is a number in the low 40's. There is some positive skew in the data. There are some potential outliers (79.5 and 80.0), and there is a reasonably large amount of variation in the data (e.g., the spread 80.0-29.8 = 50.2 is large compared with the typical values in the low 40's).

2	9
3	33
3	5566677889
4	1223
4	56689
5	1
5	
6	2
6	9
7	
7	9
8	0

stem = tens
leaf = ones

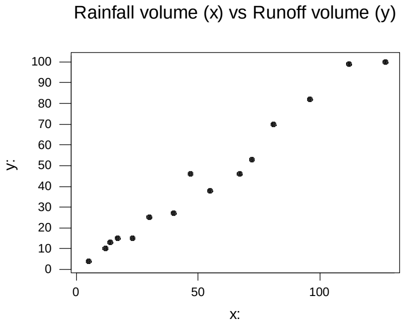
b. No, the strength values are not uniquely determined by the MoE values. For example, note that the two pairs of observations having strength values of 42.8 have different MoE values.

c. The least squares line is $\hat{y} = 3.2925 + .10748x$. For a beam whose modulus of elasticity is $x = 40$, the predicted strength would be $\hat{y} = 3.2925 + .10748(40) = 7.59$. The value $x = 100$ is far beyond the range of the x values in the data, so it would be dangerous (i.e., potentially misleading) to extrapolate the linear relationship that far.

d. From the output, SSE = 18.736, SST = 71.605, and the coefficient of determination is $r^2 = .738$ (or 73.8%). The r^2 value is large, which suggests that the linear relationship is a useful approximation to the true relationship between these two variables.

16.

a.


Rainfall volume (x) vs Runoff volume (y)

Yes, the scatterplot shows a strong linear relationship between rainfall volume and runoff volume, thus it supports the use of the simple linear regression model.

b. $\overline{x} = 53.200$, $\overline{y} = 42.867$, $S_{xx} = 63040 - \dfrac{(798)^2}{15} = 20{,}586.4$, $S_{yy} = 41{,}999 - \dfrac{(643)^2}{15} = 14{,}435.7$, and $S_{xy} = 51{,}232 - \dfrac{(798)(643)}{15} = 17{,}024.4 \Rightarrow \hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}} = \dfrac{17{,}024.4}{20{,}586.4} = .82697$ and $\hat{\beta}_0 = 42.867 - (.82697)53.2 = -1.1278$.

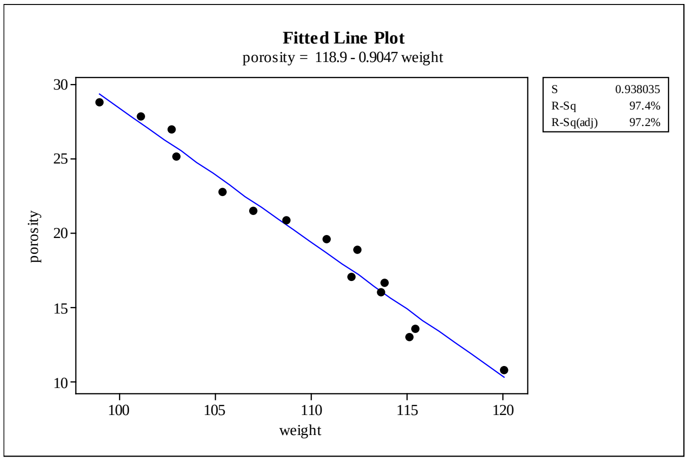
c. $\mu_{y\cdot 50} = -1.1278 + .82697(50) = 40.2207$.

d. $\text{SSE} = S_{yy} - \hat{\beta}_1 S_{xy} = 14{,}435.7 - (.82697)(17{,}324.4) = 357.07 \Rightarrow s = \sqrt{\dfrac{357.07}{15-2}} = 5.24$.

e. $r^2 = 1 - \dfrac{\text{SSE}}{\text{SST}} = 1 - \dfrac{357.07}{14{,}435.7} = .9753$. So 97.53% of the observed variation in runoff volume can be attributed to the simple linear regression relationship between runoff and rainfall.

17.

a. From software, the equation of the least squares line is $\hat{y}$ = 118.91 – .905*x*. The accompanying fitted line plot shows a very strong, linear association between unit weight and porosity. So, yes, we anticipate the linear model will explain a great deal of the variation in *y*.



b. The slope of the line is b_1 = –.905. A one-pcf increase in the unit weight of a concrete specimen is associated with a .905 percentage point decrease in the specimen's predicted porosity. (Note: slope is not ordinarily a percent decrease, but the units on porosity, *y*, are percentage points.)

c. When *x* = 135, the predicted porosity is $\hat{y}$ = 118.91 – .905(135) = –3.265. That is, we get a negative prediction for *y*, but in actuality *y* cannot be negative! This is an example of the perils of extrapolation; notice that *x* = 135 is outside the scope of the data.

d. The first observation is (99.0, 28.8). So, the actual value of *y* is 28.8, while the predicted value of *y* is 118.91 – .905(99.0) = 29.315. The residual for the first observation is $y - \hat{y}$ = 28.8 – 29.315 = –.515 ≈ –.52. Similarly, for the second observation we have $\hat{y}$ = 27.41 and residual = 27.9 – 27.41 = .49.

e. From software and the data provided, a point estimate of σ is s = .938. This represents the "typical" size of a deviation from the least squares line. More precisely, predictions from the least squares line are "typically" ± .938% off from the actual porosity percentage.

f. From software, r^2 = 97.4% or .974, the proportion of observed variation in porosity that can be attributed to the approximate linear relationship between unit weight and porosity.

18. Minitab output is provided below.

a. Using software and the data provided, the equation of the least squares regression line is given by $\hat{y}$ = –31.80 + 0.987x. So, a one-MPa increase in cube strength is associated with a 0.987 MPa increase in the predicted axial strength for these asphalt samples.

b. From software, r^2 = .630. That is, 63.0% of the observed variation in axial strength of asphalt samples of this type can be attributed to its linear relationship with cube strength.

c. From software, a point estimate of σ is s = 6.625. This is the "typical" size of a residual. That is, the model's prediction for axial strength will typically differ by ± 6.625 MPa from the specimen's actual axial strength.

Regression Analysis: y versus x

```
The regression equation is
y = - 31.8 + 0.987 x


Predictor    Coef  SE Coef      T      P
Constant   -31.80    25.87  -1.23  0.254
x          0.9870   0.2674   3.69  0.006


S = 6.62476   R-Sq = 63.0%   R-Sq(adj) = 58.4%
```

19. n = 14, $\Sigma x_i = 3300$, $\Sigma y_i = 5010$, $\Sigma x_i^2 = 913,750$, $\Sigma y_i^2 = 2,207,100$, $\Sigma x_i y_i = 1,413,500$

a. $\hat{\beta}_1 = \dfrac{3,256,000}{1,902,500} = 1.71143233$, $\hat{\beta}_0 = -45.55190543$, so the equation of the least squares line is roughly $\hat{y}$ = –45.5519 + 1.7114x.

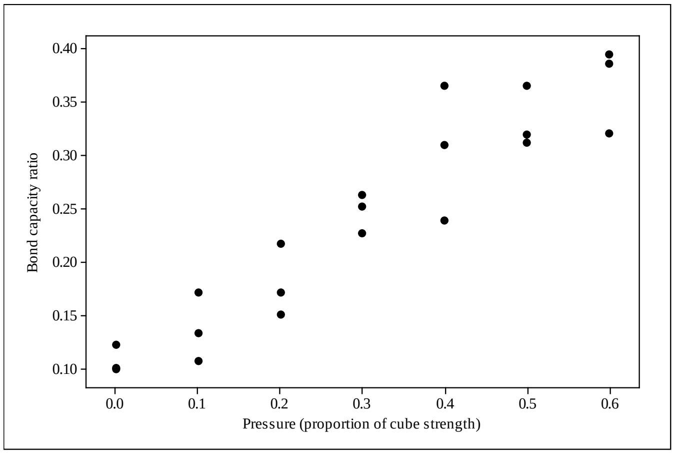
b. $\hat{\mu}_{Y\cdot 225} = -45.5519 + 1.7114(225) = 339.51$.

c. Estimated expected change $= -50\hat{\beta}_1 = -85.57$.

d. No, the value 500 is outside the range of x values for which observations were available (the danger of extrapolation).

20.

a. The accompanying scatterplot shows a reasonably strong, positive, linear relationship between pressure and the bond capacity ratio. The linearity of the graph supports the use of a linear model.



b. The slope and intercept estimates appear in the least squares equation: $\hat{\beta}_1 = .461$, $\hat{\beta}_0 = .101$. Their more precise values of .46071 and .10121, respectively, appear in the table below the equation.

c. Substitute $x = .45$ to get $\hat{y} = .10121 + .46071(.45) = .3085$.

d. From the Minitab output, an estimate of σ is $s = 0.0332397$. This represents the typical difference between a concrete specimen's actual bond capacity ratio and the ratio predicted by the least squares regression line.

e. The total variation in bond capacity ratio is SST = 0.19929. 89.5% of this variation can be explained by the model. Note: $1-\frac{\text{SSE}}{\text{SST}}=1-\frac{0.02099}{0.19929}$ = .895, which matches r^2 on output.

21.

a. Yes – a scatter plot of the data shows a strong, linear pattern, and $r^2 = 98.5\%$.

b. From the output, the estimated regression line is ŷ = 321.878 + 156.711x, where x = absorbance and y = resistance angle. For x = .300, ŷ = 321.878 + 156.711(.300) = 368.89.

c. The estimated regression line serves as an estimate both for a single y at a given x-value and for the true average μ_y at a given x-value. Hence, our estimate for μ_y when x = .300 is also 368.89.

22.

a. Software (Minitab) provides the accompanying output. The least squares regression line for this data is $\hat{y} = 11.0 - 0.448x$. The coefficient of determination is 69.4% or .694, meaning that 69.4% of the observed variation in compressive strength (y) can be attributed to a linear model relationship with fiber weight (x). Finally, a point estimate of $\sigma = V(\varepsilon)$ is $s = 1.08295$.

Regression Analysis: y versus x

```
The regression equation is
y = 11.0 - 0.448 x


Predictor       Coef  SE Coef      T      P
Constant     11.0129   0.3289  33.49  0.000
x           -0.44805  0.06073  -7.38  0.000


S = 1.08295   R-Sq = 69.4%   R-Sq(adj) = 68.1%
```

b. The accompanying Minitab output corresponds to least squares regression through the six points (0, 11.05), (1.25, 10.51), (2.50, 10.32), (5, 8.15), (7.5, 6.93), (10, 7.24). Notice the least squares regression line is barely different; however, we now have $r^2 = 90.2\% = .902$. This is markedly higher than r^2 for the original data. A linear model can explain 69.3% of the observed variation in compressive strength, but 90.2% of the observed variation in average compressive strength.

In general, averaged y-values (the six response values here) will have less variation than individual values (the 26 original observations). Therefore, there is less observed variation in the "response variable," and the least squares line can account for a larger proportion of that variation.

Regression Analysis: avg y versus x

```
The regression equation is
avg y = 11.0 - 0.445 x


Predictor       Coef  SE Coef      T      P
Constant     10.9823   0.4114  26.69  0.000
x           -0.44547  0.07330  -6.08  0.004

S = 0.631439   R-Sq = 90.2%   R-Sq(adj) = 87.8%
```

23.

a. Using the given y_i's and the formula $\hat{y}_i = -45.5519 + 1.7114x_i$,

$SSE = (150 - 125.6)^2 + ... + (670 - 639.0)^2 = 16{,}213.64$. The computation formula gives

$SSE = 2{,}207{,}100 - (-45.55190543)(5010) - (1.71143233)(1{,}413{,}500) = 16{,}205.45$

b. $SST = 2{,}207{,}100 - \frac{(5010)^2}{14} = 414{,}235.71$ so $r^2 = 1 - \frac{16{,}205.45}{414{,}235.71} = .961$.

24.

a. From software, the least squares line is $\hat{y}$ = –305.9 + 9.963x. When x = 70, $\hat{y}$ = 392; the residual corresponding to the point (70, 13) is $y - \hat{y}$ = 13 – 392 = –379. When x = 71, 402; the residual corresponding to the point (71, 1929) is $y - \hat{y}$ = 1929 – 402 = 1527. Both residuals are extraordinarily large, but the residual for the first point is large and negative (the line greatly over-estimates the true colony density) while the residual for the second point is large and positive (the line greatly under-estimates the true, enormous colony density for that observation).

b. From software, $r^2 = 1 - \frac{\text{SSE}}{\text{SST}} = 1 - \frac{2900807}{3310341}$ = .124, or 12.4%. Just 12.4% of the total variation in colony density can be explained by a linear regression model with rock surface area as the predictor.

c. The table below compares the least squares equation, s, and r^2 for the two data sets (the 15 observations provided, and the 14 points left when the outlier is removed). Everything changes radically. The slope and intercept are completely different without the outlier; the residual standard deviation decreases by a factor of 5; and r^2 decreases to an even smaller 2.4%.

	With outlier (n = 15)	Without outlier (n = 14)
Least squares line	$\hat{y}$ = –305.9 + 9.963x	$\hat{y}$ = 34.37 + 0.779x
s	472.376	87.222
r^2	12.4%	2.4%

25. Substitution of $\hat{\beta}_0 = \frac{\Sigma y_i - \hat{\beta}_1 \Sigma x_i}{n}$ and $\hat{\beta}_1$ for b_0 and b_1 on the left-hand side of the first normal equation yields $n\frac{\Sigma y_i - \hat{\beta}_1 \Sigma x_i}{n} + \hat{\beta}_1 \Sigma x_i = \Sigma y_i - \hat{\beta}_1 \Sigma x_i + \hat{\beta}_1 \Sigma x_i = \Sigma y_i$, which verifies they satisfy the first one. The same substitution in the left-hand side of the second equation gives

$$\frac{(\Sigma x_i)\left(\Sigma y_i - \hat{\beta}_1 \Sigma x_i\right)}{n} + \left(\Sigma x_i^2\right)\hat{\beta}_1 = \frac{(\Sigma x_i)(\Sigma y_i) + \hat{\beta}_1\left(n\Sigma x_i^2 - (\Sigma x_i)^2\right)}{n} = (\Sigma x_i)(\Sigma y_i)/n + \hat{\beta}_1\left(\Sigma x_i^2 - (\Sigma x_i)^2/n\right)$$

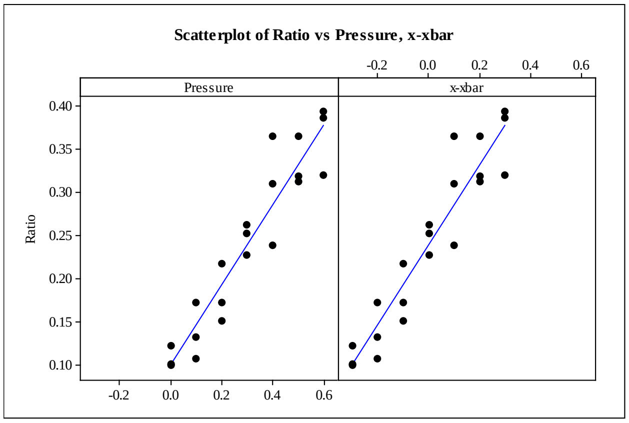
. The last term in parentheses is S_{xx}, so making that substitution along with Equation (12.2) we have $(\Sigma x_i)(\Sigma y_i)/n + \frac{S_{xy}}{S_{xx}}(S_{xx}) = (\Sigma x_i)(\Sigma y_i)/n + S_{xy}$. By the definition of S_{xy}, this last expression is exactly $\Sigma\, x_i y_i$, which verifies that the slope and intercept formulas satisfy the second normal equation.

26. We must show that when $\bar{x}$ is substituted for x in $\hat{\beta}_0 + \hat{\beta}_1 x$, $\bar{y}$ results, so that $(\bar{x}, \bar{y})$ is on the least squares line: $\hat{\beta}_0 + \hat{\beta}_1 \bar{x} = \frac{\Sigma y_i - \hat{\beta}_1 \Sigma x_i}{n} + \hat{\beta}_1 \bar{x} = \bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 \bar{x} = \bar{y}$.

27. We wish to find b_1 to minimize $f(b_1) = \sum (y_i - b_1 x_i)^2$. Equating $f'(b_1)$ to 0 yields $\sum\left[2(y_i - b_1 x_i)^1(-x_i)\right] = 0 \Rightarrow 2\sum\left[-x_i y_i + b_1 x_i^2\right] = 0 \Rightarrow \Sigma x_i y_i = b_1 \Sigma x_i^2$ and $b_1 = \frac{\Sigma x_i y_i}{\Sigma x_i^2}$. The least squares estimator of $\hat{\beta}_1$ is thus $\hat{\beta}_1 = \frac{\Sigma x_i Y_i}{\Sigma x_i^2}$.

28.

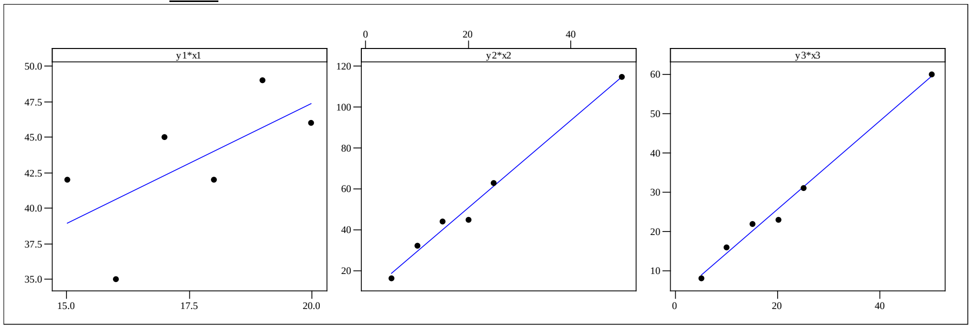
a. In the figure below, the left-hand panel shows the original y versus x, while the right-hand panel shows the new plot y versus $x-\overline{x}$. Subtracting $\overline{x}$ from each x_i shifts the plot horizontally without otherwise altering its character. The least squares line for the new plot will thus have the same slope as the one for the old plot. As for the intercept, Exercise 26 showed that $(\overline{x},\overline{y})$ lies on the least squares line in the left-hand plot; shifting that horizontally by $\overline{x}$ implies that the point $(0,\overline{y})$ lies on the least squares line in the right-hand plot. (The mean ratio is about $\overline{y}$ = .24, and the point (0, .24) does seem to lie on the right-hand line.) In other words, the new intercept is $\overline{y}$ itself.



b. We want b_0 and b_1 that minimize f(b_0, b_1) = $f(b_0,b_1)=\sum\left[y_i-\left(b_0+b_1\left(x_i-\overline{x}\right)\right)\right]^2$. Equating $\dfrac{\partial f}{\partial b_0}$ and $\dfrac{\partial f}{\partial b_1}$ to 0 yields $nb_0+b_1\Sigma\left(x_i-\overline{x}\right)=\Sigma y_i$ and $b_0\Sigma\left(x_i-\overline{x}\right)+b_1\Sigma\left(x_i-\overline{x}\right)^2=\Sigma\left(x_i-\overline{x}\right)y_i$.

Since $\Sigma\left(x_i-\overline{x}\right)=0$, the first equation implies $b_0=\overline{y}$ (as noted in **a**) and the second equation becomes $b_1=\dfrac{\Sigma\left(x_i-\overline{x}\right)y_i}{\Sigma\left(x_i-\overline{x}\right)^2}$. But this expanding the top sum provides $b_1=\dfrac{\Sigma x_iy_i-\left(\Sigma x_i\right)\left(\Sigma y_i\right)/n}{\Sigma\left(x_i-\overline{x}\right)^2}$ = $\dfrac{S_{xy}}{S_{xx}}$ = $\hat{\beta}_1$ from Equation (12.2). Therefore, the intercept and slope estimators under this modified regression model are $\hat{\beta}_0^*=\overline{Y}$ and $\hat{\beta}_1^*=\hat{\beta}_1$; again, this is what the graph in **a** implied.

29. For data set #1, r^2 = .43 and s = 4.03; for #2, r^2 = .99 and s = 4.03; for #3, r^2 = .99 and s = 1.90. In general, we hope for both large r^2 (large % of variation explained) and small s (indicating that observations don't deviate much from the estimated line). Simple linear regression would thus seem to be <u>most</u> effective for data set #3 and <u>least</u> effective for data set #1.



Section 12.3

30.

a. $\Sigma\left(x_i-\overline{x}\right)^2=7,000,000$, so $V\left(\hat{\beta}_1\right)=\dfrac{\left(350\right)^2}{7,000,000}=.0175$ and the standard deviation of $\hat{\beta}_1$ is $\sqrt{.0175}=.1323$.

b. $P\left(1.0\le\hat{\beta}_1\le1.5\right)=P\left(\dfrac{1.0-1.25}{.1323}\le Z\le\dfrac{1.5-1.25}{.1323}\right)$ $=P\left(-1.89\le Z\le1.89\right)=.9412$.

c. Although *n* = 11 here and *n* = 7 in **a**, $\Sigma\left(x_i-\overline{x}\right)^2=1{,}100{,}000$ now, which is smaller than in **a**. Because this appears in the denominator of $V\left(\hat{\beta}_1\right)$, the variance is smaller for the choice of *x* values in **a**.

31.

a. Software output from least squares regression on this data appears below. From the output, we see that r^2 = 89.26% or .8926, meaning 89.26% of the observed variation in threshold stress (*y*) can be attributed to the (approximate) linear model relationship with yield strength (*x*).

```
Regression Equation

y  =  211.655 - 0.175635 x

Coefficients

Term         Coef  SE Coef        T      P        95% CI
Constant  211.655  15.0622  14.0521  0.000  (178.503, 244.807)
x          -0.176   0.0184  -9.5618  0.000  ( -0.216,  -0.135)

Summary of Model

S = 6.80578      R-Sq = 89.26%        R-Sq(adj) = 88.28%
```

b. From the software output, $\hat{\beta}_1$ = –0.176 and $s_{\hat{\beta}_1}$ = 0.0184. Alternatively, the residual standard deviation is *s* = 6.80578, and the sum of squared deviations of the *x*-values can be calculated to equal S_{xx} = $\sum(x_i-\overline{x})^2$ = 138095. From these, $s_{\hat{\beta}_1}=\dfrac{s}{\sqrt{S_{xx}}}$ = .0183 (due to some slight rounding error).

c. From the software output, a 95% CI for β_1 is (–0.216, –0.135). This is a fairly narrow interval, so β_1 has indeed been precisely estimated. Alternatively, with *n* = 13 we may construct a 95% CI for β_1 as $\hat{\beta}_1\pm t_{.025,12}s_{\hat{\beta}_1}$ = –0.176 ± 2.179(.0184) = (–0.216, –0.136).

32. Let β_1 denote the true average change in runoff for each 1 m^3 increase in rainfall. To test the hypotheses H_0: $\beta_1 = 0$ versus H_a: $\beta_1 \ne 0$, the calculated *t* statistic is $t=\dfrac{\hat{\beta}_1-0}{s_{\hat{\beta}_1}}=\dfrac{.82697}{.03652}=22.64$ which (from the printout) has an associated *P*-value of ~0.000. Therefore, since the *P*-value is so small, H_0 is rejected and we conclude that there <u>is</u> a useful linear relationship between runoff and rainfall.

A confidence interval for β_1 is based on $n - 2 = 15 - 2 = 13$ degrees of freedom. Since $t_{.025,13} = 2.160$, the interval estimate is $\hat{\beta}_1 \pm t_{.025,13} \cdot s_{\hat{\beta}_1} = .82697 \pm (2.160)(.03652) = (.748, .906)$. Therefore, we can be confident that the true average change in runoff, for each 1 m^3 increase in rainfall, is somewhere between .748 m^3 and .906 m^3.

33.

a. Error df = $n - 2 = 25$, $t_{.025,25} = 2.060$, and so the desired confidence interval is $\hat{\beta}_1 \pm t_{.025,25} \cdot s_{\hat{\beta}_1} = .10748 \pm (2.060)(.01280) = (.081, .134)$. We are 95% confident that the true average change in strength associated with a 1 GPa increase in modulus of elasticity is between .081 MPa and .134 MPa.

b. We wish to test H_0: $\beta_1 \le .1$ versus H_a: $\beta_1 > .1$. The calculated test statistic is $t = \dfrac{\hat{\beta}_1 - .1}{s_{\hat{\beta}_1}} = \dfrac{.10748 - .1}{.01280} = .58$, which yields a P-value of .277 at 25 df. Thus, we fail to reject H_0; i.e., there is not enough evidence to contradict the prior belief.

34.

a. From the R output, the intercept and slope are 4.858691 and –0.074676, respectively. So, the equation of the least squares line is $\hat{y} = 4.858691 - 0.074676x$. According to the slope, a one-percentage-point increase in air void is associated with an estimated decrease in dielectric constant of 0.074676.

b. From the output, $r^2 = .7797$, or 77.97%.

c. Yes. The hypotheses are H_0: $\beta_1 = 0$ versus H_a: $\beta_1 \ne 0$, for which the R output provides a test statistic of $t = -7.526$ and a P-value of 1.21×10^{-6}. Based on the extremely small P-value, we strongly reject H_0 and conclude that a statistically significant linear relationship exists between dielectric constant and air void percentage.

d. Now the hypotheses of interest are H_0: $\beta_1 \ge -.05$ versus H_a: $\beta_1 < -.05$. From the R output, the new test statistic is $t = \dfrac{-0.074676 - (-.05)}{0.009923} = -2.5$. At 16 df, the lower-tailed P-value is .012 from Table A.8. Thus, we barely fail to reject H_0 at the .01 level: there is insufficient evidence (but only barely) to contradict the prior belief.

35.

a. We want a 95% CI for β_1: Using the given summary statistics, $S_{xx} = 3056.69 - \dfrac{(222.1)^2}{17} = 155.019$, $S_{xy} = 2759.6 - \dfrac{(222.1)(193)}{17} = 238.112$, and $\hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}} = \dfrac{238.112}{115.019} = 1.536$. We need $\hat{\beta}_0 = \dfrac{193 - (1.536)(222.1)}{17} = -8.715$ to calculate the SSE:

$\text{SSE} = 2975 - (-8.715)(193) - (1.536)(2759.6) = 418.2494$. Then $s = \sqrt{\dfrac{418.2494}{15}} = 5.28$ and $s_{\hat{\beta}_1} = \dfrac{5.28}{\sqrt{155.019}} = .424$. With $t_{.025,15} = 2.131$, our CI is $1.536 \pm 2.131 \cdot (.424)$ = (.632, 2.440). With 95% confidence, we estimate that the change in reported nausea percentage for every one-unit change in motion sickness dose is between .632 and 2.440.

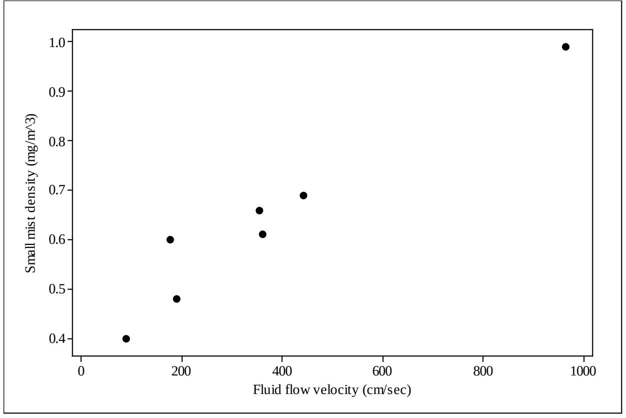
b. We test the hypotheses H_0: $\beta_1 = 0$ versus H_a: $\beta_1 \neq 0$, and the test statistic is $t = \frac{1.536}{.424} = 3.6226$. With df = 15, the two-tailed P-value = $2P(T > 3.6226) = 2(.001) = .002$. With a P-value of .002, we would reject the null hypothesis at most reasonable significance levels. This suggests that there is a useful linear relationship between motion sickness dose and reported nausea.

c. No. A regression model is only useful for estimating values of nausea % when using dosages between 6.0 and 17.6, the range of values sampled.

d. Removing the point (6.0, 2.50), the new summary stats are: $n = 16, \Sigma x_i = 216.1$, $\Sigma y_i = 191.5$, $\Sigma x_i^2 = 3020.69$, $\Sigma y_i^2 = 2968.75$, $\Sigma x_i y_i = 2744.6$, and then $\hat{\beta}_1 = 1.561$, $\hat{\beta}_0 = -9.118$, SSE = 430.5264, $s = 5.55$, $s_{\hat{\beta}_1} = .551$, and the new CI is $1.561 \pm 2.145 \cdot (.551)$, or (.379, 2.743). The interval is a little wider. But removing the one observation did not change it that much. The observation does not seem to be exerting undue influence.

36.

a. A scatter plot, generated by Minitab, supports the decision to use linear regression analysis.



b. We are asked for the coefficient of determination, r^2. From the Minitab output, $r^2 = .931$ (which is close to the hand calculated value, the difference being accounted for by round-off error.)

c. Increasing x from 100 to 1000 means an increase of 900. If, as a result, the average y were to increase by .6, the slope would be .6/900 = .0006667. We should test the hypotheses H_0: $\beta_1 = .0006667$ versus H_a: $\beta_1 < .0006667$. The test statistic is $t = \frac{.00062108 - .0006667}{.00007579} = -.601$, which is not statistically significant. There is not sufficient evidence that with an increase from 100 to 1000, the true average increase in y is less than .6.

d. We are asked for a confidence interval for β_1. Using the values from the Minitab output, we have $.00062108 \pm 2.776(.00007579) = (.00041069, .00083147)$.

37.

a. Let μ_d = the true mean difference in velocity between the two planes. We have 23 pairs of data that we will use to test H_0: $\mu_d = 0$ v. H_a: $\mu_d \neq 0$. From software, $\bar{x}_d = 0.2913$ with $s_d = 0.1748$, and so $t = \frac{0.2913 - 0}{0.1748} \approx 8$, which has a two-sided P-value of 0.000 at 22 df. Hence, we strongly reject the null hypothesis and conclude there is a statistically significant difference in true average velocity in the two planes. [*Note:* A normal probability plot of the differences shows one mild outlier, so we have slight concern about the results of the t procedure.]

b. Let β_1 denote the true slope for the linear relationship between Level – – velocity and Level – velocity. We wish to test H_0: $\beta_1 = 1$ v. H_a: $\beta_1 < 1$. Using the relevant numbers provided, $t = \frac{b_1 - 1}{s(b_1)} = \frac{0.65393 - 1}{0.05947}$ = –5.8, which has a one-sided P-value at 23–2 = 21 df of $P(T < -5.8) \approx 0$. Hence, we strongly reject the null hypothesis and conclude the same as the authors; i.e., the true slope of this regression relationship is significantly less than 1.

38.

a. From Exercise 23, which also refers to Exercise 19, SSE = 16.205.45, so $s^2 = 1350.454$, $s = 36.75$, and $s_{\hat{\beta}_1} = \frac{36.75}{368.636} = .0997$. Thus $t = \frac{1.711 - 0}{.0997} = 17.2$; at 14 df, the P-value is < .001. Because the P-value < .01, H_0: $\beta_1 = 0$ is rejected at level .01 in favor of the conclusion that the model is useful ($\beta_1 \neq 0$).

b. The CI for β_1 is $1.711 \pm (2.179)(.0997) = 1.711 \pm .217 = (1.494, 1.928)$. Thus the CI for $10\beta_1$ is (14.94, 19.28).

39. SSE = 124,039.58– (72.958547)(1574.8) – (.04103377)(222657.88) = 7.9679, and SST = 39.828

Source	**df**	**SS**	**MS**	***f***
Regr	1	31.860	31.860	18.0
Error	18	7.968	1.77	
Total	19	39.828		

At df = (1, 18), $f = 18.0 > F_{.001,1,18} = 15.38$ implies that the P-value is less than .001. So, H_0: $\beta_1 = 0$ is rejected and the model is judged useful. Also, $s = \sqrt{1.77} = 1.33041347$ and $S_{xx} = 18{,}921.8295$, so $t = \frac{.04103377}{1.33041347 / \sqrt{18{,}921.8295}} = 4.2426$ and $t^2 = (4.2426)^2 = 18.0 = f$, showing the equivalence of the two tests.

40. We use the fact that $\hat{\beta}_1$ is unbiased for β_1.

$$E\left(\hat{\beta}_0\right)=E\left(\frac{\Sigma Y_i-\hat{\beta}_1\Sigma x_i}{n}\right)=\frac{E\left(\Sigma Y_i\right)-E\left(\hat{\beta}_1\right)\Sigma x_i}{n}=\frac{\Sigma E(Y_i)-\beta_1\Sigma x_i}{n}$$
$$=\frac{\Sigma(\beta_0+\beta_1 x_i)-\beta_1\Sigma x_i}{n}=\frac{n\beta_0+\Sigma\beta_1 x_i-\beta_1\Sigma x_i}{n}=\frac{n\beta_0}{n}=\beta_0$$

41.

a. Under the regression model, $E(Y_i)=\beta_0+\beta_1 x_i$ and, hence, $E(\overline{Y})=\beta_0+\beta_1\overline{x}$. Therefore, $E(Y_i-\overline{Y})=\beta_1(x_i-\overline{x})$, and $E\left(\hat{\beta}_1\right)=E\left[\frac{\sum(x_i-\overline{x})(Y_i-\overline{Y})}{\sum(x_i-\overline{x})^2}\right]=\frac{\sum(x_i-\overline{x})E[Y_i-\overline{Y}]}{\sum(x_i-\overline{x})^2}$
$=\frac{\sum(x_i-\overline{x})\beta_1(x_i-\overline{x})}{\sum(x_i-\overline{x})^2}=\beta_1\frac{\sum(x_i-\overline{x})^2}{\sum(x_i-\overline{x})^2}=\beta_1$.

b. Here, we'll use the fact that $\sum(x_i-\overline{x})(Y_i-\overline{Y})=\sum(x_i-\overline{x})Y_i-\overline{Y}\sum(x_i-\overline{x})=\sum(x_i-\overline{x})Y_i-\overline{Y}(0)$ = $\sum(x_i-\overline{x})Y_i$. With $c=\Sigma\left(x_i-\overline{x}\right)^2$, $\hat{\beta}_1=\frac{1}{c}\sum(x_i-\overline{x})(Y_i-\overline{Y})=\sum\frac{(x_i-\overline{x})}{c}Y_i$ $\Rightarrow$ since the Y_is are independent, $V(\hat{\beta}_1)=\sum\left(\frac{x_i-\overline{x}}{c}\right)^2 V(Y_i)=\frac{1}{c^2}\sum\left(x_i-\overline{x}\right)^2\sigma^2=\frac{\sigma^2}{c}=\frac{\sigma^2}{\sum\left(x_i-\overline{x}\right)^2}$ or, equivalently, $\frac{\sigma^2}{\Sigma x_i^2-\left(\Sigma x_i\right)^2/n}$, as desired.

42. Let * indicate the rescaled test statistic (x to cx, y to dy). The revised t statistic is $t*=\frac{\hat{\beta}_1^*}{s^*/\sqrt{S_{xx}^*}}$.

First, the new slope is $\hat{\beta}_1^*=\frac{S_{xy}^*}{S_{xx}^*}=\frac{\Sigma(cx_i-c\overline{x})(dy_i-d\overline{y})}{\Sigma(cx_i-c\overline{x})^2}=\frac{cdS_{xy}}{c^2S_{xx}}=\frac{d}{c}\hat{\beta}_1$.

Second, the new residual sd is $s^*=\sqrt{\frac{\text{SSE}^*}{n-2}}=\sqrt{\frac{\Sigma(dy_i-d\hat{y}_i)^2}{n-2}}=\sqrt{\frac{d^2\Sigma(y_i-\hat{y}_i)^2}{n-2}}=d\sqrt{\frac{\text{SSE}}{n-2}}=ds$.

Third, $S_{xx}^*=\Sigma(cx_i-c\overline{x})^2=c^2\Sigma(x_i-\overline{x})^2=c^2S_{xx}$. Putting all of the rescalings together,

$t*=\frac{(d/c)\hat{\beta}_1}{ds/\sqrt{c^2S_{xx}}}=\frac{(d/c)\hat{\beta}_1}{(d/c)\cdot s/\sqrt{S_{xx}}}=\frac{\hat{\beta}_1}{s/\sqrt{S_{xx}}}=t$, where t is the original test statistic.

43. The numerator of d is |1 – 2| = 1, and the denominator is $\frac{4\sqrt{14}}{\sqrt{324.40}}=.831$, so $d=\frac{1}{.831}=1.20$. The approximate power curve is for n – 2 df = 13, and β is read from Table A.17 as approximately .1.

Section 12.4

44.

a. The mean of the x data in Exercise 12.15 is $\overline{x} = 45.11$. Since x = 40 is closer to 45.11 than is x = 60, the quantity $\left(40-\overline{x}\right)^2$ must be smaller than $\left(60-\overline{x}\right)^2$. Therefore, since these quantities are the only ones that are different in the two $s_{\hat{y}}$ values, the $s_{\hat{y}}$ value for x = 40 must necessarily be smaller than $s_{\hat{y}}$ for x = 60. Said briefly, the closer x is to $\overline{x}$, the smaller the value of $s_{\hat{y}}$.

b. From the printout in Exercise 12.15, the error degrees of freedom is df = 25. Since $t_{.025,25}$ = 2.060, the interval estimate when x = 40 is 7.592 ± 2.060(.179) = 7.592 ± .369 = (7.223, 7.961). We estimate, with a high degree of confidence, that the true average strength for all beams whose MoE is 40 GPa is between 7.223 MPa and 7.961 MPa.

c. From the printout in Exercise 12.15, s = .8657, so the 95% prediction interval is $\hat{y} \pm t_{.025,25}\sqrt{s^2+s_{\hat{y}}^2} = 7.592 \pm \left(2.060\right)\sqrt{\left(.8657\right)^2+\left(.179\right)^2}$ = 7.592 ± 1.821 = (5.771, 9.413). Note that the prediction interval is almost 5 times as wide as the confidence interval.

d. For two 95% intervals, the simultaneous confidence level is at least 100(1 – 2(.05)) = 90%.

45.

a. We wish to find a 90% CI for $\mu_{y\cdot125}$: $\hat{y}_{125} = 78.088$, $t_{.05,18} = 1.734$, and $s_{\hat{y}} = s\sqrt{\frac{1}{20}+\frac{\left(125-140.895\right)^2}{18{,}921.8295}} = .1674$. Putting it together, we get 78.088 ± 1.734(.1674) = (77.797, 78.378).

b. We want a 90% PI. Only the standard error changes: $s_{\hat{y}} = s\sqrt{1+\frac{1}{20}+\frac{\left(125-140.895\right)^2}{18{,}921.8295}} = .6860$, so the PI is 78.088 ± 1.734(.6860) = (76.898, 79.277).

c. Because the x^* of 115 is farther away from $\overline{x}$ than the previous value, the term $(x^*-\overline{x})^2$ will be larger, making the standard error larger, and thus the width of the interval is wider.

d. We would be testing to see if the filtration rate were 125 kg-DS/m/h, would the average moisture content of the compressed pellets be less than 80%. The test statistic is $t = \frac{78.088-80}{.1674} = -11.42$, and with 18 df the P-value is $P(T < -11.42) \approx 0.00$. Hence, we reject H_0. There is significant evidence to prove that the true average moisture content when filtration rate is 125 is less than 80%.

46. The accompanying Minitab output will be used throughout.

a. From software, the least squares regression line is $\hat{y} = -1.5846 + 2.58494x$. The coefficient of determination is r^2 = 83.73% or .8373.

b. From software, a 95% CI for β_1 is roughly (2.16, 3.01). We are 95% confident that a one-unit increase in tannin concentration is associated with an increase in expected perceived astringency between 2.16

units and 3.01 units. (Since a 1-unit increase is unrealistically large, it would make more sense to say a 0.1-unit increase in *x* is associated with an increase between .216 and .301 in the expected value of *y*.)

c. From software, a 95% CI for $\mu_{Y|.6}$, the mean perceived astringency when $x = x^* = .6$, is roughly (–0.125, 0.058).

d. From software, a 95% PI for *Y*|.6, a single astringency value when $x = x^* = .6$, is roughly (–0.559, 0.491). Notice the PI is much wider than the corresponding CI, since we are making a prediction for a single future value rather than an estimate for a mean.

e. The hypotheses are H_0: $\mu_{Y|.7} = 0$ versus H_a: $\mu_{Y|.7} \neq 0$, where $\mu_{Y|.7}$ is the true mean astringency when $x = x^* = .7$. Since this is a two-sided test, the simplest approach is to use the 95% CI for $\mu_{Y|.7}$ provided by software. That CI, as seen in the output is roughly (0.125, 0.325). In particular, since this interval does not include 0, we reject H_0. There is evidence at the .05 level that the true mean astringency when tannin concentration equals .7 is something other than 0.

```
Coefficients

Term          Coef   SE Coef         T      P          95% CI
Constant  -1.58460  0.133860  -11.8377  0.000  (-1.85798, -1.31122)
x          2.58494  0.208042   12.4251  0.000  ( 2.16007,  3.00982)

Summary of Model

S = 0.253259     R-Sq = 83.73%        R-Sq(adj) = 83.19%

Predicted Values for New Observations

New Obs        Fit     SE Fit           95% CI                  95% PI
      1  -0.033635  0.0447899  (-0.125108, 0.057838)  (-0.558885, 0.491615)
      2   0.224859  0.0488238  ( 0.125148, 0.324571)  (-0.301888, 0.751606)

Values of Predictors for New Observations

New Obs    x
      1  0.6
      2  0.7
```

47.

a. $\hat{y}_{(40)} = -1.128 + .82697(40) = 31.95$, $t_{.025,13} = 2.160$; a 95% PI for runoff is

$$31.95 \pm 2.160\sqrt{(5.24)^2 + (1.44)^2} = 31.95 \pm 11.74 = (20.21, 43.69).$$

No, the resulting interval is very wide, therefore the available information is not very precise.

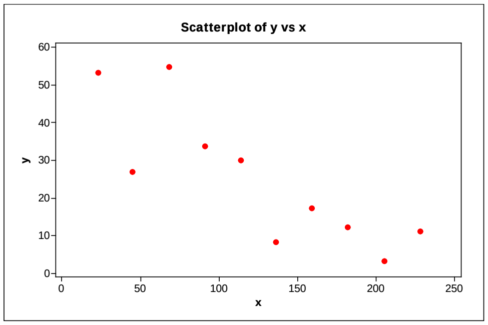
b. $\Sigma x = 798, \Sigma x^2 = 63{,}040$ which gives $S_{xx} = 20{,}586.4$, which in turn gives

$$s_{\hat{y}(50)} = 5.24\sqrt{\frac{1}{15} + \frac{(50 - 53.20)^2}{20{,}586.4}} = 1.358$$, so the PI for runoff when $x = 50$ is

$$40.22 \pm 2.160\sqrt{(5.24)^2 + (1.358)^2} = 40.22 \pm 11.69 = (28.53, 51.92).$$ The simultaneous prediction level for the two intervals is at least $100(1 - 2\alpha)\% = 90\%$.

48.

a. Yes, the scatter plot shows a reasonably linear relationship between percent of total suspended solids removed (y) and amount removed (x).



b. Using software, $\hat{y}$ = 52.63 – 0.2204x. These coefficients can also be obtained from the summary quantities provided.

c. Using software, $r^2 = \dfrac{2081.43}{2969.32}$ = 70.1%.

d. We wish to test H_0: $\beta_1 = 0$ v. H_a: $\beta_1 \neq 0$. Using software, $t = \dfrac{b_1 - 0}{s(b_1)} = \dfrac{-0.2204}{0.0509}$ = –4.33, which has a 2-sided P-value at 10–2 = 8 df of $P(|T| > 4.33) \approx 0.003 < .05$. Hence, we reject H_0 at the α = .05 level and conclude that a statistically useful linear relationship exists between x and y.

e. The hypothesized slope is (2-unit <u>decrease</u> in mean y)/(10-unit increase in x) = –0.2. Specifically, we wish to test H_0: $\beta_1 = -0.2$ v. H_a: $\beta_1 < -0.2$. The revised test statistic is $t = \dfrac{b_1 - (-0.2)}{s(b_1)} = \dfrac{-0.0204}{0.0509}$ = –0.4, and the corresponding P-value at 8 df is $P(T < -0.4) = P(T > 0.4)$ = .350 from Table A.8. Hence, we fail to reject H_0 at α = .05: the data do not provide significant evidence that the true average decrease in y associated with a 10kL increase in amount filtered exceeds 2%.

f. Plug x = 100kL into software; the resulting interval estimate for $\mu_{y.100}$ is (22.36, 38.82). We are 95% confident that the true average % of total suspended solids removed when 100,000 L are filtered is between 22.36% and 38.82%. Since x = 100 is nearer the average than x = 200, a CI for $\mu_{y.200}$ will be wider.

g. Again use software; the resulting PI for Y is (4.94, 56.24). We are 95% confident that the total suspended solids removed for one sample of 100,000 L filtered is between 4.94% and 56.24%. This interval is very wide, much wider than the CI in part **f**. However, this PI is narrower than a PI at x = 200, since x = 200 is farther from the mean than x = 100.

49. 95% CI = (462.1, 597.7) ⇒ midpoint = 529.9; $t_{.025,8} = 2.306 \Rightarrow 529.9 + (2.306)\hat{s}_{\hat{\beta}_0+\hat{\beta}_1(15)} = 597.7 \Rightarrow$

$\hat{s}_{\hat{\beta}_0+\hat{\beta}_1(15)} = 29.402 \Rightarrow$ 99% CI $= 529.9 \pm t_{.005,8}(29.402) = 529.9 \pm (3.355)(29.402) = (431.3, 628.5)$.

50.

a. Use software to find $s(b_1) = 0.065$. A 95% CI for β_1 is $b_1 \pm t_{.025,11}s(b_1) = –0.433 \pm 2.201(0.065) =$ (–0.576, –0.290). We want the effect of a .1-unit change in x, i.e. $0.1\beta_1$; the desired CI is just (–0.058, –0.029). We are 95% confident that the <u>decrease</u> in mean Fermi level position associated with a 0.1 increase in Ge concentration is between 0.029 and 0.058.

b. Using software, a 95% CI for $\mu_{y.0.50}$ is (0.4566, 0.5542). We are 95% confident that the mean Fermi position level when Ge concentration equals .50 is between 0.4566 and 0.5542.

c. Again using software, a 95% PI for Y when $x = 0.50$ is (0.3359, 0.6749). We are 95% confident that the Fermi position level for a single observation to be made at 0.50 Ge concentration will be between 0.3359 and 0.6749. As always, this prediction interval is markedly wider than the corresponding CI.

d. To obtain simultaneous confidence of at least 97% for the three intervals, we compute each one using confidence level 99%. Using software, the intervals are:
for $x = .3$, (0.3450, 0.8389);
for $x = .5$, (0.2662, 0.7446);
for $x = .7$, (0.1807, 0.6570).

51.

a. 0.40 is closer to $\bar{x}$.

b. $\hat{\beta}_0 + \hat{\beta}_1(0.40) \pm t_{\alpha/2,n-2}\hat{s}_{\hat{\beta}_0+\hat{\beta}_1(0.40)}$ or $0.8104 \pm 2.101(0.0311)$ = (0.745, 0.876).

c. $\hat{\beta}_0 + \hat{\beta}_1(1.20) \pm t_{\alpha/2,n-2} \cdot \sqrt{s^2 + s^2_{\hat{\beta}_0+\hat{\beta}_1(1.20)}}$ or $0.2912 \pm 2.101 \cdot \sqrt{(0.1049)^2 + (0.0352)^2} = (.059,.523)$.

52.

a. We wish to test H_0: $\beta_1 = 0$ v. H_a: $\beta_1 \neq 0$. The test statistic $t = \dfrac{10.6026 - 0}{.9985} = 10.62$ leads to a P-value of < .006 [$2P(T > 4.0)$ from the 7 df row of Table A.8], and H_0 is rejected since the P-value is smaller than any reasonable α. The data suggest that this model does specify a useful relationship between chlorine flow and etch rate.

b. A 95% confidence interval for β_1: 10.6026 ± 2.365(.9985) = (8.24, 12.96). We can be highly confident that when the flow rate is increased by 1 SCCM, the associated expected change in etch rate will be between 824 and 1296 A/min.

c. A 95% CI for $\mu_{Y\cdot 3.0}$: $38.256 \pm 2.365\left(2.546\sqrt{\dfrac{1}{9} + \dfrac{9(3.0 - 2.667)^2}{58.50}}\right)$ = 38.526 ± 2.365(2.546)(.35805) = 38.256 ± 2.156 = (36.100, 40.412), or 3610.0 to 4041.2 A/min.

d. The 95% PI is $38.256 \pm 2.365\left(2.546\sqrt{1 + \dfrac{1}{9} + \dfrac{9(3.0 - 2.667)^2}{58.50}}\right)$ = 38.526 ± 2.365(2.546)(1.06) = 38.256 ± 6.398 = (31.859, 44.655), or 3185.9 to 4465.5 A/min.

e. The intervals for $x^* = 2.5$ will be narrower than those above because 2.5 is closer to the mean than is 3.0.

f. No. A value of 6.0 is not in the range of observed x values, therefore predicting at that point is meaningless.

53. Choice **a** will be the smallest, with **d** being largest. The width of interval **a** is less than **b** and **c** (obviously), and **b** and **c** are both smaller than **d**. Nothing can be said about the relationship between **b** and **c**.

54.

a. From the summaries provided, the slope is $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{137.60}{36.463571}$ = 3.7736 and the intercept is $\hat{\beta}_0 = \bar{y} - \hat{\beta}_0\bar{x}$ = (2310/14) – 3.7736(346.1/14) = 71.71. So, the equation of the least squares line is $\hat{y}$ = 71.71 + 3.7736*x*. The slope indicates that a one-centimeter increase in ulna length is associated with an estimated 3.77 cm increase in (predicted mean) height.

b. From the summaries provided, SSE = $S_{yy} - \hat{\beta}_1 S_{xy}$ = 626.00 – 3.7736(137.60) = 106.75 and SST = S_{yy} = 626.00. Thus, $r^2 = 1 - \frac{\text{SSE}}{\text{SST}} = 1 - \frac{106.75}{626.00}$ = .829. That is, 82.9% of the total variation in subjects' heights can be explained by a linear regression on ulna length.

c. We wish to test H_0: $\beta_1 = 0$ v. H_a: $\beta_1 \neq 0$. Using the model utility test procedure, SSR = SST – SSE = 519.25, from which $f = \frac{\text{SSR}/1}{\text{SSE}/(n-2)} = \frac{519.25}{106.75/12}$ = 58.37. At df = (1, 12), $f = 58.37 > F_{.001,1,12}$ = 18.64 ⇒ *P*-value < .001. Hence, we strongly reject H_0 and conclude that there is a statistically significant linear relationship between ulna length and height.

d. The two 95% prediction intervals are (for x^* = 23 cm and 25 cm, respectively):

$$71.71 + 3.7736(23) \pm 2.179 \cdot 2.98257\sqrt{1 + \frac{1}{14} + \frac{(23-24.72)^2}{36.463571}} = (151.527, 165.481) \text{ and}$$

$$71.71 + 3.7736(25) \pm 2.179 \cdot 2.98257\sqrt{1 + \frac{1}{14} + \frac{(25-24.72)^2}{36.463571}} = (159.318, 172.784)$$

The *t* critical value is $t_{.025,14-2}$ = 2.179 and the residual sd is $s = \sqrt{\frac{\text{SSE}}{n-2}} = \sqrt{\frac{106.75}{12}}$ = 2.98257.

e. As always, precision is debatable. Here, the 95% PIs for heights at ulna length = 23 cm and 25 cm overlap, suggesting that height can't be predicted quite as precisely as we might like.

55. $\hat{\beta}_0 + \hat{\beta}_1 x^* = (\overline{Y} - \hat{\beta}_1\overline{x}) + \hat{\beta}_1 x^* = \overline{Y} + \left(x^* - \overline{x}\right)\hat{\beta}_1 = \frac{1}{n}\sum Y_i + \frac{\left(x^* - \overline{x}\right)\sum\left(x_i - \overline{x}\right)Y_i}{S_{xx}} = \sum d_i Y_i$, where

$d_i = \frac{1}{n} + \frac{\left(x^* - \overline{x}\right)}{S_{xx}}\left(x_i - \overline{x}\right)$. Thus, since the Y_is are independent,

$$V\left(\hat{\beta}_0 + \hat{\beta}_1 x\right) = \sum d_i^2 V\left(Y_i\right) = \sigma^2 \sum d_i^2$$

$$= \sigma^2 \sum \left[\frac{1}{n^2} + 2\frac{(x^* - \overline{x})(x_i - \overline{x})}{nS_{xx}} + \frac{(x^* - \overline{x})^2 (x_i - \overline{x})^2}{nS_{xx}^2}\right]$$

$$= \sigma^2 \left[n\frac{1}{n^2} + 2\frac{(x^* - \overline{x})\sum(x_i - \overline{x})}{nS_{xx}} + \frac{(x^* - \overline{x})^2 \sum(x_i - \overline{x})^2}{S_{xx}^2}\right]$$

$$= \sigma^2 \left[\frac{1}{n} + 2\frac{(x^* - \overline{x})\cdot 0}{nS_{xx}} + \frac{(x^* - \overline{x})^2 S_{xx}}{S_{xx}^2}\right] = \sigma^2 \left[\frac{1}{n} + \frac{(x^* - \overline{x})^2}{S_{xx}}\right]$$

56.

a. Yes: a normal probability plot of yield load (not shown) is quite linear.

b. From software, $\overline{y}$ = 498.2 and s_y = 103.6. Hence, a 95% CI for μ_y is 498.2 ± $t_{.025,14}$(103.6) = 498.2 ± (2.145)(103.6) = (440.83, 555.57). We are 95% confident that the true average yield load is between 440.83 N and 555.57 N.

c. Yes: the *t*-statistic and *P*-value associated with the hypotheses H_0: $\beta_1 = 0$ v. H_a: $\beta_1 \neq 0$ are $t = 3.88$ and $P = 0.002$, respectively. At any reasonable significance level, we reject H_0 and conclude that a useful linear relationship exists between yield load and torque.

d. Yes: prediction intervals based upon this data will be too wide to be useful. For example, a PI for *Y* when *x* = 2.0 is (345.4, 672.4), using software. This interval estimate includes the entire range of *y* values in the data.

Section 12.5

57. Most people acquire a license as soon as they become eligible. If, for example, the minimum age for obtaining a license is 16, then the time since acquiring a license, *y*, is usually related to age by the equation $y \approx x - 16$, which is the equation of a straight line. In other words, the majority of people in a sample will have y values that closely follow the line $y = x - 16$.

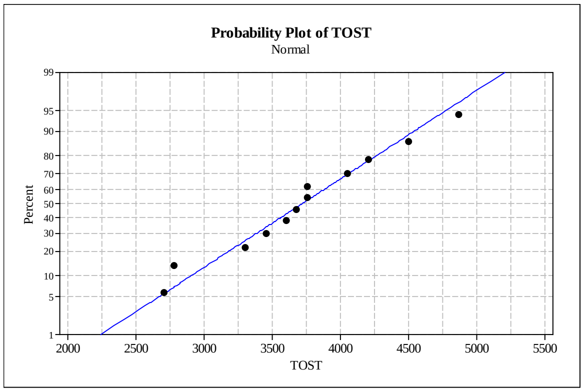
58. Summary values: $n = 12$, $\Sigma x = 44{,}615$, $\Sigma x^2 = 170{,}355{,}425$, $\Sigma y = 3{,}860$, $\Sigma y^2 = 1{,}284{,}450$, $\Sigma xy = 14{,}755{,}500 \Rightarrow S_{xx} = 4{,}480{,}572.92$, $S_{yy} = 42{,}816.67$, and $S_{xy} = 404{,}391.67$.

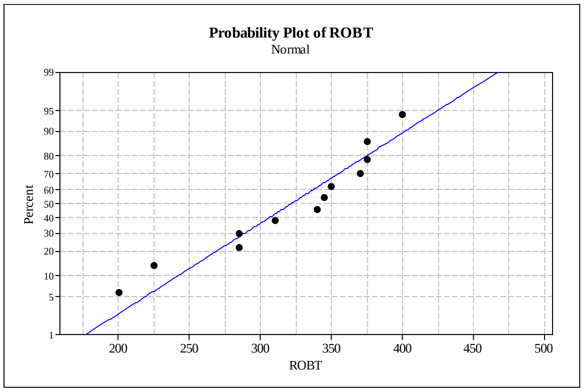
a. $r = \frac{S_{xy}}{\sqrt{S_{xx}}\sqrt{S_{yy}}} = .9233$.

b. The value of *r* does not depend on which of the two variables is labeled as the *x* variable. Thus, had we let *x* = RBOT time and *y* = TOST time, the value of *r* would have remained the same.

c. The value of r does not depend on the unit of measure for either variable. Thus, had we expressed RBOT time in hours instead of minutes, the value of r would have remained the same.

d. Based on the linearity of the accompanying plots, both TOST time and ROBT time could plausibly have come from normally distributed populations. (This is not the same as verifying that the two variables have a *bivariate* normal distribution, but it's a start.)





e. We wish to test H_0: $\rho = 0$ v. H_a: $\rho \neq 0$. The test statistic is $t = \dfrac{r\sqrt{n-2}}{\sqrt{1-r^2}} = \dfrac{.9233\sqrt{12-2}}{\sqrt{1-.9233^2}} \approx 7.6$. At 10 df, the corresponding P-value is less than 2(.001) = .002. Thus, we reject H_0 at level .05 and conclude that there is a statistically significant linear relationship between ROBT and TOST.

59.

a. $S_{xx} = 251{,}970 - \dfrac{(1950)^2}{18} = 40,720$, $S_{yy} = 130.6074 - \dfrac{(47.92)^2}{18} = 3.033711$, and $S_{xy} = 5530.92 - \dfrac{(1950)(47.92)}{18} = 339.586667$, so $r = \dfrac{339.586667}{\sqrt{40,720}\sqrt{3.033711}} = .9662$. There is a very strong, positive correlation between the two variables.

b. Because the association between the variables is positive, the specimen with the larger shear force will tend to have a larger percent dry fiber weight.

c. Changing the units of measurement on either (or both) variables will have no effect on the calculated value of r, because any change in units will affect both the numerator and denominator of r by exactly the same multiplicative constant.

d. $r^2 = .9662^2 = .933$, or 93.3%.

e. We wish to test H_0: $\rho = 0$ v. H_a: $\rho > 0$. The test statistic is $t = \dfrac{r\sqrt{n-2}}{\sqrt{1-r^2}} = \dfrac{.9662\sqrt{18-2}}{\sqrt{1-.9662^2}} = 14.94$. This is "off the charts" at 16 df, so the one-tailed P-value is less than .001. So, H_0 should be rejected: the data indicate a positive linear relationship between the two variables.

60.

a. From software, the sample correlation coefficient is $r = .722$.

b. The hypotheses are H_0: $\rho = 0$ versus H_a: $\rho \neq 0$. Assuming bivariate normality, the test statistic value is $t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} = \frac{.722\sqrt{14-2}}{\sqrt{1-.722^2}}$ = 3.61. At df = 14 – 2 = 12, the two-tailed P-value for this t test is roughly 2(.002) = .004. Hence, we reject H_0 at the .01 level and conclude that the population correlation coefficient between clockwise and counterclockwise rotation is not zero. We would not make the same conclusion at the .001 level, however, since P-value = .004 > .001.

61.

a. We are testing H_0: $\rho = 0$ v. H_a: $\rho > 0$. The correlation is $r = \frac{7377.704}{\sqrt{36.9839}\sqrt{2,628,930.359}} = .7482$, and the test statistic is $t = \frac{.7482\sqrt{12}}{\sqrt{1-.7482^2}} \approx 3.9$. At 14 df, the P-value is roughly .001. Hence, we reject H_0: there is evidence that a positive correlation exists between maximum lactate level and muscular endurance.

b. We are looking for r^2, the coefficient of determination: $r^2 = (.7482)^2 = .5598$, or about 56%. It is the same no matter which variable is the predictor.

62.

a. We are testing H_0: $\rho = 0$ v. H_a: $\rho > 0$. The test statistic is $t = \frac{.853\sqrt{20-2}}{\sqrt{1-.853^2}} \approx 6.93$. At 18 df, this is “off the charts,” so the one-tailed P-value is < .001. Hence, we strongly reject H_0 and conclude that a statistically significant positive association exists between cation exchange capacity and specific surface area for this class of soils.

b. In **a**, the null hypothesis would <u>not</u> be rejected if the test statistic were <u>less</u> than $t_{.01,18} = 2.552$, since such a test statistic would result in a P-value greater than .01 (remember, this is a one-sided test). That is, we fail to reject H_0 with $n = 20$ iff $\frac{r\sqrt{20-2}}{\sqrt{1-r^2}} < 2.552$. Changing the < to = and solving for r, we get $r \approx .3524$. So, we would fail to reject H_0 if $r < .3524$ and reject H_0 otherwise. [*Note*: Because this is a one-sided test, H_0 would not be rejected if r were <u>any</u> negative number, since a negative sample correlation would certainly not provide evidence in favor of H_a.]

c. First, $v = \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right) = \frac{1}{2}\ln\left(\frac{1+.853}{1-.853}\right)$ = 1.267. Next, a 95% CI for μ_V is $v \pm \frac{z_{.025}}{\sqrt{n-3}} = 1.267 \pm \frac{1.96}{\sqrt{20-3}}$ = (.792, 1.742). Finally, a 95% CI for ρ is $\left(\frac{e^{2(.792)}-1}{e^{2(.792)}+1}, \frac{e^{2(1.742)}-1}{e^{2(1.742)}+1}\right)$ = (.659, .941).

63. With the aid of software, the sample correlation coefficient is $r = .7729$. To test H_0: $\rho = 0$ v. H_a: $\rho \neq 0$, the test statistic is $t = \frac{(.7729)\sqrt{6-2}}{\sqrt{1-(.7729)^2}}$ = 2.44. At 4 df, the 2-sided P-value is about 2(.035) = .07 (software gives a P-value of .072). Hence, we fail to reject H_0: the data do not indicate that the population correlation coefficient differs from 0. This result may seem surprising due to the relatively large size of r (.77), however, it can be attributed to a small sample size (n = 6).

64.

a. From the summary quantities provided, $r = \frac{S_{xy}}{\sqrt{S_{xx}}\sqrt{S_{yy}}}$ = .700. This indicates a moderate-to-strong, direct/positive association between UV transparency index and maximum prevalence of infection.

b. This is asking for the coefficient of determination: $r^2 = .700^2 = .49$, or 49%.

c. Interchanging the roles of x and y does not affect r, and so it does not affect r^2, either. The answer is still 49%.

d. Since $\rho_0 \neq 0$, we must use the z procedure here. First, $v = \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right) = \frac{1}{2}\ln\left(\frac{1+.7}{1-.7}\right)$ = 0.8673. Next, the test statistic is $z = \frac{v - \frac{1}{2}\ln[(1+\rho_0)/(1-\rho_0)]}{1/\sqrt{n-3}} = \frac{0.8673 - \frac{1}{2}\ln[(1+.5)/(1-.5)]}{1/\sqrt{17-3}}$ = 1.19. The corresponding upper-tailed P-value is $P(Z \geq 1.19) = 1 - \Phi(1.19) = .1170$. Since .1170 > .05, H_0 cannot be rejected at the α = .05 level. We do not have sufficient evidence to conclude that the population correlation coefficient for these two variables exceeds .5.

65.

a. From the summary statistics provided, a point estimate for the population correlation coefficient ρ is $r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2}} = \frac{44{,}185.87}{\sqrt{(64{,}732.83)(130{,}566.96)}}$ = .4806.

b. The hypotheses are H_0: $\rho = 0$ versus H_a: $\rho \neq 0$. Assuming bivariate normality, the test statistic value is $t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} = \frac{.4806\sqrt{15-2}}{\sqrt{1-.4806^2}}$ = 1.98. At df = 15 – 2 = 13, the two-tailed P-value for this t test is $2P(T_{13} \geq 1.98) \approx 2P(T_{13} \geq 2.0) = 2(.033) = .066$. Hence, we fail to reject H_0 at the .01 level; there is not sufficient evidence to conclude that the population correlation coefficient between internal and external rotation velocity is not zero.

c. If we tested H_0: $\rho = 0$ versus H_a: $\rho > 0$, the one-sided P-value would be .033. We would still fail to reject H_0 at the .01 level, lacking sufficient evidence to conclude a positive true correlation coefficient. However, for a one-sided test at the .05 level, we would reject H_0 since P-value = .033 < .05. We have evidence at the .05 level that the true population correlation coefficient between internal and external rotation velocity is positive.

66.

a. We used Minitab to calculate the r_is: $r_1 = 0.192$, $r_2 = 0.382$, and $r_3 = 0.183$. It appears that the lag 2 correlation is best, but all of them are weak, based on the definitions given in the text.

b. We reject H_0 if $|r_i| \geq 2/\sqrt{100} = .2$. For all three lags specified in **b**, r_i does not fall in the rejection region, so we cannot reject H_0. There is not evidence of theoretical autocorrelation at the first 3 lags.

c. If we want an approximate .05 significance level for three simultaneous hypotheses, we would have to use smaller individual significance levels to control the global Type I error rate. (This is similar to what Tukey's method addresses in ANOVA.) Increasing the numerator from 2 would make it more difficult to reject H_0, which is equivalent to a lower significance level (lower α).

67.

a. Because P-value = .00032 < α = .001, H_0 should be rejected at this significance level.

b. Not necessarily. For such a large n, the test statistic t has approximately a standard normal distribution when H_0: $\rho = 0$ is true, and a P-value of .00032 corresponds to $z = \pm 3.60$. Solving $\pm 3.60 = \dfrac{r\sqrt{500-2}}{\sqrt{1-r^2}}$ for r yields $r = \pm .159$. That is, with $n = 500$ we'd obtain this P-value with $r = \pm .159$. Such an r value suggests only a weak linear relationship between x and y, one that would typically have little practical importance.

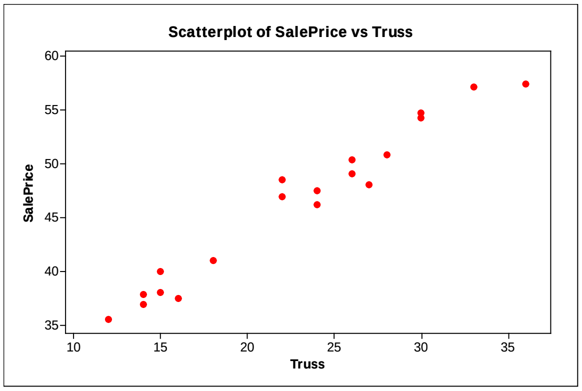
c. The test statistic value would be $t = \dfrac{.022\sqrt{10{,}000-2}}{\sqrt{1-.022^2}} = 2.20$; since the test statistic is again approximately normal, the 2-sided P-value would be roughly $2[1 - \Phi(2.20)] = .0278 < .05$, so H_0 is rejected in favor of H_a at the .05 significance level. The value $t = 2.20$ is statistically significant — it cannot be attributed just to sampling variability in the case $\rho = 0$. But with this enormous n, $r = .022$ implies $\rho \approx .022$, indicating an extremely weak relationship.

Supplementary Exercises

68.

a. Clearly not: for example, the two observations with truss height = 14 ft have different sale prices (37.82 and 36.90).

b. The scatter plot below suggests a strong, positive, linear relationship.



c. Using software, the least squares line is $\hat{y}$ = 23.8 + 0.987x, where x = truss height and y = sale price.

d. At x = 27, $\hat{y}$ = 23.8 + 0.987(27) = \$50.43 per square foot. The observed value at x = 27 is y = 48.07, and the corresponding residual is $y - \hat{y}$ = 48.07 – 50.43 = –2.36.

e. Using software, r^2 = SSR/SST = 890.36/924.44 = 96.3%.

69. Use available software for all calculations.

a. We want a confidence interval for β_1. From software, b_1 = 0.987 and $s(b_1)$ = 0.047, so the corresponding 95% CI is 0.987 ± $t_{.025,17}$(0.047) = 0.987 ± 2.110(0.047) = (0.888, 1.086). We are 95% confident that the true average change in sale price associated with a one-foot increase in truss height is between \$0.89 per square foot and \$1.09 per square foot.

b. Using software, a 95% CI for $\mu_{y.25}$ is (47.730, 49.172). We are 95% confident that the true average sale price for all warehouses with 25-foot truss height is between \$47.73/ft^2 and \$49.17/ft^2.

c. Again using software, a 95% PI for Y when x = 25 is (45.378, 51.524). We are 95% confident that the sale price for a single warehouse with 25-foot truss height will be between \$45.38/ft^2 and \$51.52/ft^2.

d. Since x = 25 is nearer the mean than x = 30, a PI at x = 30 would be wider.

e. From software, r^2 = SSR/SST = 890.36/924.44 = .963. Hence, $r = \sqrt{.963}$ = .981.

70. First, use software to get the least squares relation $\hat{y}$ = 0.5817 + 0.049727x, where x = age and y = %DDA. From this, $\hat{\beta}_1$ = 0.049727. Also from software, s = 0.244, $\bar{x}$ = 29.73, and S_{xx} = 5390.2. Finally, if y^* = 2.01, then 2.01 = 0.5817 + 0.049727 $\hat{x}$, whence $\hat{x}$ = 28.723.
Therefore, an approximate 95% CI for X when $y = y^*$ = 2.01 is

$$28.723 \pm t_{.025,9}\,\frac{0.244}{0.049727}\left\{1+\frac{1}{11}+\frac{(28.723-29.73)^2}{5390.2}\right\}^{1/2} = 28.723 \pm 2.262(5.125) = (17.130,\ 40.316).$$

Since this interval straddles 22, we cannot say with confidence whether the individual was older or younger than 22 — ages both above and below 22 are plausible based on the data.

71. Use software whenever possible.

a. From software, the estimated coefficients are $\hat{\beta}_1$ = 16.0593 and $\hat{\beta}_0$ = 0.1925.

b. Test H_0: $\beta_1 = 0$ versus H_a: $\beta_1 \neq 0$. From software, the test statistic is $t = \dfrac{16.0593-0}{0.2965}$ = 54.15; even at just 7 df, this is "off the charts" and the P-value is ≈ 0. Hence, we strongly reject H_0 and conclude that a statistically significant relationship exists between the variables.

c. From software or by direct computation, residual sd = s = .2626, $\bar{x}$ = .408 and S_{xx} = .784. When $x = x^*$ = .2, $\hat{y}$ = 0.1925 + 16.0593(.2) = 3.404 with an estimated standard deviation of

$$s_{\hat{Y}} = s\sqrt{\frac{1}{n}+\frac{(x^*-\bar{x})^2}{S_{xx}}} = .2626\sqrt{\frac{1}{9}+\frac{(.2-.408)^2}{.784}} = .107.$$ The analogous calculations when $x = x^*$ = .4 result in $\hat{y}$ = 6.616 and $s_{\hat{Y}}$ = .088, confirming what's claimed. Prediction error is larger when x = .2 because .2 is farther from the sample mean of .408 than is x = .4.

d. A 95% CI for $\mu_{Y\cdot .4}$ is $\hat{y} \pm t_{.025,9-2} s_{\hat{Y}}$ = 6.616 ± 2.365(.088) = (6.41, 6.82).

e. A 95% PI for Y when x = .4 is $\hat{y} \pm t_{.025,9-2}\sqrt{s^2 + s_{\hat{Y}}^2}$ = (5.96, 7.27).

72.

a. df(SSE) = 6 = n – 2, so sample size = 8.

b. $\hat{y}$ = 326.76038 – 8.403964x. When x = 35.5, $\hat{y}$ = 28.64.

c. The P-value for the model utility test is 0.0002 according to the output. The model utility test is statistically significant at the level .01.

d. $r = (\text{sign of slope})\cdot\sqrt{r^2} = -\sqrt{0.9134} = –0.9557$

e. First check to see if the value x = 40 falls within the range of x values used to generate the least-squares regression equation. If it does not, this equation should not be used. Furthermore, for this particular model an x value of 40 yields a y value of –9.18mcM/L, which is an impossible value for y.

73.

a. From the output, r^2 = .5073.

b. $r = (\text{sign of slope})\cdot\sqrt{r^2} = +\sqrt{.5073}$ = .7122.

c. We test H_0: $\beta_1 = 0$ versus H_a: $\beta_1 \neq 0$. The test statistic t = 3.93 gives P-value = .0013, which is < .01, the given level of significance, therefore we reject H_0 and conclude that the model is useful.

d. We use a 95% CI for $\mu_{Y\cdot 50}$. $\hat{y}$(50) = .787218 + .007570(50) = 1.165718; t.025,15 = 2.131;

s = “Root MSE” = .20308 ⇒ $s_{\hat{Y}} = .20308\sqrt{\frac{1}{17} + \frac{17(50-42.33)^2}{17(41{,}575)-(719.60)^2}} = .051422$. The resulting 95% CI is 1.165718 ± 2.131(.051422) = 1.165718 ± .109581 = (1.056137, 1.275299).

e. Our prediction is $\hat{y}$(30) = .787218 + .007570(30) = 1.0143, with a corresponding residual of $y - \hat{y}$ = .80 – 1.0143 = –.2143.

74.

a. A scatterplot shows a reasonably strong, positive linear relationship between ΔCO and ΔNO_y. The least squares line is $\hat{y}$ = –.220 + .0436x. A test of H_0: $\beta_1 = 0$ versus H_a: $\beta_1 \neq 0$ in Minitab gives t = 12.72 and a P-value of ≈ 0. Hence, we have sufficient evidence to conclude that a statistically significant relationship exists between ΔCO and ΔNO_y.

b. The point prediction is $\hat{y}$ = –.220 + .0436(400) = 17.228. A 95% prediction interval produced by Minitab is (11.953, 22.503). Since this interval is so wide, it does not appear that ΔNO_y is accurately predicted.

c. While the large ΔCO value appears to be “near” the least squares regression line, the value has extremely high “leverage.” The least squares line that is obtained when excluding the value is $\hat{y}$ = 1.00 + .0346x, and the r^2 value of 96% is reduced to 75% when the value is excluded. The value of s with the value included is 2.024, and with the value excluded is 1.96. So the large ΔCO value does appear to affect our analysis in a substantial way.

75.

a. With y = stride rate and x = speed, we have $\hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}} = \dfrac{660.130-(205.4)(35.16)/11}{3880.08-(205.4)^2/11} = \dfrac{3.597}{44.702} =$ 0.080466 and $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$ = (35.16/11) – 0.080466(205.4)/11 = 1.694. So, the least squares line for predicting stride rate from speed is $\hat{y}$ = 1.694 + 0.080466x.

b. With y = speed and x = stride rate, we have $\hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}} = \dfrac{660.130-(35.16)(205.4)/11}{112.681-(35.16)^2/11} = \dfrac{3.597}{0.297} =$ 12.117 and $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$ =(205.4)/11 – 12.117(35.16/11) = –20.058. So, the least squares line for predicting speed from stride rate is $\hat{y}$ = –20.058 + 12.117x.

c. The fastest way to find r^2 from the available information is $r^2 = \hat{\beta}_1^2 \dfrac{S_{xx}}{S_{yy}}$. For the first regression, this gives $r^2 = (0.080466)^2 \dfrac{44.702}{0.297} \approx$ .97. For the second regression, $r^2 = (12.117)^2 \dfrac{0.297}{44.702} \approx$ .97 as well. In fact, rounding error notwithstanding, these two r^2 values should be exactly the same.

76.

a. Minitab output appears below. The equation of the estimated regression line is $\hat{y}$ = –115.2 + 38.068x. A test of H_0: $\beta_1 = 0$ versus H_a: $\beta_1 \neq 0$ gives t = 3.84 and P-value = .002, suggesting we reject H_0 and conclude that a statistically significant relationship exists between these variables. It is reasonable to use a linear regression model to predict fracture toughness from mode-mixity angle.

```
The regression equation is
y = - 115 + 38.1 x


Predictor    Coef  SE Coef      T      P
Constant   -115.2    226.7  -0.51  0.619
x          38.068    9.924   3.84  0.002
```

b. A formal test really isn't necessary, since the sample slope is less than 50. But the formal hypothesis test is of H_0: $\beta_1 \leq 50$ versus H_a: $\beta_1 > 50$; the test statistic is $t = \dfrac{38.068-50}{9.924}$ = –1.2; and the upper-tailed P-value is $P(T \geq -1.2$ when $T \sim t_{14})$ = 1 – $P(T \geq -1.2$ when $T \sim t_{14})$ = 1 – .124 = .876. With such a large P-value we fail to reject H_0 at any reasonable significance level. The data provide no evidence that the average change in fracture toughness associated with a one-degree increase in mode-mixity angle exceeds 50 N/m.

c. Looking at the formula for the standard deviation of the slope, better precision (lower se) corresponds to greater variability in the x values, since S_{xx} is in the denominator. For the 16 actual x values from the study, S_{xx} = 148.9; for the 16 x values suggested in **c**, S_{xx} = 144. Since 144 < 148.9, the answer is no: using the 16 x values suggested in **c** would actually result in a larger standard error for the slope.

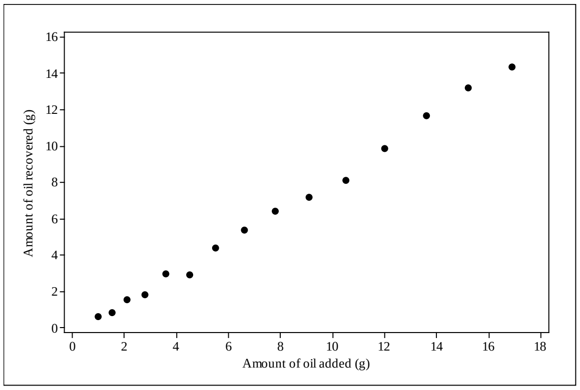
d. Minitab provides the 95% CIs and PIs below; the top row is for $x = 18$ degrees and the bottom row is for $x = 22$ degrees. Thus, we are 95% confident that the average fracture toughness at 18 degrees is between 451.9 N/m and 688.2 N/m, while at 22 degrees the average is between 656.0 N/m and 788.6 N/m. The two intervals overlap slightly, owing to the small sample size and large amount of natural variation in fracture toughness.

At a 95% prediction level, the fracture toughness of a single specimen at 18 degrees will fall between 284.7 N/m and 855.4 N/m, while the fracture toughness of a single specimen at 22 degrees is predicted to fall between 454.2 N/m and 990.4 N/m. Not only do these two intervals overlap, they are also very wide — much wider than the associated CIs, which will always be the case. In particular, the available data suggest that we cannot predict fracture toughness very precisely based on mode-mixity angle.

```
Predicted Values for New Observations

New Obs    Fit  SE Fit       95% CI           95% PI
      1  570.0    55.1  (451.9, 688.2)  (284.7, 855.4)
      2  722.3    30.9  (656.0, 788.6)  (454.2, 990.4)
```

77.

a. Yes: the accompanying scatterplot suggests an extremely strong, positive, linear relationship between the amount of oil added to the wheat straw and the amount recovered.



b. Pieces of Minitab output appear below. From the output, $r^2 = 99.6\%$ or .996. That is, 99.6% of the total variation in the amount of oil recovered in the wheat straw can be explained by a linear regression on the amount of oil added to it.

```
Predictor      Coef  SE Coef      T      P
Constant    -0.5234   0.1453  -3.60  0.003
x           0.87825  0.01610  54.56  0.000

S = 0.311816   R-Sq = 99.6%   R-Sq(adj) = 99.5%

Predicted Values for New Observations

New Obs     Fit  SE Fit       95% CI            95% PI
      1  3.8678  0.0901  (3.6732, 4.0625)  (3.1666, 4.5690)
```

c. Refer to the preceding Minitab output. A test of H_0: $\beta_1 = 0$ versus H_a: $\beta_1 \neq 0$ returns a test statistic of $t = 54.56$ and a P-value of ≈ 0, from which we can strongly reject H_0 and conclude that a statistically significant linear relationship exists between the variables. (No surprise, based on the scatterplot!)

d. The last line of the preceding Minitab output comes from requesting predictions at $x = 5.0$ g. The resulting 95% PI is (3.1666, 4.5690). So, at a 95% prediction level, the amount of oil recovered from wheat straw when the amount added was 5.0 g will fall between 3.1666 g and 4.5690 g.

e. A formal test of H_0: $\rho = 0$ versus H_a: $\rho \neq 0$ is completely equivalent to the t test for slope conducted in **c**. That is, the test statistic and P-value would once again be $t = 54.56$ and $P \approx 0$, leading to the conclusion that $\rho \neq 0$.

78. Substituting $x^* = 0$ into the variance formula for $\hat{Y}$ gives $\sigma^2_{\hat{\beta}_0} = \sigma^2 \cdot \left[\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right]$, from which the estimated standard deviation of $\hat{\beta}_0$ is $s_{\hat{\beta}_0} = s\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}}$. For the data in Example 12.11, $n = 10$, $s = 2661.33$, $S_{xx} =$ 18,000, and $\bar{x} = 110$ (most of these are provided in the example). Hence, the estimated standard deviation of $\hat{\beta}_0$ is $s_{\hat{\beta}_0} = 2661.33\sqrt{\frac{1}{10} + \frac{110^2}{18,000}} = 2338.675$. A 95% CI for β_0, the true y-intercept, is given by

$\hat{\beta}_0 \pm t_{.025,n-2} s_{\hat{\beta}_0}$ = 10,698.33 ± 2.306(2338.675) = (5305.35, 16,091.31).

The interval is very wide because we only have $n = 10$ observations, there is a lot of variability in the response values (rupture load), <u>and</u> the value $x = 0$ is quite far from the x values within the data set (specifically, 110 away from the mean).

79. Start with the alternative formula $\text{SSE} = \Sigma y^2 - \hat{\beta}_0 \Sigma y - \hat{\beta}_1 \Sigma xy$. Substituting $\hat{\beta}_0 = \frac{\Sigma y - \hat{\beta}_1 \Sigma x}{n}$,

$$\text{SSE} = \Sigma y^2 - \frac{\Sigma y - \hat{\beta}_1 \Sigma x}{n}\Sigma y - \hat{\beta}_1 \Sigma xy = \Sigma y^2 - \frac{(\Sigma y)^2}{n} + \frac{\hat{\beta}_1 \Sigma x \Sigma y}{n} - \hat{\beta}_1 \Sigma xy = \left[\Sigma y^2 - \frac{(\Sigma y)^2}{n}\right] - \hat{\beta}_1\left[\Sigma xy - \frac{\Sigma x \Sigma y}{n}\right]$$

$$= S_{yy} - \hat{\beta}_1 S_{xy}$$

80. The value of the sample correlation coefficient using the squared y values would not necessarily be approximately 1. If the y values are greater than 1, then the squared y values would differ from each other by more than the y values differ from one another. Hence, the relationship between x and y^2 would be less like a straight line, and the resulting value of the correlation coefficient would decrease.

81.

a. Recall that $r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$, $s_x^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1} = \frac{S_{xx}}{n-1}$, and similarly $s_y^2 = \frac{S_{yy}}{n-1}$. Using these formulas,

$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{r \cdot \sqrt{S_{xx}S_{yy}}}{S_{xx}} = r \cdot \sqrt{\frac{S_{yy}}{S_{xx}}} = r \cdot \sqrt{\frac{(n-1)s_y^2}{(n-1)s_x^2}} = r \cdot \frac{s_y}{s_x}$. Using the fact that $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$, the least squares equation becomes $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x = \bar{y} + \hat{\beta}_1(x - \bar{x}) = \bar{y} + r \cdot \frac{s_y}{s_x}(x - \bar{x})$, as desired.

b. In Exercise 64, $r = .700$. So, a specimen whose UV transparency index is 1 standard deviation below average is predicted to have a maximum prevalence of infection that is .7 standard deviations below average.

82. We need to show that $\frac{\hat{\beta}_1}{s_{\hat{\beta}_1}} = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$, where $s_{\hat{\beta}_1} = \frac{s}{\sqrt{S_{xx}}}$. We'll rely on the SSE formula of Exercise 79 and the fact that $s = \sqrt{\frac{\text{SSE}}{n-2}}$. Combining these with the formula for $\hat{\beta}_1$, we have

$$\frac{\hat{\beta}_1}{s_{\hat{\beta}_1}} = \frac{S_{xy}/S_{xx}}{s/\sqrt{S_{xx}}} = \frac{S_{xy}/\sqrt{S_{xx}}}{s} = \frac{S_{xy}/\sqrt{S_{xx}}}{\sqrt{[S_{yy}-\hat{\beta}_1 S_{xy}]/(n-2)}} = \frac{\left[S_{xy}/\sqrt{S_{xx}}\right]\cdot\sqrt{n-2}}{\sqrt{S_{yy}-\hat{\beta}_1 S_{xy}}} = \frac{\left[S_{xy}/\sqrt{S_{xx}}\right]\cdot\sqrt{n-2}}{\sqrt{S_{yy}-S_{xy}^2/S_{xx}}}$$

Divide numerator and denominator by $\sqrt{S_{yy}}$, and remember that $r = S_{xy}/\sqrt{S_{xx}S_{yy}}$:

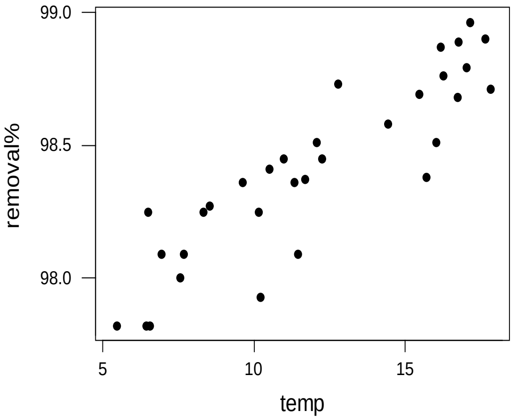
$$\frac{\hat{\beta}_1}{s_{\hat{\beta}_1}} = \frac{\left[S_{xy}/\sqrt{S_{xx}}\right]\cdot\sqrt{n-2}\div\sqrt{S_{yy}}}{\sqrt{S_{yy}-S_{xy}^2/S_{xx}}\div\sqrt{S_{yy}}} = \frac{\left[S_{xy}/\sqrt{S_{xx}S_{yy}}\right]\cdot\sqrt{n-2}}{\sqrt{1-S_{xy}^2/S_{xx}S_{yy}}} = \frac{r\cdot\sqrt{n-2}}{\sqrt{1-r^2}}$$, completing the proof.

83. Remember that SST = S_{yy} and use Exercise 79 to write SSE = $S_{yy}-\hat{\beta}_1 S_{xy}$ = $S_{yy}-S_{xy}^2/S_{xx}$. Then

$$r^2 = \frac{S_{xy}^2}{S_{xx}S_{yy}} = \frac{S_{xy}^2/S_{xx}}{S_{yy}} = \frac{S_{yy}-\text{SSE}}{S_{yy}} = 1-\frac{\text{SSE}}{S_{yy}} = 1-\frac{\text{SSE}}{\text{SST}}.$$

84.

a. A scatterplot suggests the linear model is appropriate.



b. From the accompanying Minitab output, the least squares line is $\hat{y}$ = 97.4986 + 0.075691x. A point prediction of removal efficiency when x = temperature = 10.50 is 97.4986 + 0.075691(10.50) = 98.29. For the observation (10.50, 98.41), the residual is 98.41 – 98.29 = 0.12.

```
The regression equation is
removal% = 97.5 + 0.0757 temp

Predictor        Coef       StDev          T        P
Constant      97.4986      0.0889    1096.17    0.000
temp         0.075691    0.007046      10.74    0.000

S = 0.1552      R-Sq = 79.4%     R-Sq(adj) = 78.7%
```

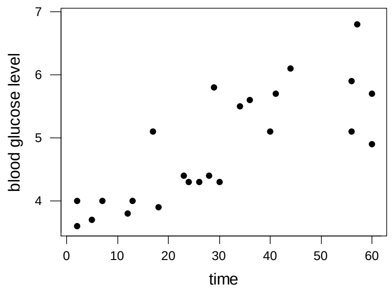
c. From the Minitab output, s = .1552.

d. From the Minitab output, $r^2 = 79.4\%$.

e. A 95% CI for β_1, using the Minitab output and $t_{.025,30} = 2.042$, is $.075691 \pm 2.042(.007046) = (.061303, .090079)$.

f. Re-running the regression, the slope of the regression line is steeper. The value of *s* is almost doubled (to 0.291), and the value of r^2 drops correspondingly to 61.6%.

85. Using Minitab, we create a scatterplot to see if a linear regression model is appropriate.



A linear model is reasonable; although it appears that the variance in y gets larger as x increases. The Minitab output follows:

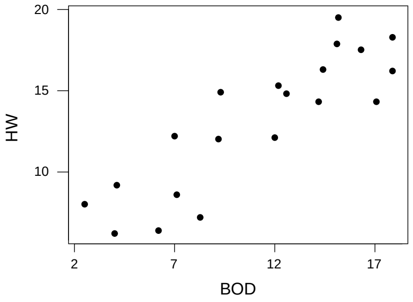
```
The regression equation is
blood glucose level = 3.70 + 0.0379 time

Predictor        Coef       StDev          T        P
Constant       3.6965      0.2159      17.12    0.000
time         0.037895    0.006137       6.17    0.000

S = 0.5525      R-Sq = 63.4%     R-Sq(adj) = 61.7%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1      11.638      11.638     38.12    0.000
Residual Error    22       6.716       0.305
Total             23      18.353
```

The coefficient of determination of 63.4% indicates that only a moderate percentage of the variation in y can be explained by the change in x. A test of model utility indicates that time is a significant predictor of blood glucose level. ($t = 6.17$, $P \approx 0$). A point estimate for blood glucose level when time = 30 minutes is 4.833%. We would expect the average blood glucose level at 30 minutes to be between 4.599 and 5.067, with 95% confidence.

86.

a. Using the techniques from a previous chapter, we can perform a *t* test for the difference of two means based on paired data. Minitab's paired t test for equality of means gives $t = 3.54$, with a *P*-value of .002, which suggests that the average bf% reading for the two methods is not the same.

b. Using linear regression to predict HW from BOD POD seems reasonable after looking at the scatterplot, below.



The least squares linear regression equation, as well as the test statistic and *P*-value for a model utility test, can be found in the Minitab output below. We see that we do have significance, and the coefficient of determination shows that about 75% of the variation in HW can be explained by the variation in BOD.

```
The regression equation is
HW = 4.79 + 0.743 BOD

Predictor        Coef       StDev          T        P
Constant        4.788       1.215       3.94    0.001
BOD            0.7432      0.1003       7.41    0.000

S = 2.146       R-Sq = 75.3%     R-Sq(adj) = 73.9%

Analysis of Variance

Source           DF          SS          MS         F        P
Regression        1      252.98      252.98     54.94    0.000
Residual Error   18       82.89        4.60
Total            19      335.87
```

87. From the SAS output in Exercise 73, $n_1 = 17$, $\text{SSE}_1 = 0.61860$, $\hat{\beta}_1 = 0.007570$; by direct computation, $\text{SS}_{x1} = 11{,}114.6$. The pooled estimated variance is $\hat{\sigma}^2 = \dfrac{.61860 + .51350}{17 + 15 - 4} = .040432$, and the calculated test statistic for testing H_0: $\beta_1 = \gamma_1$ is

$$t = \frac{.007570 - .006845}{\sqrt{.040432}\sqrt{\dfrac{1}{11114.6} + \dfrac{1}{7152.5578}}} \approx 0.24.$$ At 28 df, the two-tailed *P*-value is roughly 2(.39) = .78.

With such a large *P*-value, we do not reject H_0 at any reasonable level (in particular, .78 > .05). The data do not provide evidence that the expected change in wear loss associated with a 1% increase in austentite content is different for the two types of abrasive — it is plausible that $\beta_1 = \gamma_1$.

CHAPTER 13

Section 13.1

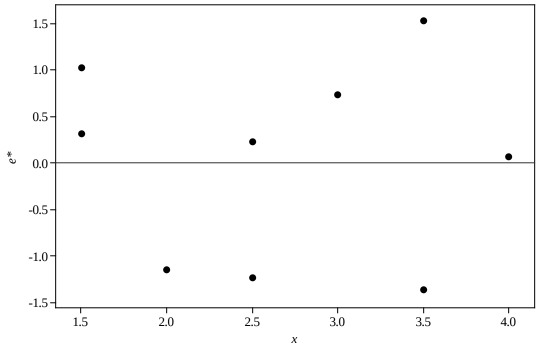
1.

a. $\overline{x} = 15$ and $\sum(x_i - \overline{x})^2 = 250$, so the standard deviation of the residual $Y_i - \hat{Y}_i$ is $10\sqrt{1 - \frac{1}{5} - \frac{(x_i - 15)^2}{250}}$ = 6.32, 8.37, 8.94, 8.37, and 6.32 for i = 1, 2, 3, 4, 5.

b. Now $\overline{x} = 20$ and $\sum(x_i - \overline{x})^2 = 1250$, giving residual standard deviations 7.87, 8.49, 8.83, 8.94, and 2.83 for i = 1, 2, 3, 4, 5.

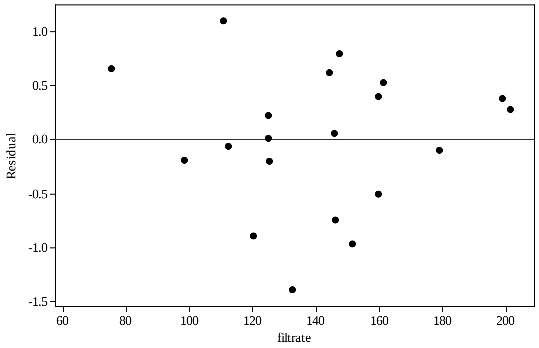
c. The deviation from the estimated line is likely to be much smaller for the observation made in the experiment of **b** for x = 50 than for the experiment of **a** when x = 25. That is, the observation (50, Y) is more likely to fall close to the least squares line than is (25, Y).

2. The pattern gives no cause for questioning the appropriateness of the simple linear regression model, and no observation appears unusual.



3.

a. This plot indicates there are no outliers, the variance of ε is reasonably constant, and the ε are normally distributed. A straight-line regression function is a reasonable choice for a model.

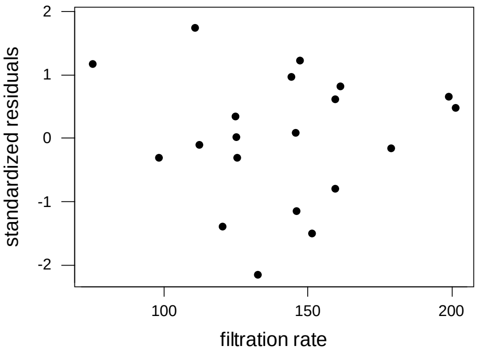


b. We need $S_{xx} = \sum\left(x_i - \overline{x}\right)^2 = 415{,}914.85 - \frac{\left(2817.9\right)^2}{20} = 18{,}886.8295$. Then each e_i^* can be calculated as follows: $e_i^* = \frac{e_i}{.4427\sqrt{1 + \frac{1}{20} + \frac{\left(x_i - 140.895\right)^2}{18{,}886.8295}}}$. The table below shows the values:

standardized residuals	e_i / e_i^*	standardized residuals	e_i / e_i^*
–0.31064	0.644053	0.6175	0.64218
–0.30593	0.614697	0.09062	0.64802
0.4791	0.578669	1.16776	0.565003
1.2307	0.647714	–1.50205	0.646461
–1.15021	0.648002	0.96313	0.648257
0.34881	0.643706	0.019	0.643881
–0.09872	0.633428	0.65644	0.584858
–1.39034	0.640683	–2.1562	0.647182
0.82185	0.640975	–0.79038	0.642113
–0.15998	0.621857	1.73943	0.631795

Notice that if $e_i^* \approx e_i / s$, then $e_i / e_i^* \approx$ s. All of the e_i / e_i^*'s range between .57 and .65, which are close to *s*.

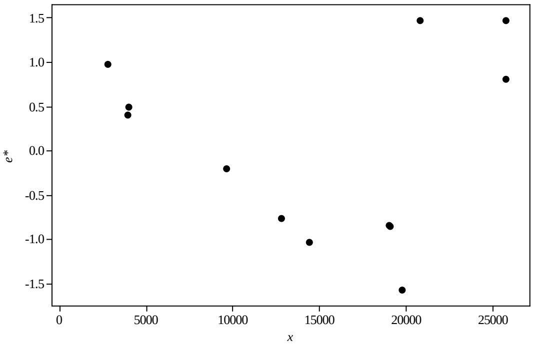
c. This plot looks very much the same as the one in part a.



4.

a. Yes: with $R^2 = 90.2\%$ and a t-test P-value of 0.000, the output indicates a useful relationship between normalized energy and interocular pressure.

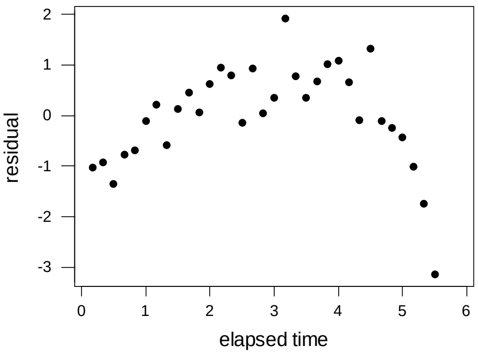
b. The unusual curvature in the residual plot might indicate that a straight-line model is not appropriate for these two variables. (A scatterplot of y versus x also exhibits curvature.)



5.

a. 97.7% of the variation in ice thickness can be explained by the linear relationship between it and elapsed time. Based on this value, it is tempting to assume an approximately linear relationship; however, r^2 does <u>not</u> measure the aptness of the linear model.

b. The residual plot shows a curve in the data, suggesting a non-linear relationship exists. One observation (5.5, –3.14) is extreme.



6. Yes, the outlying observation does seem to have a substantial effect. The slope without that observation is roughly 8.8, compared to 9.9 with the point included (more than a 10% increase!). Notice also that the estimated standard deviation of the slope is decreased substantially by the inclusion of this outlying value (from .47 to .38, almost a 20% decrease). The outlier gives a false impression about the quality of the fit of the least squares line.

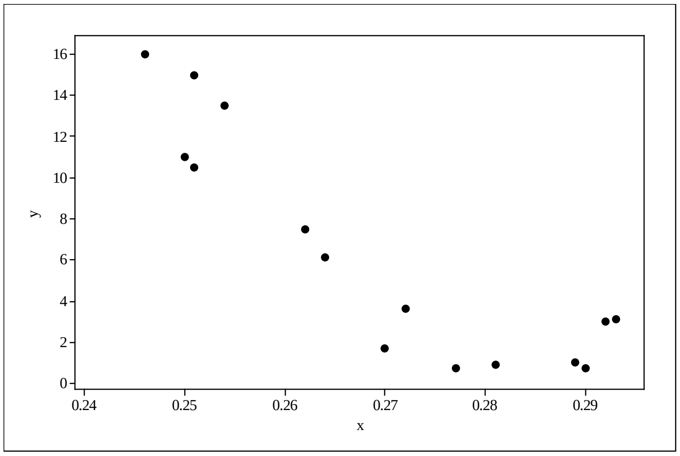
7.

a. From software and the data provided, the least squares line is $\hat{y} = 84.4 - 290x$. Also from software, the coefficient of determination is $r^2 = 77.6\%$ or .776.

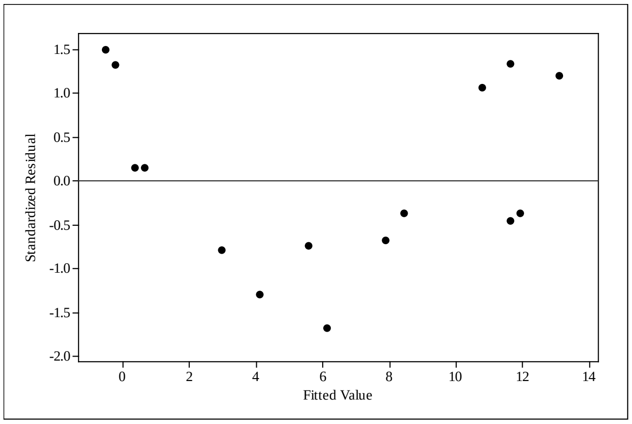
Regression Analysis: y versus x

```
The regression equation is
y = 84.4 - 290 x

Predictor     Coef  SE Coef      T      P
Constant     84.38    11.64   7.25  0.000
x          -289.79    43.12  -6.72  0.000

S = 2.72669   R-Sq = 77.6%   R-Sq(adj) = 75.9%
```

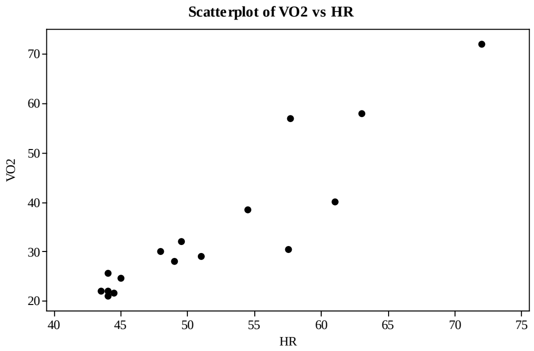
b. The accompanying scatterplot exhibits substantial curvature, which suggests that a straight-line model is not actually a good fit.

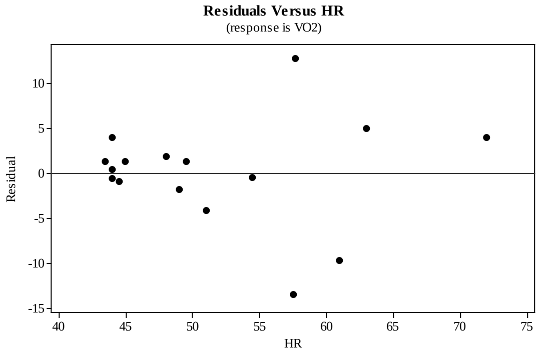


c. Fits, residuals, and standardized residuals were computed using software and the accompanying plot was created. The residual-versus-fit plot indicates very strong curvature but not a lack of constant variance. This implies that a linear model is inadequate, and a quadratic (parabolic) model relationship might be suitable for *x* and *y*.



8. The scatter plot below appears fairly linear, but at least one of the points (72.0,72.0) is potentially influential.



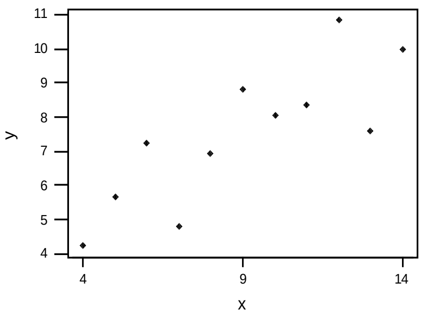


Minitab flags three potential problems: besides the possible influence of (72.0, 72.0), the points (57.5, 30.5) and (57.7, 57.0) have large standardized residuals (–2.31 and 2.18, respectively). These observations also give the appearance of non-constant variance (larger for middle values of *x*), but this is difficult to assess with so few data. Other Minitab output follows; the significance tests should be taken with caution.

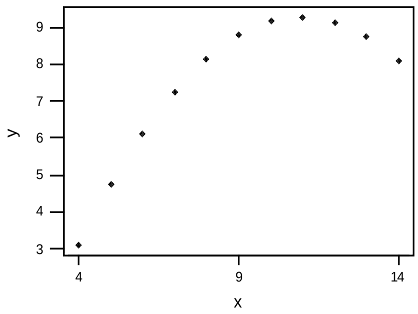
```
The regression equation is
VO2 = - 51.4 + 1.66 HR

Predictor     Coef  SE Coef      T      P
Constant   -51.355    9.795  -5.24  0.000
HR          1.6580   0.1869   8.87  0.000


S = 6.11911   R-Sq = 84.9%   R-Sq(adj) = 83.8%

Unusual Observations

Obs    HR    VO2    Fit  SE Fit  Residual  St Resid
 12  57.5  30.50  43.98    1.87    -13.48     -2.31R
 13  57.7  57.00  44.31    1.89     12.69      2.18R
 16  72.0  72.00  68.02    4.08      3.98      0.87 X
```

9. Both a scatter plot and residual plot (based on the simple linear regression model) for the first data set suggest that a simple linear regression model is reasonable, with no pattern or influential data points which would indicate that the model should be modified. However, scatter plots for the other three data sets reveal difficulties.

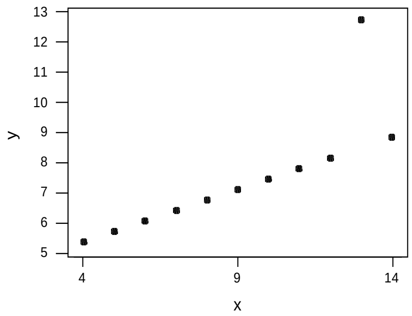
Scatter Plot for Data Set #1



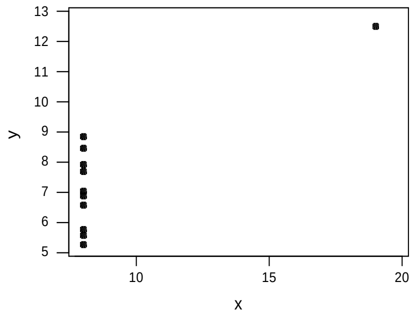
Scatter Plot for Data Set #2



Scatter Plot for Data Set #3



Scatter Plot for Data Set #4



For data set #2, a quadratic function would clearly provide a much better fit. For data set #3, the relationship is perfectly linear except one outlier, which has obviously greatly influenced the fit even though its *x* value is not unusually large or small. One might investigate this observation to see whether it was mistyped and/or it merits deletion. For data set #4 it is clear that the slope of the least squares line has been determined entirely by the outlier, so this point is extremely influential. A linear model is completely inappropriate for data set #4.

10.

a. $e_i = y_i - \left(\hat{\beta}_0 - \hat{\beta}_1 x_i\right) = y_i - \overline{y} - \hat{\beta}_1\left(x_i - \overline{x}\right)$, so $\Sigma e_i = \Sigma\left(y_i - \overline{y}\right) - \hat{\beta}_1 \Sigma\left(x_i - \overline{x}\right) = 0 + \hat{\beta}_1 \cdot 0 = 0$.

b. Since $\Sigma e_i = 0$ always, the residuals cannot be independent. There is clearly a linear relationship between the residuals. If one e_i is large positive, then at least one other e_i would have to be negative to preserve $\Sigma e_i = 0$. This suggests a negative correlation between residuals (for fixed values of any $n - 2$, the other two obey a negative linear relationship).

c. $\Sigma x_i e_i = \Sigma x_i y_i - \Sigma x_i \bar{y} - \hat{\beta}_1 \Sigma x_i \left(x_i - \bar{x}\right) = \left[\Sigma x_i y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n}\right] - \hat{\beta}_1 \left[\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}\right]$ but the first term in brackets is the numerator of $\hat{\beta}_1$, while the second term is the denominator of $\hat{\beta}_1$, so the difference becomes (numerator of $\hat{\beta}_1$) – (numerator of $\hat{\beta}_1$) = 0.

d. The five e_i^*'s from Exercise 7 above are –1.55, .68, 1.25, –.05, and –1.06, which sum to –.73. This sum differs too much from 0 to be explained by rounding. In general it is not true that $\Sigma e_i^* = 0$.

11.

a. $Y_i - \hat{Y}_i = Y_i - \bar{Y} - \hat{\beta}_1\left(x_i - \bar{x}\right) = Y_i - \frac{1}{n}\sum_j Y_j - \frac{\left(x_i - \bar{x}\right)\Sigma_j\left(x_j - \bar{x}\right)Y_j}{\Sigma_j\left(x_j - \bar{x}\right)^2} = \sum_j c_j Y_j$, where

$c_j = 1 - \frac{1}{n} - \frac{\left(x_i - \bar{x}\right)^2}{n\Sigma\left(x_j - \bar{x}\right)^2}$ for $j = i$ and $c_j = 1 - \frac{1}{n} - \frac{\left(x_i - \bar{x}\right)\left(x_j - \bar{x}\right)}{\Sigma\left(x_j - \bar{x}\right)^2}$ for $j \neq i$. Thus

$V\left(Y_i - \hat{Y}_i\right) = \Sigma V\left(c_j Y_j\right)$ (since the Y_j's are independent) = $\sigma^2 \Sigma c_j^2$ which, after some algebra, gives Equation (13.2).

b. $\sigma^2 = V(Y_i) = V\left(\hat{Y}_i + \left(Y_i - \hat{Y}_i\right)\right) = V(\hat{Y}_i) + V\left(Y_i - \hat{Y}_i\right)$, so

$V\left(Y_i - \hat{Y}_i\right) = \sigma^2 - V(\hat{Y}_i) = \sigma^2 - \sigma^2\left[\frac{1}{n} + \frac{\left(x_i - \bar{x}\right)^2}{\Sigma\left(x_j - \bar{x}\right)^2}\right]$, which is exactly (13.2).

c. As x_i moves further from $\bar{x}$, $\left(x_i - \bar{x}\right)^2$ grows larger, so $V(\hat{Y}_i)$ increases since $\left(x_i - \bar{x}\right)^2$ has a positive sign in $V(\hat{Y}_i)$, but $V\left(Y_i - \hat{Y}_i\right)$ decreases since $\left(x_i - \bar{x}\right)^2$ has a negative sign in that expression.

12.

a. $\Sigma e_i = 34$, which is not = 0, so these cannot be the residuals.

b. Each $x_i e_i$ is positive (since x_i and e_i have the same sign) so $\Sigma x_i e_i > 0$, which contradicts the result of exercise 10**c**, so these cannot be the residuals for the given x values.

13. The distribution of any particular standardized residual is also a t distribution with n – 2 d.f., since e_i^* is obtained by taking standard normal variable $\frac{Y_i - \hat{Y}_i}{\sigma_{Y_i - \hat{Y}_i}}$ and substituting the estimate of σ in the denominator (exactly as in the predicted value case). With E_i^* denoting the ith standardized residual as a random variable, when n = 25 E_i^* has a t distribution with 23 df and $t_{.01,23} = 2.50$, so $P($ E_i^* outside (–2.50, 2.50)) = $P\left(E_i^* \geq 2.50\right) + P\left(E_i^* \leq -2.50\right) = .01 + .01 = .02$.

14.

a. $n_1 = n_2 = 3$ (3 observations at 110 and 3 at 230), $n_3 = n_4 = 4$, $\bar{y}_{1.} = 202.0$, $\bar{y}_{2.} = 149.0$, $\bar{y}_{3.} = 110.5$, $\bar{y}_{4.} = 107.0$, $\Sigma\Sigma y_{ij}^2 = 288{,}013$, so

$$\text{SSPE} = 288{,}013 - \left[3(202.0)^2 + 3(149.0)^2 + 4(110.5)^2 + 4(107.0)^2\right] = 4361.$$

With $\Sigma x_i = 4480$, $\Sigma y_i = 1923$, $\Sigma x_i^2 = 1{,}733{,}500$, $\Sigma y_i^2 = 288{,}013$ (as above), and $\Sigma x_i y_i = 544{,}730$, SSE = 7241 so SSLF = 7241 – 4361=2880.
With $c - 2 = 2$ and $n - c = 10$, MSLF = 2880/2 = 1440 and MSPE = 4361/10 = 436.1, so the computed test statistic value is f = 1440/436.1 = 3.30. Looking at df = (2, 10) in Table A.9, 2.92 < 3.30 < 4.10 $\Rightarrow$ the P-value is between .10 and .05. In particular, since P-value > .05, we fail to reject H_0. This formal test procedure does not suggest that a linear model is inappropriate.

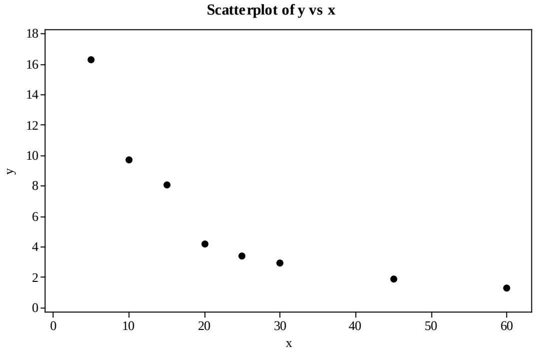
b. The scatter plot clearly reveals a curved pattern which suggests that a nonlinear model would be more reasonable and provide a better fit than a linear model. The contradiction between the scatterplot and the formal test can best be attributed to the small sample size (low power to detect a violation of H_0).

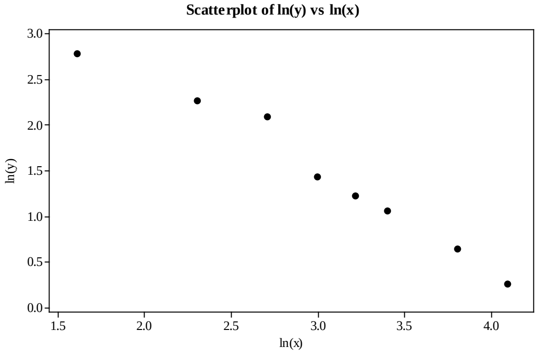
Section 13.2

15.

a. The scatterplot of y versus x below, left has a curved pattern. A linear model would not be appropriate.

b. The scatterplot of ln(y) versus ln(x) below, right exhibits a strong linear pattern.

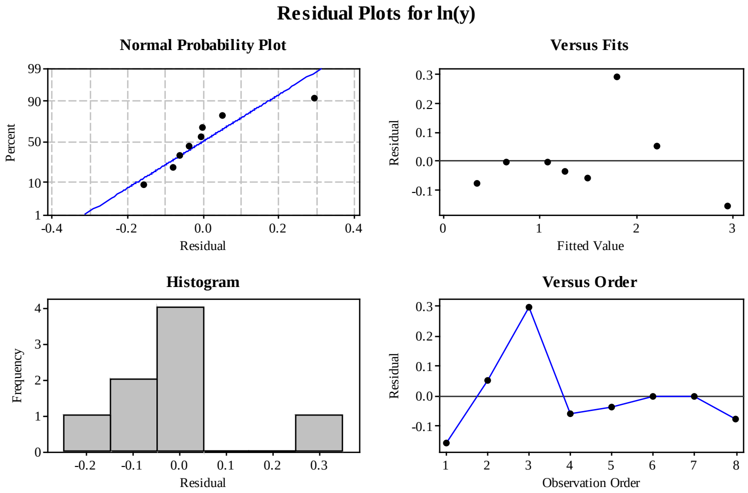




c. The linear pattern in **b** above would indicate that a transformed regression using the natural log of both x and y would be appropriate. The probabilistic model is then $y = \alpha x^{\beta} \cdot \varepsilon$, the power function with an error term.

d. A regression of ln(y) on ln(x) yields the equation ln(y) = 4.6384 – 1.04920 ln(x). Using Minitab we can get a PI for y when x = 20 by first transforming the x value: ln(20) = 2.996. The computer generated 95% PI for ln(y) when ln(x) = 2.996 is (1.1188, 1.8712). We must now take the antilog to return to the original units of y: ($e^{1.1188}$, $e^{1.8712}$) = (3.06, 6.50).

e. A computer generated residual analysis:



Looking at the residual vs. fits (bottom right), one standardized residual, corresponding to the third observation, is a bit large. There are only two positive standardized residuals, but two others are essentially 0. The patterns in the residual plot and the normal probability plot (upper left) are marginally acceptable.

16. A scatter plot of log(time) versus load shows a reasonably linear pattern. From the computer output below, r^2 = 57.7%, so the linear model does an adequate job. With y = log(time) and x = load, the regression equation is $\hat{y} = 18.305 – 0.21421x$; in particular, $\hat{\beta}_1 = – 0.21421$. A 95% PI for log(time) when x = 80.0 is given below as (–0.923, 3.258). Transform to create a PI on the original scale: $(10^{–0.923}, 10^{3.258})$ = (0.119, 1811.34). That is, for a rope at 80% of breaking load, we're 95% confident the failure time will be between 0.119 hours and 1811.34 hours. (The PI is not very precise, due to small sample size and relatively large s.)

Regression Analysis: log(Time) versus Load

```
The regression equation is
log(Time) = 18.3 - 0.214 Load

Predictor       Coef   SE Coef      T      P
Constant      18.305     3.836   4.77  0.000
Load        -0.21421   0.04582  -4.68  0.000

S = 0.946479   R-Sq = 57.7%   R-Sq(adj) = 55.1%

Predicted Values for New Observations

New
Obs    Fit  SE Fit       95% CI             95% PI
  1  1.168   0.277  (0.580, 1.755)  (-0.923, 3.258)

Values of Predictors for New Observations

New
Obs  Load
  1  80.0
```

17.

a. $\Sigma x_i' = 15.501$, $\Sigma y_i' = 13.352$, $\Sigma x_i'^2 = 20.228$, $\Sigma y_i'^2 = 16.572$, $\Sigma x_i' \; y_i' = 18.109$, from which $\hat{\beta}_1 = 1.254$ and $\hat{\beta}_0 = -.468$ so $\hat{\beta} = \hat{\beta}_1 = 1.254$ and $\hat{\alpha} = e^{-.468} = .626$.

b. The plots give strong support to this choice of model; in addition, $r^2 = .960$ for the transformed data.

c. SSE = .11536 (computer printout), $s = .1024$, and the estimated sd of $\hat{\beta}_1$ is .0775, so $t = \frac{1.25 - \frac{4}{3}}{.0775} = -1.02$, for a P-value at 11 df of $P(T \le -1.02) \approx .16$. Since .16 > .05, H_0 cannot be rejected in favor of H_a.

d. The claim that $\mu_{Y \cdot 5} = 2\mu_{Y \cdot 2.5}$ is equivalent to $\alpha \cdot 5^{\beta} = 2\alpha(2.5)^{\beta}$, or that $\beta = 1$. Thus we wish test H_0: $\beta = 1$versus H_a: $\beta \ne 1$. With $t = \frac{1.25 - 1}{.0775}$ = 3.28, the 2-sided P-value at 11 df is roughly 2(.004) = .008. Since .008 ≤ .01, H_0 is rejected at level .01.

18. A scatterplot suggests making a logarithmic transformation of x. We transform $x' = \ln(x)$, so $y = \alpha + \beta \ln(x)$. This transformation yields a linear regression equation $y = .0197 - .00128x'$ or $y = .0197 - .00128\ln(x)$. Minitab output follows:

```
The regression equation is
y = 0.0197 - 0.00128 ln(x)

Predictor        Coef       StDev          T        P
Constant     0.019709    0.002633       7.49    0.000
ln(x)      -0.0012805   0.0003126      -4.10    0.001

S = 0.002668    R-Sq = 49.7%     R-Sq(adj) = 46.7%

Predicted Values for New Observations

New Obs       Fit    SE Fit         95% CI                95% PI
      1  0.008803  0.000621  (0.007494, 0.010112)  (0.003023, 0.014582)


Values of Predictors for New Observations

New Obs  ln(x)
      1   8.52
```

The model is useful, based on a t test, with a P-value of .001. But r^2 = 49.7, so only 49.7% of the variation in y can be explained by its relationship with ln(x). [This is slightly better than a regression of ln(y) on ln(x) corresponding to a power model, which we also examined.]

To estimate y_{5000}, we need ln(x) = ln(5000) = 8.51719. A point estimate for y_{5000} is .0197 – .00128(8.51719) = .0088. A 95% prediction interval for y_{5000} is (0.003023, 0.014582).

19.

a. No, there is definite curvature in the plot.

b. With x = temperature and y = lifetime, a linear relationship between ln(lifetime) and 1/temperature implies a model $y = \exp(\alpha + \beta/x + \varepsilon)$. Let x' = 1/temperature and y' = ln(lifetime). Plotting y' vs. x' gives a plot which has a pronounced linear appearance (and, in fact, $r^2 = .954$ for the straight line fit).

c. $\Sigma x_i' = .082273$, $\Sigma y_i' = 123.64$, $\Sigma x_i'^2 = .00037813$, $\Sigma y_i'^2 = 879.88$, $\Sigma x_i' y_i' = .57295$, from which $\hat{\beta} = 3735.4485$ and $\hat{\alpha} = -10.2045$ (values read from computer output). With $x = 220$, $x' = .004545$ so $\hat{y}' = -10.2045 + 3735.4485(.004545) = 6.7748$ and thus $\hat{y} = e^{\hat{y}'} = 875.50$.

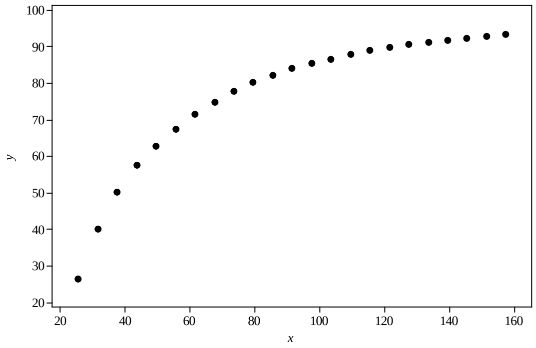
d. For the transformed data, SSE = 1.39857, and $n_1 = n_2 = n_3 = 6$, $\bar{y}_{1.}' = 8.44695$, $\bar{y}_{2.}' = 6.83157$, $\bar{y}_{3.}' = 5.32891$, from which SSPE = 1.36594, SSLF = .02993, $f = \dfrac{.02993/1}{1.36594/15} = .33$. Comparing this to the F distribution with df = (1, 15), it is clear that H_0 cannot be rejected.

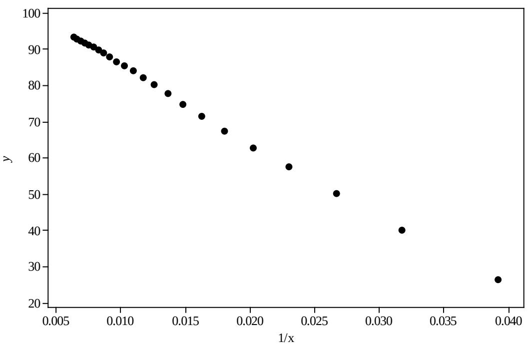
20. After examining a scatter plot and a residual plot for each of the five suggested models as well as for y vs. x, it appears that the power model $y = \alpha x^{\beta} \cdot \varepsilon$, i.e., $y' = \ln(y)$ vs. $x' = \ln(x)$, provided the best fit. The transformation seemed to remove most of the curvature from the scatter plot, the residual plot appeared quite random, $\left| e_i'^{*} \right| < 1.65$ for every i, there was no indication of any influential observations, and $r^2 = .785$ for the transformed data.

21.

a. The accompanying scatterplot, left, shows a very strong non-linear association between the variables. The corresponding residual plot would look somewhat like a downward-facing parabola.

b. The right scatterplot shows y versus $1/x$ and exhibits a much more linear pattern. We'd anticipate an r^2 value very near 1 based on the plot. (In fact, $r^2 = .998$.)





c. With the aid of software, a 95% PI for y when $x = 100$, aka $x' = 1/x = 1/100 = .01$, can be generated. Using Minitab, the 95% PI is (83.89, 87.33). That is, at a 95% prediction level, the nitrogen extraction percentage for a single run when leaching time equals 100 h is between 83.89 and 87.33.

22.

a. $\frac{1}{y} = \alpha + \beta x$, so with $y' = \frac{1}{y}$, $y' = \alpha + \beta x$. The corresponding probabilistic model is $\frac{1}{Y} = \alpha + \beta x + \varepsilon$.

b. $\frac{1}{y} - 1 = e^{\alpha+\beta x}$, so $\ln\left(\frac{1}{y} - 1\right) = \alpha + \beta x$. Thus with $y' = \ln\left(\frac{1}{y} - 1\right)$, $y' = \alpha + \beta x$. The corresponding probabilistic model is $Y' = \alpha + \beta x + \varepsilon'$, or equivalently $Y = \frac{1}{1 + e^{\alpha+\beta x} \cdot \varepsilon}$ where $\varepsilon = e^{\varepsilon'}$.

c. $\ln(y) = e^{\alpha+\beta x} = \ln(\ln(y)) = \alpha + \beta x$. Thus with $y' = \ln(\ln(y)), y' = \alpha + \beta x$. The probabilistic model is $Y' = \alpha + \beta x + \varepsilon'$, or equivalently, $Y = e^{e^{\alpha+\beta x}} \cdot \varepsilon$ where $\varepsilon = e^{\varepsilon'}$.

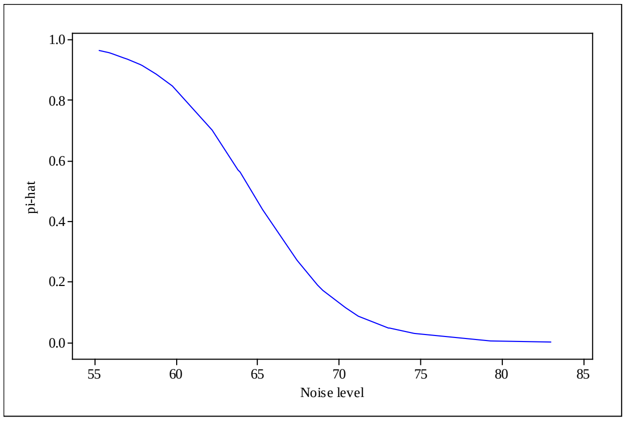
d. This function cannot be linearized.

23. $V(Y) = V(\alpha e^{\beta x} \cdot \varepsilon) = \left[\alpha e^{\beta x}\right]^2 \cdot V(\varepsilon) = \alpha^2 e^{2\beta x} \cdot \tau^2$ where we have set $V(\varepsilon) = \tau^2$. If $\beta > 0$, this is an increasing function of x so we expect more spread in y for large x than for small x, while the situation is reversed if $\beta < 0$. It is important to realize that a scatter plot of data generated from this model will not spread out uniformly about the exponential regression function throughout the range of x values; the spread will only be uniform on the transformed scale. Similar results hold for the multiplicative power model.

24. The hypotheses are H_0: $\beta_1 = 0$ versus H_a: $\beta_1 \neq 0$, where β_1 represents the model coefficient on x in the logistic regression model. The value of the test statistic is $z = .73$, with a corresponding P-value of .463. Since the P-value is greater than any reasonable significance level, we do not reject H_0. There is insufficient evidence to claim that age has a significant impact on the presence of kyphosis.

25. First, the test statistic for the hypotheses H_0: $\beta_1 = 0$ versus H_a: $\beta_1 \neq 0$ is $z = –4.58$ with a corresponding P-value of .000, suggesting noise level has a highly statistically significant relationship with people's perception of the acceptability of the work environment. The negative value indicates that the likelihood of finding work environment acceptable <u>decreases</u> as the noise level increases (not surprisingly). We estimate that a 1 dBA increase in noise level decreases the <u>odds</u> of finding the work environment acceptable by a multiplicative factor of .70 (95% CI: .60 to .81).

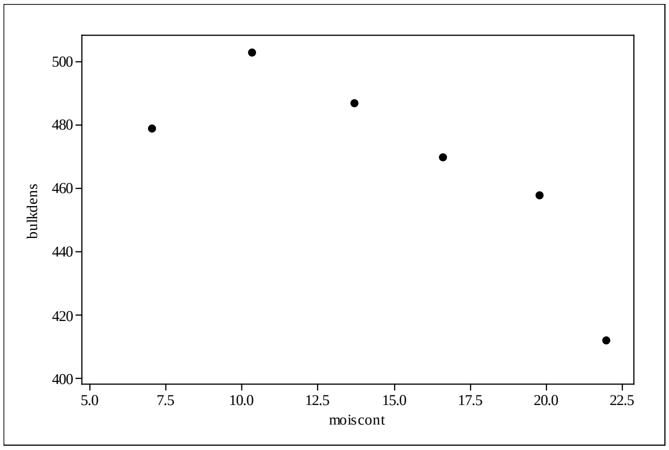
The accompanying plot shows $\hat{\pi} = \frac{e^{b_0+b_1 x}}{1+e^{b_0+b_1 x}} = \frac{e^{23.2-.359x}}{1+e^{23.2-.359x}}$. Notice that the estimate probability of finding work environment acceptable decreases as noise level, x, increases.



Section 13.3

26.

a. The accompanying scatterplot suggests that a quadratic model is indeed a good fit to this sample data.



b. From the output, the coefficient of multiple determination is R^2 = 93.8%. That is, 93.8% of observed variation in bulk density can be attributed to the model relationship.

c. From software, a 95% CI for μ_Y when x = 13.7 (and $x^2 = 13.7^2$) is (471.475, 512.760).

d. We want a 99% confidence interval, but the output gives us a 95% confidence interval of (452.71, 529.48), which can be rewritten as 491.10 ± 38.38. At df = 6 – 3 = 3, $t_{.025,3} = 3.182 \Rightarrow$

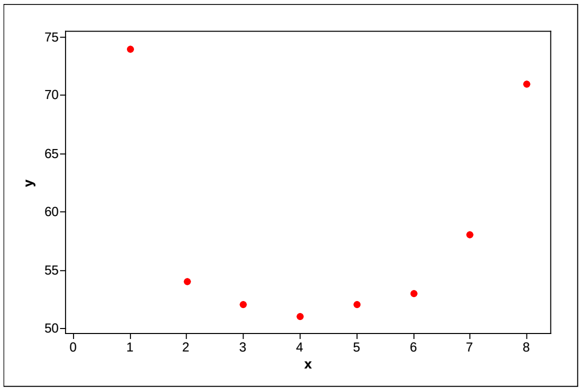
$s_{\hat{y}\cdot 14} = \dfrac{38.38}{3.182} = 12.06$. Finally, $t_{.005,3}$ = 5.841, so the 99% PI is

$491.10 \pm 5.841(12.06) = 491.10 \pm 70.45 = (420.65, 561.55)$.

e. To test the utility of the quadratic term, the hypotheses are H_0: $\beta_2 = 0$ versus H_a: $\beta_2 \neq 0$. The test statistic is t = –3.81, with a corresponding P-value of .032. At the .05 level, we reject H_0: quadratic term appears to be useful in this model.

27.

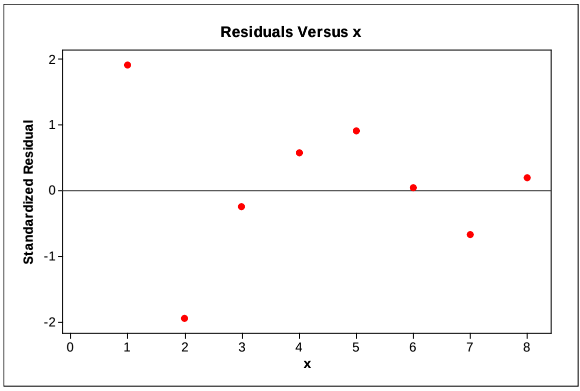
a. A scatter plot of the data indicated a quadratic regression model might be appropriate.

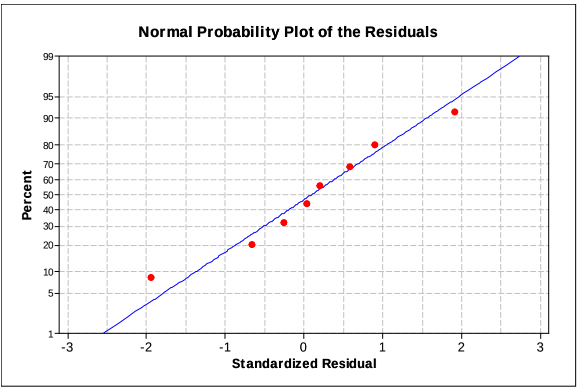


b. $\hat{y} = 84.482 - 15.875(6) + 1.7679(6)^2 = 52.88$; residual = $y_6 - \hat{y}_6 = 53 - 52.88 = .12$

c. $SST = \Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n} = 586.88$, so $R^2 = 1 - \frac{61.77}{586.88} = .895$.

d. None of the standardized residuals exceeds 2 in magnitude, suggesting none of the observations are outliers. The ordered *z* percentiles needed for the normal probability plot are –1.53, –.89, –.49, –.16, .16, .49, .89, and 1.53. The normal probability plot below does not exhibit any troublesome features.





e. $\hat{\mu}_{Y\cdot 6} = 52.88$ (from **b**) and $t_{.025,n-3} = t_{.025,5} = 2.571$, so the CI is $52.88 \pm (2.571)(1.69) = 52.88 \pm 4.34 = (48.54, 57.22)$.

f. SSE = 61.77, so $s^2 = \frac{61.77}{5} = 12.35$ and s\{pred\} $\sqrt{12.35 + (1.69)^2} = 3.90$. The PI is $52.88 \pm (2.571)(3.90) = 52.88 \pm 10.03 = (42.85, 62.91)$.

28.

a. $\hat{\mu}_{Y\cdot 75} = \hat{\beta}_0 + \hat{\beta}_1(75) + \hat{\beta}_2(75)^2 = -113.0937 + 3.36684(75) - .01780(75)^2 = 39.41$

b. $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1(60) + \hat{\beta}_2(60)^2 = 24.93$.

c. $\text{SSE} = \Sigma y_i^2 - \hat{\beta}_0 \Sigma y_i - \hat{\beta}_1 \Sigma x_i y_i - \hat{\beta}_2 \Sigma x_i^2 y_i = 8386.43 - (-113.0937)(210.70)$
$- (3.3684)(17{,}002) - (-.0178)(1{,}419{,}780) = 217.82$, $s^2 = \frac{SSE}{n-3} = \frac{217.82}{3} = 72.61$, $s = 8.52$

d. $R^2 = 1 - \frac{217.82}{987.35} = .779$

e. The computed value of the test statistic is $t = \frac{-.01780}{.00226} = -7.88$. Even at 3 df, the *P*-value is much less than .01. Hence, we reject H_0. The quadratic term is statistically significant in this model.

29.

a. The table below displays the y-values, fits, and residuals. From this, SSE = $\sum e^2 = 16.8$, s^2 = SSE/(n – 3) = 4.2, and s = 2.05.

y	$\hat{y}$	$e = y - \hat{y}$
81	82.1342	–1.13420
83	80.7771	2.22292
79	79.8502	–0.85022
75	72.8583	2.14174
70	72.1567	–2.15670
43	43.6398	–0.63985
22	21.5837	0.41630

b. SST = $\sum (y - \bar{y})^2 = \sum (y - 64.71)^2 = 3233.4$, so R^2 = 1 – SSE/SST = 1 – 16.8/3233.4 = .995, or 99.5%. 995% of the variation in free–flow can be explained by the quadratic regression relationship with viscosity.

c. We want to test the hypotheses H_0: $\beta_2 = 0$ v. H_a: $\beta_2 \neq 0$. Assuming all inference assumptions are met, the relevant t statistic is $t = \dfrac{-.0031662 - 0}{.0004835} = -6.55$. At n – 3 = 4 df, the corresponding P-value is $2P(T > 6.55) < .004$. At any reasonable significance level, we would reject H_0 and conclude that the quadratic predictor indeed belongs in the regression model.

d. Two intervals with at least 95% simultaneous confidence requires individual confidence equal to 100% – 5%/2 = 97.5%. To use the t-table, round up to 98%: $t_{.01,4} = 3.747$. The two confidence intervals are 2.1885 ± 3.747(.4050) = (.671, 3.706) for β_1 and –.0031662 ± 3.747(.0004835) = (–.00498, –.00135) for β_2. [In fact, we are at least 9<u>6</u>% confident β_1 and β_2 lie in these intervals.]

e. Plug into the regression equation to get $\hat{y} = 72.858$. Then a 95% CI for $\mu_{Y.400}$ is 72.858 ± 3.747(1.198) = (69.531, 76.186). For the PI, $s\{\text{pred}\} = \sqrt{s^2 + s_{\hat{Y}}^2} = \sqrt{4.2 + (1.198)^2} = 2.374$, so a 95% PI for Y when x = 400 is 72.858 ± 3.747(2.374) = (66.271, 79.446).

30.

a. R^2 = 98.1% or .981. This means 98.1% of the observed variation in yield strength can be attributed to the model relationship.

b. To test the utility of the quadratic term, the hypotheses are H_0: $\beta_2 = 0$ versus H_a: $\beta_2 \neq 0$. The test statistic is $t = -8.29$, with a corresponding P-value of .014. At the .05 level, we reject H_0: quadratic term appears to be useful in this model.

c. From software, a 95% CI for $\mu_{Y \cdot 100}$ is (123.848, 144.297). Alternatively, using the information provided, a 95% CI is $\hat{y} \pm t_{.025,2} s_{\hat{Y}}$ = 134.07 ± 4.303(2.38) = (123.8, 144.3).

d. From software, a 95% PI for Y·100 is (116.069, 152.076). Alternatively, using the information provided, a 95% PI is $\hat{y} \pm t_{.025,2}\sqrt{s^2 + s_{\hat{Y}}^2}$ = $134.07 \pm 4.303\sqrt{(3.444)^2 + (2.38)^2}$ = (116.06, 152.08). The value s = 3.444 comes from the output provided, where s is 3.44398. As always, the PI for a single future value of Y when x = 100 is much wider that the CI for the true <u>mean</u> value of Y when x = 100.

31.

a. R^2 = 98.0% or .980. This means 98.0% of the observed variation in energy output can be attributed to the model relationship.

b. For a quadratic model, adjusted $R^2 = \frac{(n-1)R^2-k}{n-1-k} = \frac{(24-1)(.780)-2}{24-1-2}$ = .759, or 75.9%. (A more precise answer, from software, is 75.95%.) The adjusted R^2 value for the cubic model is 97.7%, as seen in the output. This suggests that the cubic term greatly improves the model: the cost of adding an extra parameter is more than compensated for by the improved fit.

c. To test the utility of the cubic term, the hypotheses are H_0: $\beta_3 = 0$ versus H_0: $\beta_3 \neq 0$. From the Minitab output, the test statistic is $t = 14.18$ with a P-value of .000. We strongly reject H_0 and conclude that the cubic term is a statistically significant predictor of energy output, even in the presence of the lower terms.

d. Plug $x = 30$ into the cubic estimated model equation to get $\hat{y} = 6.44$. From software, a 95% CI for $\mu_{Y\cdot 30}$ is (6.31, 6.57). Alternatively, $\hat{y} \pm t_{.025,20}s_{\hat{Y}} = 6.44 \pm 2.086(.0611)$ also gives (6.31, 6.57). Next, a 95% PI for $Y\cdot 30$ is (6.06, 6.81) from software. Or, using the information provided, $\hat{y} \pm t_{.025,20}\sqrt{s^2+s_{\hat{Y}}^2}$ = $6.44 \pm 2.086\sqrt{(.1684)^2+(.0611)^2}$ also gives (6.06, 6.81). The value of s comes from the Minitab output, where $s = .168354$.

e. The null hypothesis states that the true mean energy output when the temperature difference is 35°K is equal to 5W; the alternative hypothesis says this isn't true.
Plug $x = 35$ into the cubic regression equation to get $\hat{y} = 4.709$. Then the test statistic is $t = \frac{4.709-5}{.0523} \approx -5.6$, and the two-tailed P-value at df = 20 is approximately 2(.000) = .000. Hence, we strongly reject H_0 (in particular, .000 < .05) and conclude that $\mu_{Y\cdot 35} \neq 5$.

Alternatively, software or direct calculation provides a 95% CI for $\mu_{Y\cdot 35}$ of (4.60, 4.82). Since this CI does not include 5, we can reject H_0 at the .05 level.

32.

a. $.3463 - 1.2933(x-\bar{x}) + 2.3964(x-\bar{x})^2 - 2.3968(x-\bar{x})^3$.

b. From **a**, the coefficient of x^3 is –2.3968, so $\hat{\beta}_3 = -2.3968$. There will be a contribution to x^2 both from $2.3964(x-4.3456)^2$ and from $-2.3968(x-4.3456)^3$. Expanding these and adding yields 33.6430 as the coefficient of x^2, so $\hat{\beta}_2 = 33.6430$.

c. $x = 4.5 \Rightarrow x' = x - \bar{x} = .1544$; substituting into **a** yields $\hat{y} = .1949$.

d. $t = \frac{-2.3968}{2.4590} = -.97$, which is not significant (H_0: $\beta_3 = 0$ cannot be rejected), so the inclusion of the cubic term is not justified.

33.

a. $\overline{x} = 20$ and s_x = 10.8012 so $x' = \dfrac{x-20}{10.8012}$. For x = 20, x' = 0, and $\hat{y} = \hat{\beta}_0^* = .9671$. For x = 25, x' = .4629, so $\hat{y} = .9671 - .0502(.4629) - .0176(.4629)^2 + .0062(.4629)^3 = .9407$.

b. $\hat{y} = .9671 - .0502\left(\dfrac{x-20}{10.8012}\right) - .0176\left(\dfrac{x-20}{10.8012}\right)^2 + .0062\left(\dfrac{x-20}{10.8012}\right)^3$
$= .00000492x^3 - .000446058x^2 + .007290688x + .96034944$.

c. $t = \dfrac{.0062}{.0031} = 2.00$. At df = n – 4 = 3, the P-value is 2(.070) = .140 > .05. Therefore, we cannot reject H_0; the cubic term should be deleted.

d. $\text{SSE} = \Sigma\left(y_i - \hat{y}_i\right)^2$ and the $\hat{y}_i$'s are the same from the standardized as from the unstandardized model, so SSE, SST, and R^2 will be identical for the two models.

e. $\Sigma y_i^2 = 6.355538$, $\Sigma y_i = 6.664$, so SST = .011410. For the quadratic model, R^2 = .987, and for the cubic model, R^2 = .994. The two R^2 values are very close, suggesting intuitively that the cubic term is relatively unimportant.

34.

a. $\overline{x} = 49.9231$ and s_x = 41.3652 so for x = 50, $x' = \dfrac{x-49.9231}{41.3652} = .001859$ and
$\hat{\mu}_{Y\cdot 50} = .8733 - .3255(.001859) + .0448(.001859)^2 = .873$.

b. SST = 1.456923 and SSE = .117521, so R^2 = .919.

c. $.8733 - .3255\left(\dfrac{x-49.9231}{41.3652}\right) + .0448\left(\dfrac{x-49.9231}{41.3652}\right)^2 = 1.3314 - .01048314x + .00002618x^2$.

d. $\hat{\beta}_2 = \dfrac{\hat{\beta}_2^*}{s_x^2}$ so the estimated sd of $\hat{\beta}_2$ is the estimated sd of $\hat{\beta}_2^*$ multiplied by $\dfrac{1}{s_x^2}$:
$s_{\hat{\beta}_2} = (.0319)\left(\dfrac{1}{41.3652^2}\right) = .0000186$.

e. The test statistic is t = .0448/.0319 = 1.40; at 9 df, the P-value is 2(.098) = .196 > .05, so the quadratic term should not be retained. The result is the same in both cases.

35. $Y' = \ln(Y) = \ln\alpha + \beta x + \gamma x^2 + \ln(\varepsilon) = \beta_0 + \beta_1 x + \beta_2 x^2 + \varepsilon'$ where $\varepsilon' = \ln(\varepsilon)$, $\beta_0 = \ln(\alpha)$, $\beta_1 = \beta$, and $\beta_2 = \gamma$. That is, we should fit a quadratic to (x, ln(y)). The resulting estimated quadratic (from computer output) is $2.00397 + .1799x - .0022x^2$, so $\hat{\beta} = .1799$, $\hat{\gamma} = -.0022$, and $\hat{\alpha} = e^{2.0397} = 7.6883$. [The ln($y$)'s are 3.6136, 4.2499, 4.6977, 5.1773, and 5.4189, and the summary quantities can then be computed as before.]

Section 13.4

36.

a. Holding age, time, and heart rate constant, maximum oxygen uptake will increase by .01 L/min for each 1 kg increase in weight. Similarly, holding weight, age, and heart rate constant, the maximum oxygen uptake decreases by .13 L/min with every 1 minute increase in the time necessary to walk 1 mile.

b. $\hat{y}_{76,20,12,140} = 5.0 + .01(76) - .05(20) - .13(12) - .01(140) = 1.8$ L/min.

c. $\hat{y}$ = 1.8 from **b**, and σ = .4, so, assuming Y follows a normal distribution,
$P(1.00 < Y < 2.60) = P\left(\frac{1.00 - 1.8}{.4} < Z < \frac{2.6 - 1.8}{.4}\right) = P(-2.0 < Z < 2.0) = .9544$.

37.

a. The mean value of y when x_1 = 50 and x_2 = 3 is $\mu_{y \cdot 50,3} = -.800 + .060(50) + .900(3) = 4.9$ hours.

b. When the number of deliveries (x_2) is held fixed, then average change in travel time associated with a one-mile (i.e., one unit) increase in distance traveled (x_1) is .060 hours. Similarly, when distance traveled (x_1) is held fixed, then the average change in travel time associated with on extra delivery (i.e., a one unit increase in x_2) is .900 hours.

c. Under the assumption that Y follows a normal distribution, the mean and standard deviation of this distribution are 4.9 (because x_1 = 50 and x_2 = 3) and σ = .5 (since the standard deviation is assumed to be constant regardless of the values of x_1 and x_2). Therefore,
$P(Y \le 6) = P\left(Z \le \frac{6 - 4.9}{.5}\right) = P(Z \le 2.20) = .9861$. That is, in the long run, about 98.6% of all days will result in a travel time of at most 6 hours.

38.

a. mean life $=125+7.75(40)+.0950(1100)-.009(40)(1100)=143.50$

b. First, the mean life when x_1 = 30 is equal to $125+7.75(30)+.0950x_2-.009(30)x_2=357.50-.175x_2$. So when the load increases by 1, the mean life decreases by .175. Second, the mean life when x_1 =40 is equal to $125+7.75(40)+.0950x_2-.009(40)x_2=435-.265xa$. So when the load increases by 1, the mean life decreases by .265.

39.

a. For $x_1 = 2$, $x_2 = 8$ (remember the units of x_2 are in 1000s), and $x_3 = 1$ (since the outlet has a drive-up window), the average sales are $\hat{y}=10.00-1.2(2)+6.8(8)+15.3(1)=77.3$ (i.e., $77,300).

b. For $x_1 = 3$, $x_2 = 5$, and $x_3 = 0$ the average sales are $\hat{y}=10.00-1.2(3)+6.8(5)+15.3(0)=40.4$ (i.e., $40,400).

c. When the number of competing outlets (x_1) and the number of people within a 1-mile radius (x_2) remain fixed, the expected sales will increase by $15,300 when an outlet has a drive-up window.

40.

a. For testing H_0: $\beta_1=\beta_2=\beta_3=0$ vs. H_a: at least one among β_1,β_2,β_3 is not zero, the test statistic is $F=\dfrac{R^2/k}{(1-R^2)/(n-k-1)}=\dfrac{.91/3}{(1-.91)/(11-3-1)}$ = 23.6. Comparing this to the F distribution with df = (3,7), 23.6 > 18.77 ⇒ P-value < .001 < .01 ⇒ we reject H_0 at the .01 level. We conclude that at least one β is non-zero; i.e., the three predictors as a set are useful for predicting power output (y).

b. For fixed values of x_2 (excess post-exercise oxygen consumption) and x_3 (immediate posttest lactate), a one-centimeter increase in arm girth is associated with an estimated increase in predicted/mean power output of 14.06 W.

c. $\hat{y}$ = –408.20 + 14.06(36) + .76(120) – 3.64(10.0) = 152.76 W.

d. Our point estimate for $\mu_{Y\cdot 36,120,10.0}$ is the same value as in **c**, 152.76 W.

e. What's being described is the coefficient on x_3. Our point estimate of β_3 is $\hat{\beta}_3$ = –3.64.

41.

a. R^2 = .834 means that 83.4% of the total variation in cone cell packing density (y) can be explained by a linear regression on eccentricity (x_1) and axial length (x_2). For H_0: $\beta_1 = \beta_2 = 0$ vs. H_a: at least one $\beta \neq 0$, the test statistic is $F = \dfrac{R^2/k}{(1-R^2)/(n-k-1)}=\dfrac{.834/2}{(1-.834)/(192-2-1)}\approx 475$, and the associated P-value at df = (2, 189) is essentially 0. Hence, H_0 is rejected and the model is judged useful.

b. $\hat{y}$ = 35821.792 – 6294.729(1) – 348.037(25) = 20,826.138 cells/mm^2.

c. For a fixed axial length (x_2), a 1-mm increase in eccentricity is associated with an estimated <u>decrease</u> in mean/predicted cell density of 6294.729 cells/mm^2.

d. The error df = $n - k - 1 = 192 - 3 = 189$, so the critical CI value is $t_{.025,189} \approx z_{.025} = 1.96$. A 95% CI for β_1 is –6294.729 ± 1.96(203.702) = (–6694.020, –5895.438).

e. The test statistic is $t = \dfrac{-348.037 - 0}{134.350} = -2.59$; at 189 df, the 2-tailed P-value is roughly $2P(T \le -2.59) \approx 2\Phi(-2.59) = 2(.0048) \approx .01$. Since .01 < .05, we reject H_0. After adjusting for the effect of eccentricity (x_1), there is a statistically significant relationship between axial length (x_2) and cell density (y). Therefore, we should retain x_2 in the model.

42.

a. To test H_0: $\beta_1 = \beta_2 = 0$ vs. H_a: at least one $\beta \neq 0$, the test statistic is $f = \dfrac{\text{MSR}}{\text{MSE}} = 319.31$ (from output). The associated P-value is ≈ 0, so at any reasonable level of significance, H_0 should be rejected. There does appear to be a useful linear relationship between temperature difference and at least one of the two predictors.

b. The degrees of freedom for SSE = $n - (k + 1) = 9 - (2 + 1) = 6$ (which you could simply read in the DF column of the printout), and $t_{.025,6} = 2.447$, so the desired confidence interval is $3.000 \pm (2.447)(.4321) = 3.000 \pm 1.0573$, or about (1.943, 4.057). Holding furnace temperature fixed, we estimate that the average change in temperature difference on the die surface will be somewhere between 1.943 and 4.057.

c. When $x_1 = 1300$ and $x_2 = 7$, the estimated average temperature difference is $\hat{y} = -199.56 + .2100x_1 + 3.000x_2 = -199.56 + .2100(1300) + 3.000(7) = 94.44$. The desired confidence interval is then $94.44 \pm (2.447)(.353) = 94.44 \pm .864$, or (93.58, 95.30).

d. From the printout, $s = 1.058$, so the prediction interval is $94.44 \pm (2.447)\sqrt{(1.058)^2 + (.353)^2} = 94.44 \pm 2.729 = (91.71, 97.17)$.

43.

a. $\hat{y} = 185.49 - 45.97(2.6) - 0.3015(250) + 0.0888(2.6)(250) = 48.313$.

b. No, it is not legitimate to interpret β_1 in this way. It is not possible to increase the cobalt content, x_1, while keeping the interaction predictor, x_3, fixed. When x_1 changes, so does x_3, since $x_3 = x_1x_2$.

c. Yes, there appears to be a useful linear relationship between y and the predictors. We determine this by observing that the P-value corresponding to the model utility test is < .0001 (F test statistic = 18.924).

d. We wish to test H_0: $\beta_3 = 0$ vs. H_a: $\beta_3 \neq 0$. The test statistic is $t = 3.496$, with a corresponding P-value of .0030. Since the P-value is < α = .01, we reject H_0 and conclude that the interaction predictor does provide useful information about y.

e. A 95% CI for the mean value of surface area under the stated circumstances requires the following quantities: $\hat{y} = 185.49 - 45.97(2) - 0.3015(500) + 0.0888(2)(500) = 31.598$. Next, $t_{.025,16} = 2.120$, so the 95% confidence interval is $31.598 \pm (2.120)(4.69) = 31.598 \pm 9.9428 = (21.6552, 41.5408)$.

44.

a. Since the total degrees of freedom equals 80, there were 81 observations.

b. R^2 = 88.6%, so 88.6% of the observed variation in surface roughness can be explained by the model relationship with vibration amplitude, depth of cut, feed rate, and cutting speed as explanatory variables.

c. This is a model utility test. The hypotheses are H_0: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ versus H_a: at least one $\beta_i \neq 0$. From the output, the F-statistic is $f = 148.35$ with a P-value of .000. Thus, we strongly reject H_0 and conclude that at least one of the explanatory variables is a significant predictor of surface roughness.

d. 18.2602 is the coefficient on "f" (i.e., feed rate). After adjusting for the effects of vibration amplitude, depth of cut, and cutting speed, a 1 mm/rev increase in the feed rate is associated with an estimated increase of 18.2602 μm in expected surface roughness.

e. Yes: the P-value for variable "v" (cutting speed, aka velocity) is 0.480 > .10, so that variable is not a statistically significant predictor of surface roughness at the .10 level in the presence of the other three explanatory variables. None of the other variables have P-values above .10 (although "a," vibration amplitude, is close).

f. Substitute the prescribed values into the regression equation to get $\hat{y}$ = 3.7015. Using the information provided, a 95% CI for the mean response at those settings is given by $\hat{y} \pm t_{.025,76}s_{\hat{Y}} \approx 3.7015 \pm 1.99(.1178) = (3.47, 3.94)$. Next, a 95% PI for a single roughness measurement at those settings is $\hat{y} \pm t_{.025,76}\sqrt{s^2 + s_{\hat{Y}}^2} \approx 3.7015 \pm 1.99\sqrt{(.822)^2 + (.1178)^2} = (2.05, 5.35)$. As always, the PI is much wider than the CI.

45.

a. The hypotheses are H_0: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ vs. H_a: at least one $\beta_i \neq 0$. The test statistic is $f = \dfrac{R^2/k}{(1-R^2)/(n-k-1)} = \dfrac{.946/4}{(1-.946)/20} = 87.6 \geq F_{.001,4,20} = 7.10$ (the smallest available F-value from Table A.9), so the P-value is < .001 and we can reject H_0 at any significance level. We conclude that at least one of the four predictor variables appears to provide useful information about tenacity.

b. The adjusted R^2 value is $1 - \dfrac{n-1}{n-(k+1)}\left(\dfrac{SSE}{SST}\right) = 1 - \dfrac{n-1}{n-(k+1)}\left(1-R^2\right) = 1 - \dfrac{24}{20}(1-.946) = .935$, which does not differ much from R^2 = .946.

c. The estimated average tenacity when x_1 = 16.5, x_2 = 50, x_3 = 3, and x_4 = 5 is $\hat{y} = 6.121 - .082(16.5) + .113(50) + .256(3) - .219(5) = 10.091$. For a 99% CI, $t_{.005,20} = 2.845$, so the interval is $10.091 \pm 2.845(.350) = (9.095, 11.087)$. Therefore, when the four predictors are as specified in this problem, the true average tenacity is estimated to be between 9.095 and 11.087.

46.

a. Yes, there does appear to be a useful linear relationship between repair time and the two model predictors. We determine this through a model utility test H_0: $\beta_1 = \beta_2 = 0$ vs. H_a: at least one $\beta \neq 0$. The calculated statistic is $f = \dfrac{\text{SSR}/k}{\text{SSE}/(n-k-1)} = \dfrac{10.63/2}{20.9/9}$ = 22.91. At df = (2, 9), 22.91 > 16.39 ⇒ P-value < .001 < .05 ⇒ we reject H_0 and conclude that at least one of the two predictor variables is useful.

b. We test H_0: $\beta_2 = 0$ v. H_a: $\beta_2 \neq 0$. The test statistic is t = 1.250/.312 = 4.01; at 9 df, the two-tailed P-value is less than 2(.002) = .004. Hence, we reject H_0 and conclude that the "type of repair" variable does provide useful information about repair time, given that the "elapsed time since the last service" variable remains in the model.

c. A 95% confidence interval for β_2 is: 1.250 ± 2.262(.312) = (.5443, 1.9557). We estimate, with a high degree of confidence, that when an electrical repair is required the repair time will be between .54 and 1.96 hours longer than when a mechanical repair is required, while the "elapsed time" predictor remains fixed.

d. $\hat{y} = .950 + .400(6) + 1.250(1) = 4.6$, $s^2 = MSE = .23222$, and $t_{.005,9} = 3.25$, so the 99% PI is $4.6 \pm (3.25)\sqrt{(.23222) + (.192)^2} = 4.6 \pm 1.69 = (2.91, 6.29)$. The prediction interval is quite wide, suggesting a variable estimate for repair time under these conditions.

47.

a. For a 1% increase in the percentage plastics, we would expect a 28.9 kcal/kg increase in energy content. Also, for a 1% increase in the moisture, we would expect a 37.4 kcal/kg decrease in energy content. Both of these assume we have accounted for the linear effects of the other three variables.

b. The appropriate hypotheses are H_0: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ vs. H_a: at least one $\beta \neq 0$. The value of the F-test statistic is 167.71, with a corresponding P-value that is ≈ 0. So, we reject H_0 and conclude that at least one of the four predictors is useful in predicting energy content, using a linear model.

c. H_0: $\beta_3 = 0$ v. H_a: $\beta_3 \neq 0$. The value of the t test statistic is t = 2.24, with a corresponding P-value of .034, which is less than the significance level of .05. So we can reject H_0 and conclude that percentage garbage provides useful information about energy consumption, given that the other three predictors remain in the model.

d. $\hat{y} = 2244.9 + 28.925(20) + 7.644(25) + 4.297(40) - 37.354(45) = 1505.5$, and $t_{.025,25}$ = 2.060. So a 95% CI for the true average energy content under these circumstances is $1505.5 \pm (2.060)(12.47) = 1505.5 \pm 25.69 = (1479.8, 1531.1)$. Because the interval is reasonably narrow, we would conclude that the mean energy content has been precisely estimated.

e. A 95% prediction interval for the energy content of a waste sample having the specified characteristics is $1505.5 \pm (2.060)\sqrt{(31.48)^2 + (12.47)^2} = 1505.5 \pm 69.75 = (1435.7, 1575.2)$.

48.

a. We wish to test H_0: $\beta_1 = \beta_2 = ... = \beta_9 = 0$ vs. H_a: at least one $\beta \neq 0$. The model utility test statistic value is $f = \frac{R^2 / k}{(1-R^2)/(n-k-1)} = \frac{.938/9}{(1-.938)/(15-9-1)}$ = 8.41. At df = (9, 5), 4.77 < 8.41 < 10.16 ⇒ the *P*-value is between .05 and .01. In particular, *P*-value > .01 ⇒ we fail to reject H_0 at the .01 level. At this significance level, the model does not appear to specify a statistically useful relationship (though it does at $\alpha = .05$).

b. $\hat{\mu}_y = 21.967$, $t_{\alpha/2,n-(k+1)} = t_{.025,5} = 2.571$, so the CI is $21.967 \pm (2.571)(1.248) = (18.76, 25.18)$.

c. $s^2 = \frac{\text{SSE}}{n-(k+1)} = \frac{23.379}{5} = 4.6758$, and the CI is $21.967 \pm (2.571)\sqrt{4.6758 + (1.248)^2} = (15.55, 28.39)$.

d. Now we're testing H_0: $\beta_4 = \beta_5 = ... = \beta_9 = 0$ vs. H_a: at least one of $\beta_4, \beta_5, ..., \beta_9 \neq 0$. The required sums of squares are SSE_k = 23.379, SSE_l = 203.82, from which $f = \frac{(203.82 - 23.379)/(9-3)}{23.379/5}$ = 6.43. Using Table A.9 with df = (6, 5), 4.95 < 6.43 < 10.67 ⇒ the *P*-value is between .05 and .01. In particular, *P*-value < .05 ⇒ we reject H_0 at the .05 level and conclude that at least one of the second-order predictors appears useful.

49.

a. Use the ANOVA table in the output to test H_0: $\beta_1 = \beta_2 = \beta_3 = 0$ vs. H_a: at least one $\beta_j \neq 0$. With f = 17.31 and *P*-value = 0.000, so we reject H_0 at any reasonable significance level and conclude that the model is useful.

b. Use the *t* test information associated with x_3 to test H_0: $\beta_3 = 0$ vs. H_a: $\beta_3 \neq 0$. With $t = 3.96$ and *P*-value = .002 < .05, we reject H_0 at the .05 level and conclude that the interaction term should be retained.

c. The predicted value of y when $x_1 = 3$ and $x_2 = 6$ is $\hat{y}$ = 17.279 – 6.368(3) – 3.658(6) + 1.7067(3)(6) = 6.946. With error df = 11, $t_{.025,11}$ = 2.201, and the CI is 6.946 ± 2.201(.555) = (5.73, 8.17).

d. Our point prediction remains the same, but the SE is now $\sqrt{s^2 + s_{\hat{Y}}^2} = \sqrt{1.72225^2 + .555^2}$ = 1.809. The resulting 95% PI is 6.946 ± 2.201(1.809) = (2.97, 10.93).

50. Using the notation in the model, we wish to test H_0: $\beta_3 = \beta_4 = 0$ vs. H_a: $\beta_3 \neq 0$ or $\beta_4 \neq 0$. The full model has $k = 5$ and SSE_k = 28.947, while the reduced model (i.e., when H_0 is true) has $l = 3$ predictors and, from the Minitab output, SSE_l = 32.627. Thus, the test statistic value is $f = \frac{(32.627 - 28.947)/(5-3)}{28.947/[15-(5+1)]}$ = 0.572. This is a very small test statistic value; in particular, at df = (2, 9), 0.572 < 3.01 ⇒ *P*-value > .10, and so we fail to reject H_0 at any reasonable significance level. With x_1, x_2, and the interaction term already in the model, the two quadratic terms add no statistically significant predictive ability. The quadratic terms should be removed from the model.

51.

a. Associated with x_3 = drilling depth are the test statistic $t = 0.30$ and P-value = .777, so we certainly do not reject H_0: $\beta_3 = 0$ at any reasonably significance level. Thus, we should remove x_3 from the model.

b. To test H_0: $\beta_1 = \beta_2 = 0$ vs. H_a: at least one $\beta \neq 0$, use R^2: $f = \dfrac{R^2/k}{(1-R^2)/(n-k-1)} = \dfrac{.836/2}{(1-.836)/(9-2-1)}$ = 15.29; at df = (2, 6), 10.92 < 15.29 < 27.00 ⇒ the P-value is between .001 and .01. (Software gives .004.) In particular, P-value ≤ .05 ⇒ reject H_0 at the $\alpha = .05$ level: the model based on x_1 and x_2 is useful in predicting y.

c. With error df = 6, $t_{.025,6} = 2.447$, and from the Minitab output we can construct a 95% CI for β_1: –0.006767 ± 2.447(0.002055) = (–0.01180, –0.00174). Hence, after adjusting for feed rate (x_2), we are 95% confident that the true change in mean surface roughness associated with a 1rpm increase in spindle speed is between –.01180 µm and –.00174 µm.

d. The point estimate is $\hat{y} = 0.365 – 0.006767(400) + 45.67(.125) = 3.367$. With the standard error provided, the 95% CI for μ_Y is 3.367 ± 2.447(.180) = (2.93, 3.81).

e. A normal probability plot of the e^* values is quite straight, supporting the assumption of normally distributed errors. Also, plots of the e^* values against x_1 and x_2 show no discernible pattern, supporting the assumptions of linearity and equal variance. Together, these validate the regression model.

52.

a. The complete 2nd order model obviously provides a better fit, so there is a need to account for interaction between the three predictors.

b. A 95% CI for μ_Y when $x_1 = x_2 = 30$ and $x_3 = 10$ is $.66573 \pm 2.120(.01785) = (.6279, .7036)$.

53. Some possible questions might be:

(1) Is this model useful in predicting deposition of poly-aromatic hydrocarbons? A test of model utility gives us an $F = 84.39$, with a P-value of 0.000. Thus, the model is useful.

(2) Is x_1 a significant predictor of y in the presence of x_2? A test of H_0: $\beta_1 = 0$ v. H_a: $\beta_1 \neq 0$ gives us a t = 6.98 with a P-value of 0.000, so this predictor is significant.

(3) A similar question, and solution for testing x_2 as a predictor yields a similar conclusion: with a P-value of 0.046, we would accept this predictor as significant if our significance level were anything larger than 0.046.

54.

a. The variable “supplier” has three categories, so we need two indicator variables to code “supplier,” such as

$$x_2 = \begin{cases} 1 & \text{supplier 1} \\ 0 & \text{otherwise} \end{cases} \qquad x_3 = \begin{cases} 1 & \text{supplier 2} \\ 0 & \text{otherwise} \end{cases}$$

Similarly, the variable “lubrication” has three categories, so we need two more indicator variables, such as

$$x_4 = \begin{cases} 1 & \text{lubricant \#1} \\ 0 & \text{otherwise} \end{cases} \qquad x_5 = \begin{cases} 1 & \text{lubricant \#2} \\ 0 & \text{otherwise} \end{cases}$$

b. This is a model utility test. The hypotheses are H_0: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$ versus H_a: at least one $\beta_i \neq 0$. From the output, the *F*-statistic is $f = 20.67$ with a *P*-value of .000. Thus, we strongly reject H_0 and conclude that at least one of the explanatory variables is a significant predictor of springback.

c. First, find $\hat{y}$ for those settings: $\hat{y} = 21.5322 - 0.0033680(1000) - 1.7181(1) - 1.4840(0) - 0.3036(0) + 0.8931(0) = 21.5322 - 0.0033680(1000) - 1.7181 = 16.4461$. The error df is 30, so a 95% PI for a new value at these settings is $\hat{y} \pm t_{.025,30}\sqrt{s^2 + s_{\hat{Y}}^2} = 16.4461 \pm 2.042\sqrt{(1.18413)^2 + (.524)^2} = (13.80, 19.09)$.

d. The coefficient of determination in the absence of the lubrication indicators is $R^2 = 1 - \dfrac{\text{SSE}}{\text{SST}} = 1 - \dfrac{48.426}{186.980} = .741$ or 74.1%. That's a negligible drop in R^2, so we suspect keeping the indicator variables for lubrication regimen is not worthwhile.

More formally, we can test H_0: $\beta_4 = \beta_5 = 0$ versus H_a: $\beta_4 \neq 0$ or $\beta_5 \neq 0$. The "partial *F* test" statistic is $f = \dfrac{(\text{SSE}_l - \text{SSE}_k)/(k-l)}{\text{SSE}_k/[n-(k+1)]} = \dfrac{(48.426 - 42.065)/(5-3)}{42.065/30} = 2.27$. This test statistic is less than $F_{.10,2,30} = 2.49$, so the *P*-value is > .10 and we fail to reject H_0 at the .10 level. The data does not suggest that lubrication regimen needs to be included so long as BHP and supplier are retained in the model.

e. R^2 has certainly increased, but that will always happen with more predictors. Let's test the null hypothesis that the interaction terms are not statistically significant contributors to the model. The larger model contributes 4 additional variables: x_1x_2, x_1x_3, x_1x_4, x_1x_5. So, the larger model has 30 – 4 = 26 error df, and the "partial *F* test" statistic is $f = \dfrac{(\text{SSE}_l - \text{SSE}_k)/\Delta\text{df}}{\text{SSE}_k/[\text{error df}]} = \dfrac{(42.065 - 28.216)/4}{28.216/26} = 3.19 > F_{.05,4,26} = 2.74$. Therefore, the *P*-value is less than .05 and we should reject H_0 at the .05 level and conclude that the interaction terms, as a group, do contribute significantly to the regression model.

Section 13.5

55.

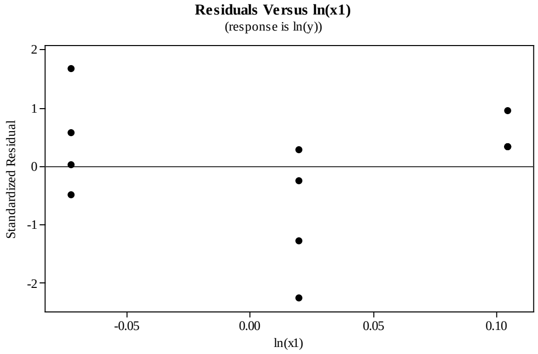
a. To test H_0: $\beta_1 = \beta_2 = \beta_3 = 0$ vs. H_a: at least one $\beta \neq 0$, use R^2:
$f = \dfrac{R^2/k}{(1-R^2)/(n-k-1)} = \dfrac{.706/3}{(1-.706)/(12-3-1)} = 6.40$. At df = (3, 8), $4.06 < 6.40 < 7.59 \Rightarrow$ the *P*-value is between .05 and .01. In particular, *P*-value < .05 $\Rightarrow$ reject H_0 at the .05 level. We conclude that the given model is statistically useful for predicting tool productivity.

b. No: the large *P*-value (.510) associated with $\ln(x_3)$ implies that we should not reject H_0: $\beta_3 = 0$, and hence we need not retain $\ln(x_3)$ in the model that already includes $\ln(x_1)$.

c. Part of the Minitab output from regression $\ln(y)$ on $\ln(x_1)$ appears below. The estimated regression equation is $\ln(y) = 3.55 + 0.844 \ln(x_1)$. As for utility, $t = 4.69$ and *P*-value = .001 imply that we should reject H_0: $\beta_1 = 0$ — the stated model is useful.

```
The regression equation is
ln(y) = 3.55 + 0.844 ln(x1)

Predictor     Coef  SE Coef       T      P
Constant   3.55493  0.01336  266.06  0.000
ln(x1)      0.8439   0.1799    4.69  0.001
```

d. The residual plot shows pronounced curvature, rather than "random scatter." This suggests that the functional form of the relationship might not be correctly modeled — that is, ln(y) might have a <u>non</u>-linear relationship with ln(x_1). [Obviously, one should investigate this further, rather than blindly continuing with the given model!]



e. First, for the model utility test of ln(x_1) and ln^2(x_1) as predictors, we again rely on R^2:

$$f = \frac{R^2/k}{(1-R^2)/(n-k-1)} = \frac{.819/2}{(1-.819)/(12-2-1)}$$ = 20.36. Since this is greater than $F_{.001,2,9}$ = 16.39, the P-value is < .001 and we strongly reject the null hypothesis of no model utility (i.e., the utility of this model is confirmed). Notice also the P-value associated with ln^2(x_1) is .031, indicating that this "quadratic" term adds to the model.

Next, notice that when x_1 = 1, ln(x_1) = 0 [and ln^2(x_1) = 0^2 = 0], so we're really looking at the information associated with the intercept. Using that plus the critical value $t_{.025,9}$ = 2.262, a 95% PI for the response, ln(Y), when x_1 = 1 is 3.5189 ± 2.262 $\sqrt{.0361358^2 + .0178^2}$ = (3.4277, 3.6099). Lastly, to create a 95% PI for Y itself, exponentiate the endpoints: at the 95% prediction level, a new value of Y when x_1 = 1 will fall in the interval ($e^{3.4277}$, $e^{3.6099}$) = (30.81, 36.97).

56.

a. To test H_0: $\beta_1 = ... = \beta_5 = 0$ versus H_a: at least one $\beta \neq 0$, use R^2:

$$f = \frac{R^2/k}{(1-R^2)/(n-k-1)} = \frac{.769/5}{(1-.769)/(20-5-1)}$$ = 9.32; at df = (5, 14), 9.32 > 7.92 ⇒ the P-value is < .001, so H_0 is rejected. Wood specific gravity appears to be linearly related to at least one of the five carriers (i.e., this set of predictors).

b. For the full model, $R_a^2 = \frac{19(.769)-5}{14} = .687$, while for the reduced model, $R_a^2 = \frac{19(.769)-4}{15} = .707$.

c. From **a**, $\text{SSE}_k = (1-.769)(.0196610) = .004542$, and $\text{SSE}_l = (1-.654)(.0196610) = .006803$, so $f = \frac{.002261/3}{.004542/14} = 2.32$. At df = (3, 14), 2.32 < 2.52 ⇒ P-value > .10 ⇒ we fail to reject the null hypothesis that $\beta_1 = \beta_2 = \beta_4 = 0$ at the .05 level. These three variables can be removed from the model.

d. $x_3' = \dfrac{x_3 - 52.540}{5.4447} = -.4665$ and $x_5' = \dfrac{x_5 - 89.195}{3.6660} = .2196$, so $\hat{y} = .5255 - (.0236)(-.4665) + (.0097)(.2196) = .5386$.

e. Error df = 20 – (2 + 1) = 17 for the two-variable model, $t_{.025,17}$ = 2.110, and so the desired CI is $-.0236 \pm 2.110(.0046) = (-.0333, -.0139)$.

f. $y = .5255 - .0236\left(\dfrac{x_3 - 52.540}{5.4447}\right) + .0097\left(\dfrac{x_5 - 89.195}{3.6660}\right)$, so $\hat{\beta}_3$ for the unstandardized model is $\dfrac{-.0236}{5.447} = -.004334$. The estimated sd of the unstandardized $\hat{\beta}_3$ is $\dfrac{.0046}{5.447} = -.000845$.

g. $\hat{y}$ = .532 and $\sqrt{s^2 + s_{\hat{\beta}_0 + \hat{\beta}_3 x_3' + \hat{\beta}_5 x_5'}} = .02058$, so the PI is $.532 \pm (2.110)(.02058) = .532 \pm .043 = (.489, .575)$.

57.

k	R^2	R_a^2	$C_k = \dfrac{SSE_k}{s^2} + 2(k+1) - n$
1	.676	.647	138.2
2	.979	.975	2.7
3	.9819	.976	3.2
4	.9824		4

where s^2 = 5.9825

a. Clearly the model with k = 2 is recommended on all counts.

b. No. Forward selection would let x_4 enter first and would not delete it at the next stage.

58.

a. $R^2 = 1 - \dfrac{\text{SSE}}{\text{SST}} = 1 - \dfrac{10.5513}{30.4395}$ = .653 or 65.3%, while adjusted R^2 = $1 - \dfrac{\text{MSE}}{\text{MST}}$ = $1 - \dfrac{10.5513/24}{30.4395/28}$ = .596 or 59.6%. Yes, the model appears to be useful.

b. The null hypothesis is that none of the 10 second-order terms is statistically significant. The "partial F test" statistic is $f = \dfrac{(\text{SSE}_l - \text{SSE}_k)/(k-l)}{\text{SSE}_k/[n-(k+1)]} = \dfrac{(10.5513 - 1.0108)/10}{1.0108/14}$ = 13.21 > $F_{.01,10,14}$. Hence, the P-value is less than .01 and we strongly reject H_0 and conclude that at least one of the second-order terms is a statistically significant predictor of protein yield.

c. We want to compare the "full" model with 14 predictors in (b) to a "reduced" model with 5 fewer predictors (x_1, x_1^2, x_1x_2, x_1x_3, x_1x_4) . As in (b), we have $f = \dfrac{(1.1887 - 1.0108)/4}{1.0108/14}$ = 0.62 < $F_{.10,4,14}$. Hence, the P-value is > .10 and we fail to reject H_0 at any reasonable significance level; therefore, it indeed appears that the five predictors involving x_1 could all be removed.

d. The "best" models seem to be the 7-, 8-, 9-, and 10-variable models. All of these models have high adjusted R^2 values, low Mallows' C_p values, and low s values compared to the other models. The 6-

variable model is notably worse than the 7-variable model; the 11-variable model is "on the cusp," in that its properties are slightly worse than the 10-variable model, but only slightly so.

59.

a. The choice of a "best" model seems reasonably clear–cut. The model with 4 variables including all but the summerwood fiber variable would seem best. R^2 is as large as any of the models, including the 5-variable model. R^2 adjusted is at its maximum and CP is at its minimum. As a second choice, one might consider the model with $k = 3$ which excludes the summerwood fiber and springwood % variables.

b. Backwards Stepping:

Step 1: A model with all 5 variables is fit; the smallest t-ratio is $t = .12$, associated with variable x_2 (summerwood fiber %). Since $t = .12 < 2$, the variable x_2 was eliminated.
Step 2: A model with all variables except x_2 was fit. Variable x_4 (springwood light absorption) has the smallest t-ratio ($t = –1.76$), whose magnitude is smaller than 2. Therefore, x_4 is the next variable to be eliminated.
Step 3: A model with variables x_3 and x_5 is fit. Both t-ratios have magnitudes that exceed 2, so both variables are kept and the backwards stepping procedure stops at this step. The final model identified by the backwards stepping method is the one containing x_3 and x_5.

Forward Stepping:

Step 1: After fitting all five 1-variable models, the model with x_3 had the t-ratio with the largest magnitude ($t = –4.82$). Because the absolute value of this t-ratio exceeds 2, x_3 was the first variable to enter the model.
Step 2: All four 2-variable models that include x_3 were fit. That is, the models $\{x_3, x_1\}$, $\{x_3, x_2\}$, $\{x_3, x_4\}$, $\{x_3, x_5\}$ were all fit. Of all 4 models, the t-ratio 2.12 (for variable x_5) was largest in absolute value. Because this t-ratio exceeds 2, x_5 is the next variable to enter the model.
Step 3: (not printed): All possible 3-variable models involving x_3 and x_5 and another predictor. None of the t-ratios for the added variables has absolute values that exceed 2, so no more variables are added. There is no need to print anything in this case, so the results of these tests are not shown.

Note: Both the forwards and backwards stepping methods arrived at the same final model, $\{x_3, x_5\}$, *in this problem. This often happens, but not always. There are cases when the different stepwise methods will arrive at slightly different collections of predictor variables.*

60.

a. To have a global Type I error rate no more than $\alpha = .1$, we should use a .05 significance level for each of the two individual test (since .1/2 = .05). For x_1, P-value = .0604 > .05, so we would fail to reject H_0: $\beta_1 = 0$. For x_2, P-value = .0319 < .05, so we would reject H_0: $\beta_2 = 0$. That is, using 5% individual significance levels, we would remove x_1 from the model and retain x_2 in the model.

b. These are the estimated odds ratios. For x_1, $e^{2.774} \approx 16$ means that a one-unit increase in the pillar height-to-width ratio is associated with an estimated 16-fold increase in the odds of stability. For x_2, $e^{5.668} \approx 289$ means that a one-unit increase in the pillar strength-to-stress ratio is associated with an estimated 289-fold increase in the odds of stability. [*Note:* If these odds ratios seem comically large, it's because a one-unit increase in these variables, especially x_2, isn't that realistic. It might make more sense to ask what, for example, a 1/2-unit or 0.1-unit increase in the variables does to the estimated odds of stability.]

61. If multicollinearity were present, at least one of the four R^2 values would be very close to 1, which is not the case. Therefore, we conclude that multicollinearity is not a problem in this data.

62. Looking at the h_{ii} column and using $2(k + 1)/n = 8/19 = .421$ for the criteria, three observations appear to have large influence. With h_{ii} values of .712933, .516298, and .513214, observations 14, 15, 16, correspond to response (y) values 22.8, 41.8, and 48.6.

63. Before removing any observations, we should investigate their source (e.g., were measurements on that observation misread?) <u>and</u> their impact on the regression. To begin, Observation #7 deviates significantly from the pattern of the rest of the data (standardized residual = –2.62); if there's concern the PAH deposition was not measured properly, we might consider removing that point to improve the overall fit. If the observation was <u>not</u> mis–recorded, we should <u>not</u> remove the point.

We should also investigate Observation #6: Minitab gives $h_{66} = .846 > 3(2+1)/17$, indicating this observation has very high leverage. However, the standardized residual for #6 is not large, suggesting that it follows the regression pattern specified by the other observations. Its "influence" only comes from having a comparatively large x_1 value.

64.

a. Use $2(k + 1)/n = 2(2 + 1)/10 = .6$: since $h_{44} > .6$, data point #4 is potentially influential (more accurately, it has high leverage, meaning its x-values are unusual).

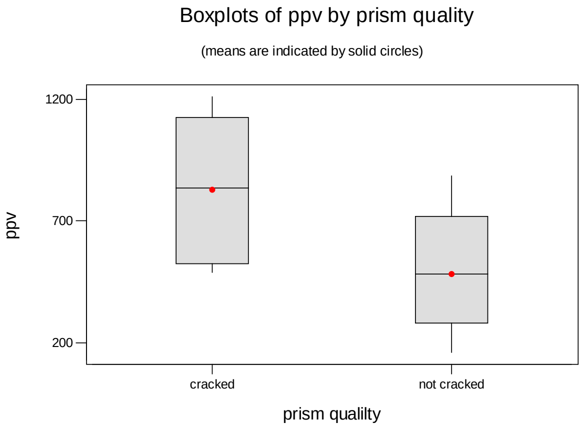
b. The standardized residual larger than 2 suggests that point #2 has an unusually large residual (it's more than 2 standard deviations away from the regression surface). So, at the very least #2 is an outlier with respect to response (discharge amount). Next, let's consider the standardized changes in slope:
$\hat{\beta}_0 : \frac{1.5652-1.8982}{.7328} = -0.45$; $\hat{\beta}_1 : \frac{.9450-1.025}{.1528} = -0.52$; $\hat{\beta}_2 : \frac{.1815-.3085}{.1752} = -0.72$
These are all very small standardized changes: none of the coefficients changes more than 1 standard deviation when point #2 is deleted. Thus, point #2 might be an outlier, but it is not especially influential.

c. Let's repeat the process for point #4:
$\hat{\beta}_0 : \frac{1.5652-1.4592}{.7328} = +0.14$; $\hat{\beta}_1 : \frac{.9450-.9850}{.1528} = -0.26$; $\hat{\beta}_2 : \frac{.1815-.1515}{.1752} = +0.17$
These standardized changes are even smaller than those for point #2. Thus, although h_{44} is large, indicating a potential high influence of point #4 on the fit, the actual influence does not appear to be great.

Supplementary Exercises

65.

a.



A two–sample t confidence interval, generated by Minitab:

```
Two sample T for ppv

prism qu     N      Mean     StDev   SE Mean
cracked     12       827       295        85
not cracke  18       483       234        55

95% CI for mu (cracked   ) – mu (not cracke): ( 132,  557)
```

b. The simple linear regression results in a significant model, r^2 is .577, but we have an extreme observation, with std resid = –4.11. Minitab output is below. Also run, but not included here was a model with an indicator for cracked/ not cracked, and for a model with the indicator and an interaction term. Neither improved the fit significantly.

```
The regression equation is
ratio = 1.00 –0.000018 ppv

Predictor        Coef       StDev          T        P
Constant      1.00161     0.00204     491.18    0.000
ppv       –0.00001827  0.00000295      –6.19    0.000

S = 0.004892    R–Sq = 57.7%     R–Sq(adj) = 56.2%

Analysis of Variance

Source            DF          SS          MS         F        P
Regression         1  0.00091571  0.00091571     38.26    0.000
Residual Error    28  0.00067016  0.00002393
Total             29  0.00158587

Unusual Observations
Obs        ppv      ratio         Fit   StDev Fit    Residual    St Resid
 29       1144   0.962000    0.980704    0.001786   –0.018704       –4.11R

R denotes an observation with a large standardized residual
```

66.

a. Our goals are to achieve a large adjusted R^2, small Mallows' C_p, and small residual standard deviation. By those criteria, the best two models appear to be the top 2-variable model (x_3 and x_5) and the top 3-variable model (x_2, x_3, and x_5).

b. Yes: with $f = 121.74$ and P-value ≈ 0, we strongly reject H_0: $\beta_3 = \beta_5 = 0$ and conclude that the model using x_3 and x_5 as predictors is statistically useful.

c. No: the variable t-test statistics are –4.20 (P-value ≈ 0) for x_3 and 15.31 (P-value ≈ 0) for x_5. Therefore, each of these two variables is highly individually significant.

d. With error df = 114 – (2 + 1) = 111 and $t_{.025,111} \approx 1.98$, the Minitab output provides the following CIs:
for β_3: $-0.00004639 \pm 1.98(0.00001104) = (-0.000068, -0.000024)$
for β_5: $0.73710 \pm 1.98(0.04813) = (0.64173, 0.83247)$
We are 95% confident that a 1-gram increase in dye weight is associated with a decrease in pre-dye pH between .000024 and .000068, after adjusting for after-dye pH.
We are 95% confident that a 1-unit increase in after-dye pH is associated with an increase in pre-dye pH between .64173 and .83247, after adjusting for dye weight.

e. The point estimate is $\hat{y} = 0.9402 - 0.00004639(1000) + 0.73710(6) = 5.31637$. Using the same t-value as in **d** and the standard error provided, a 95% CI for $\mu_{\hat{Y}}$ is $5.31637 \pm 1.98(.0336) = (5.250, 5.383)$.

67.

a. After accounting for all the other variables in the regression, we would expect the VO_2max to decrease by .0996, on average for each one-minute increase in the one-mile walk time.

b. After accounting for all the other variables in the regression, we expect males to have a VO_2max that is .6566 L/min higher than females, on average.

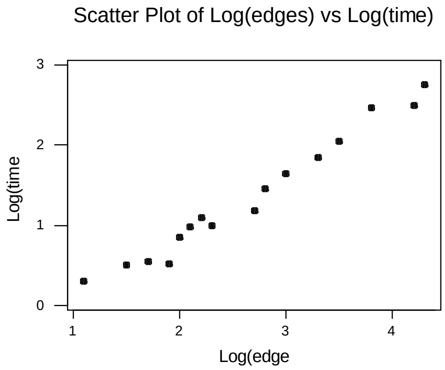
c. $\hat{y} = 3.5959 + .6566(1) + .0096(170) - .0996(11) - .0880(140) = 3.67$. The residual is $\hat{y} = (3.15 - 3.67) = -.52$.

d. $R^2 = 1 - \dfrac{SSE}{SST} = 1 - \dfrac{30.1033}{102.3922} = .706$, or 70.6% of the observed variation in VO_2max can be attributed to the model relationship.

e. To test H_0: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ vs. H_a: at least one $\beta \neq 0$, use R^2:
$f = \dfrac{R^2 / k}{(1-R^2)/(n-k-1)} = \dfrac{.706/4}{(1-.706)/(20-4-1)} = 9.005$. At df = (4, 15), $9.005 > 8.25 \Rightarrow$ the P-value is less than .05, so H_0 is rejected. It appears that the model specifies a useful relationship between VO_2max and at least one of the other predictors.

68.

a. Yes, the scatter plot of the two transformed variables appears quite linear, and thus suggests a linear relationship between the two.

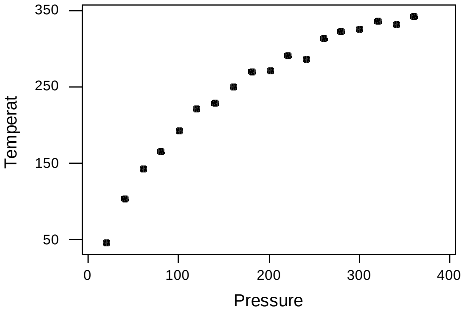


b. Letting *y* denote the variable *time*, the regression model for the variables y' and x' is $\log_{10}(y) = y' = \alpha + \beta x' + \varepsilon'$. Solving for *y* gives $y = 10^{\alpha + \beta \log(x) + \varepsilon'} = \left(10^{\alpha}\right)\left(x^{\beta}\right)10^{\varepsilon'} = \gamma_0 x^{\gamma_1} \cdot \varepsilon$; i.e., the model is $y = \gamma_0 x^{\gamma_1} \cdot \varepsilon$ where $\gamma_0 = 10^{\alpha}$ and $\gamma_1 = \beta$.

c. Using the transformed variables y' and x', the necessary sums of squares are $S_{x'y'} = 68.640 - \frac{(42.4)(21.69)}{16} = 11.1615$ and $S_{x'x'} = 126.34 - \frac{(42.4)^2}{16} = 13.98$. Therefore $\hat{\beta}_1 = \frac{S_{x'y'}}{S_{x'x'}} = \frac{11.1615}{13.98} = .79839$ and $\hat{\beta}_0 = \frac{21.69}{16} - (.79839)\left(\frac{42.4}{16}\right) = -.76011$. The estimate of γ_1 is $\hat{\gamma}_1 = .7984$ and $\gamma_0 = 10^{\alpha} = 10^{-.76011} = .1737$. The estimated power function model is then $y = .1737x^{.7984}$. For *x* = 300, the predicted value of *y* is $\hat{y} = .1737(300)^{.7984} = 16.502$, or about 16.5 seconds.

69.

a. Based on a scatter plot (below), a simple linear regression model would not be appropriate. Because of the slight, but obvious curvature, a quadratic model would probably be more appropriate.



b. Using a quadratic model, a Minitab generated regression equation is $\hat{y} = 35.423 + 1.7191x - .0024753x^2$, and a point estimate of temperature when pressure is 200 is $\hat{y} = 280.23$. Minitab will also generate a 95% prediction interval of (256.25, 304.22). That is, we are confident that when pressure is 200 psi, a single value of temperature will be between 256.25 and 304.22°F.

70.

a. For the model excluding the interaction term, $R^2 = 1 - \frac{5.18}{8.55} = .394$, or 39.4% of the observed variation in lift/drag ratio can be explained by the model without the interaction accounted for. However, including the interaction term increases the amount of variation in lift/drag ratio that can be explained by the model to $R^2 = 1 - \frac{3.07}{8.55} = .641$, or 64.1%.

b. Without interaction, we are testing H_0: $\beta_1 = \beta_2 = 0$ vs. H_a: β_1 or $\beta_2 \neq 0$. The calculated test statistic is $f = \frac{.394/2}{(1-.394)/(9-2-1)}$ = 1.95; at df = (2, 6), 1.95 yields a *P*-value > .10 > .05, so we fail to reject H_0. This model is not useful. With the interaction term, we are testing H_0: $\beta_1 = \beta_2 = \beta_3 = 0$ vs. H_a: at least one of these three β's is $\neq 0$. The new *F*-value is $f = \frac{.641/3}{(1-.641)/(9-3-1)}$ = 2.98; at df = (3, 5), this still gives a *P*-value > .10, and so we still fail to reject the null hypothesis. Even with the interaction term, there is not enough of a significant relationship between lift/drag ratio and the two predictor variables to make the model useful (a bit of a surprise!).

71.

a. Using Minitab to generate the first order regression model, we test the model utility (to see if any of the predictors are useful), and with *f* = 21.03and a *P*-value of .000, we determine that at least one of the predictors is useful in predicting palladium content. Looking at the individual predictors, the *P*-value associated with the pH predictor has value .169, which would indicate that this predictor is unimportant in the presence of the others.

b. We wish to test H_0: $\beta_1 = ... = \beta_{20} = 0$ vs. H_a: at least one $\beta \neq 0$. With calculated statistic *f* = 6.29 and *P*-value .002, this model is also useful at any reasonable significance level.

c. Testing $H_0 : \beta_6 = ... = \beta_{20} = 0$ vs. H_a: at least one of the listed β's $\neq 0$, the test statistic is $f = \frac{(716.10 - 290.27)/(20-5)}{290.27(32-20-1)} = 1.07 < F_{.05,15,11} = 2.72$. Thus, *P*-value > .05, so we fail to reject H_0 and conclude that all the quadratic and interaction terms should not be included in the model. They do not add enough information to make this model significantly better than the simple first order model.

d. Partial output from Minitab follows, which shows all predictors as significant at level .05:

```
The regression equation is
pdconc = – 305 + 0.405 niconc + 69.3 pH – 0.161 temp + 0.993 currdens
           + 0.355 pallcont – 4.14 pHsq

Predictor        Coef       StDev          T        P
Constant      -304.85       93.98      -3.24    0.003
niconc        0.40484     0.09432       4.29    0.000
pH              69.27       21.96       3.15    0.004
temp         -0.16134     0.07055      -2.29    0.031
currdens       0.9929      0.3570       2.78    0.010
pallcont      0.35460     0.03381      10.49    0.000
pHsq           -4.138       1.293      -3.20    0.004
```

72.

a. $R^2 = 1 - \frac{SSE}{SST} = 1 - \frac{.80017}{16.18555} = .9506$, or 95.06% of the observed variation in weld strength can be attributed to the given model.

b. The complete second order model consists of nine predictors and nine corresponding coefficients. The hypotheses are H_0: $\beta_1 = ... = \beta_9 = 0$ vs. H_a: at least one of the β's. The test statistic value is $f = \frac{.9506/9}{(1-.9506)/(37-9-1)}$ = 57.68, with a corresponding *P*-value of ≈ 0. We strongly reject the null hypothesis. The complete second order model is useful.

c. To test H_0: $\beta_7 = 0$ vs. H_a: $\beta_7 \neq 0$ (the coefficient corresponding to the wc*wt predictor), use the test statistic value $t = \sqrt{f} = \sqrt{2.32} = 1.52$. With df = 27, *P*-value ≈ 2(.073) = .146 from Table A.8. With such a large *P*-value, this predictor is not useful in the presence of all the others, so it can be eliminated.

d. The point estimate is $\hat{y} = 3.352 + .098(10) + .222(12) + .297(6) - .0102(10^2) - .037(6^2)$ $+.0128(10)(12) = 7.962$. With $t_{.025,27}$ = 2.052, the 95% PI would be $7.962 \pm 2.052(.0750) = 7.962 \pm .154 = (7.808, 8.116)$. Because of the narrowness of the interval, it appears that the value of strength can be accurately predicted.

73.

a. We wish to test H_0: $\beta_1 = \beta_2 = 0$ vs. H_a: either β_1 or $\beta_2 \neq 0$. With $R^2 = 1 - \frac{.29}{202.88} = .9986$, the test statistic is $f = \frac{R^2/k}{(1-R^2)/(n-k-1)} = \frac{.9986/2}{(1-.9986)/(8-2-1)}$ = 1783, where k = 2 for the quadratic model. Clearly the *P*-value at df = (2,5) is effectively zero, so we strongly reject H_0 and conclude that the quadratic model is clearly useful.

b. The relevant hypotheses are H_a: $\beta_2 = 0$ vs. H_a: $\beta_2 \neq 0$. The test statistic value is $t = \frac{\hat{\beta}_2}{s_{\hat{\beta}_2}} = \frac{-.00163141 - 0}{.00003391} = -48.1$; at 5 df, the *P*-value is $2P(T \geq |-48.1|) \approx 0$. Therefore, H_0 is rejected. The quadratic predictor should be retained.

c. No. R^2 is extremely high for the quadratic model, so the marginal benefit of including the cubic predictor would be essentially nil – and a scatter plot doesn't show the type of curvature associated with a cubic model.

d. $t_{.025,5} = 2.571$, and $\hat{\beta}_0 + \hat{\beta}_1(100) + \hat{\beta}_2(100)^2 = 21.36$, so the CI is 21.36 ± 2.571(.1141) = 21.36 ± .29 = (21.07,21.65).

e. First, we need to figure out s^2 based on the information we have been given: s^2 = MSE = SSE/df = .29/5 = .058. Then, the 95% PI is $21.36 \pm 2.571\sqrt{.058 + (.1141)^2} = 21.36 \pm 0.685 = (20.675, 22.045)$.

74.

a. Our goals are to achieve large adjusted R^2, small Mallows' C_p and small residual standard deviation. Based on those criteria, the "best" models appear to be the two 3-variable models (x_1, x_3, x_3^2 and x_3, x_1^2, x_3^2) and the top 4-variable model (x_1, x_3, x_3^2, x_1x_3). *Note:* In practice, we might be leery of the two 3-variable models, because negative C_p can sometimes indicate a biased model. Also, we would traditionally reject the second 3-variable model, because we would not include x_1^2 without x_1.

b. With $n = 15$ and $k = 3$ in this case, error df = 15 – 3 – 1 = 11. To test H_0: $\beta_1 = 0$ vs. H_a: $\beta_1 \neq 0$, the test statistic –1.32 is compared to the t distribution with 11 df: P-value = $2P(T \geq |-1.32|) \approx 2(.11) = .22$. With such a large P-value, we fail to reject H_0 and conclude that x_1 is not a statistically significant predictor of tenacity with x_3 and x_3^2 already in the model.

c. Yes to both questions. Although model utility is obvious from the individual P-values listed, we can test H_0: $\beta_1 = \beta_2 = 0$ vs. H_a: either β_1 or $\beta_2 \neq 0$ as follows. $f = \dfrac{.734/2}{(1-.734)/(15-2-1)} = 16.56 > 12.97 = F_{.001,2,12} \Rightarrow$ P-value < .001 $\Rightarrow$ reject H_0 (the model is useful). Next, the variable utility test for the quadratic term yields t = –5.46 and P-value = 0.000, so again we reject the null hypothesis, meaning here that the quadratic term should be retained.

d. Let's construct a 95% PI for Y when $x_3 = 6$. First, a point prediction is $\hat{y} = -24.743 + 14.457(6) - 1.2284(6)^2 = 17.7766$. Next, the prediction SE is $\sqrt{s^2 + s_{\hat{Y}}^2} = \sqrt{.435097^2 + .164^2} = .465$. Finally, with error df = 15 – (2 + 1) = 12, $t_{.025,12} = 2.179$. Putting it all together, a 95% PI for tenacity when the number of draw frame doubling equals 6 is 17.7766 ± 2.179(.465) = (16.76, 18.79).

75.

a. To test H_0: $\beta_1 = \beta_2 = 0$ vs. H_a: either β_1 or $\beta_2 \neq 0$, first find R^2: SST = $\Sigma y^2 - (\Sigma y)^2/n = 264.5 \Rightarrow R^2$ = 1 – SSE/SST = 1 – 26.98/264.5 = .898. Next, $f = \dfrac{.898/2}{(1-.898)/(10-2-1)} = 30.8$, which at df = (2,7) corresponds to a P-value of ≈ 0. Thus, H_0 is rejected at significance level .01 and the quadratic model is judged useful.

b. The hypotheses are H_0: $\beta_2 = 0$ vs. H_a: $\beta_2 \neq 0$. The test statistic value is $t = (-2.3621 - 0)/.3073 = -7.69$, and at 7 df the P-value is $2P(T \geq |-7.69|) \approx 0$. So, H_0 is rejected at level .001. The quadratic predictor should not be eliminated.

c. $x = 1$ here, $\hat{\mu}_{Y \cdot 1} = \hat{\beta}_0 + \hat{\beta}_1(1) + \hat{\beta}_2(1)^2 = 45.96$, and $t_{.025,7} = 1.895$, giving the CI $45.96 \pm (1.895)(1.031) = (44.01, 47.91)$.

76.

a. 80.79

b. Yes, *P*-value = .007 which is less than .01.

c. No, *P*-value = .043 which is less than .05.

d. $.14167 \pm (2.447)(.03301) = (.0609, .2224)$

e. $\hat{\mu}_{y\cdot 9,66} = 6.3067$, using $\alpha = .05$, the interval is $6.3067 \pm (2.447)\sqrt{(.4851)^2 + (.162)^2} = (5.06, 7.56)$

77.

a. The hypotheses are H_0: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ versus H_a: at least one $\beta_i \neq 0$. From the output, the *F*-statistic is $f = 4.06$ with a *P*-value of .029. Thus, at the .05 level we reject H_0 and conclude that at least one of the explanatory variables is a significant predictor of power.

b. Yes, a model with $R^2 = .834$ would appear to be useful. A formal model utility test can be performed: $f = \dfrac{R^2/k}{(1-R^2)/[n-(k+1)]} = \dfrac{.834/3}{(1-.834)/[16-4]}$ = 20.1, which is much greater than $F_{.05,3,12} = 3.49$. Thus, the mode including $\{x_3, x_4, x_3x_4\}$ is useful.

We <u>cannot</u> use an *F* test to compare this model with the first-order model in (a), because neither model is a "subset" of the other. Compare $\{x_1, x_2, x_3, x_4\}$ to $\{x_3, x_4, x_3x_4\}$.

c. The hypotheses are H_0: $\beta_5 = \ldots = \beta_{10} = 0$ versus H_a: at least one of these $\beta_i \neq 0$, where β_5 through β_{10} are the coefficients for the six interaction terms. The "partial *F* test" statistic is $f = \dfrac{(\text{SSE}_l - \text{SSE}_k)/(k-l)}{\text{SSE}_k/[n-(k+1)]} = \dfrac{(R_k^2 - R_l^2)/(k-l)}{(1-R_k^2)/[n-(k+1)]} = \dfrac{(.960-.596)/(10-4)}{(1-.960)/[16-(10+1)]}$ = 7.58, which is greater than $F_{.05,6,5} = 4.95$. Hence, we reject H_0 at the .05 level and conclude that at least one of the interaction terms is a statistically significant predictor of power, in the presence of the first-order terms.

78.

a. After adjusting for the effects of fiber content and hydraulic gradient, a 1 mm increase in fiber length is associated with an estimated .0003020 cm/sec *decrease* in expected seepage velocity.

b. The hypotheses are H_0: $\beta_1 = 0$ versus H_a: $\beta_1 \neq 0$. From the output, the test statistic and *P*-value are $t = -1.63$ and $P = .111$. Since this *P*-value is larger than any reasonable significance level, we fail to reject H_0. In the presence of fiber length and hydraulic gradient, fiber content is not a statistically significant predictor of seepage velocity.

c. Before we begin the hypothesis test, let's calculate $\hat{y}$ for these settings: $\hat{y} = -.005315 - .0004968(25) + .102204(1.2) = .1049$. Let μ_Y denote the true mean seepage velocity at these settings; the hypotheses of interest are H_0: $\mu_Y = .1$ versus H_a: $\mu_Y \neq .1$. The test statistic is $t = \dfrac{.1049 - .1}{.00286}$ = 1.71; at df = 49 – (2 + 1) = 46, the *P*-value is roughly 2(.048) = .096. At the .05 level, we fail to reject H_0; there is not significant evidence that the true mean at these settings differs from .1.

d. The hypotheses are H_0: $\beta_4 = \beta_5 = \beta_6 = 0$ versus H_a: at least one of these $\beta_i \neq 0$, where β_4 through β_6 are the coefficients for the three interaction terms (x_1x_2, x_1x_3, x_2x_3). The "partial F test" statistic is $f = \frac{(\text{SSE}_l - \text{SSE}_k)/(k-l)}{\text{SSE}_k/[n-(k+1)]} = \frac{(.011862 - .003579)/(6-3)}{.003579/[49-(6+1)]} = 16.2 > F_{.01,3,42} = 4.29$. Hence, we reject H_0 and conclude the "full" model, i.e. the model with interaction terms, should be retained over the first-order model.

79. There are obviously several reasonable choices in each case. In **a**, the model with 6 carriers is a defensible choice on all three grounds, as are those with 7 and 8 carriers. The models with 7, 8, or 9 carriers in **b** merit serious consideration. These models merit consideration because R_k^2, MSE_k, and C_k meet the variable selection criteria given in Section 13.5.

80.

a. $f = \frac{.90/15}{(1-.90)/4} = 2.4$, for a P-value > .100 at df = (15,4). Hence, $H_0 : \beta_1 = ... = \beta_{15} = 0$ cannot be rejected. There does not appear to be a useful linear relationship.

b. The high R^2 value resulted from saturating the model with predictors. In general, one would be suspicious of a model yielding a high R^2 value when k is large relative to n.

c. We get a P-value ≤ .05 iff $f \geq F_{.05,15,4} = 5.86$. $\frac{R^2/15}{(1-R^2)/4} \geq 5.86$ iff $\frac{R^2}{1-R^2} \geq 21.975$ iff $R^2 \geq \frac{21.975}{22.975} = .9565$.

81.

a. The relevant hypotheses are $H_0 : \beta_1 = ... = \beta_5 = 0$ vs. H_a: at least one among $\beta_1, ..., \beta_5 \neq 0$. $f = \frac{.827/5}{.173/11} = 106.1 \geq F_{.05,5,111} \approx 2.29$, so P-value < .05. Hence, H_0 is rejected in favor of the conclusion that there is a useful linear relationship between Y and at least one of the predictors.

b. $t_{.05,111} = 1.66$, so the CI is $.041 \pm (1.66)(.016) = .041 \pm .027 = (.014, .068)$. β_1 is the expected change in mortality rate associated with a one-unit increase in the particle reading when the other four predictors are held fixed; we can be 90% confident that $.014 < \beta_1 < .068$.

c. In testing H_0: $\beta_4 = 0$ versus H_a: $\beta_4 \neq 0$, $t = \frac{\hat{\beta}_4 - 0}{s_{\hat{\beta}_4}} = \frac{.047}{.007} = 5.9$, with an associated P-value of ≈ 0. So, H_0 is rejected and this predictor is judged important.

d. $\hat{y} = 19.607 + .041(166) + .071(60) + .001(788) + .041(68) + .687(.95) = 99.514$, and the corresponding residual is 103 – 99.514 = 3.486.

82.

a. The set $x_1, x_3, x_4, x_5, x_6, x_8$ includes both x_1, x_4, x_5, x_8 and x_1, x_3, x_5, x_6, so $R^2_{1,3,4,5,6,8} \geq \max\left(R^2_{1,4,5,8}, R^2_{1,3,5,6}\right) = .723$.

b. $R^2_{1,4} \leq R^2_{1,4,5,8} = .723$, but it is not necessarily ≤ .689 since x_1, x_4 is not a subset of x_1, x_3, x_5, x_6.

83. Taking logs, the regression model is ln(Y) = $\beta_0 + \beta_1$ ln(x_1) + β_2 ln(x_2) + ε', where β_0 = ln(α). Relevant Minitab output appears below.

a. From the output, $\hat{\beta}_0 = 10.8764, \hat{\beta}_1 = -1.2060, \hat{\beta}_2 = -1.3988$. In the original model, solving for α returns $\hat{\alpha} = \exp(\hat{\beta}_0) = e^{10.8764}$ = 52,912.77.

b. From the output, R^2 = 78.2%, so 78.2% of the total variation in ln(wear life) can be explained by a linear regression on ln(speed) and ln(load). From the ANOVA table, a test of H_0: $\beta_1 = \beta_2 = 0$ versus H_a: at least one of these β's ≠ 0 produces f = 42.95 and P-value = 0.000, so we strongly reject H_0 and conclude that the model is useful.

c. Yes: the variability utility t-tests for the two variables have t = –7.05, P = 0.000 and t = –6.01, P = 0.000. These indicate that each variable is highly statistically significant.

d. With ln(50) ≈ 3.912 and ln(5) ≈ 1.609 substituted for the transformed x values, Minitab produced the accompanying output. A 95% PI for ln(Y) at those settings is (2.652, 5.162). Solving for Y itself, the 95% PI of interest is ($e^{2.652}$, $e^{5.162}$) = (14.18, 174.51).

```
The regression equation is
ln(y) = 10.9 - 1.21 ln(x1) - 1.40 ln(x2)

Predictor     Coef  SE Coef      T      P
Constant   10.8764   0.7872  13.82  0.000
ln(x1)     -1.2060   0.1710  -7.05  0.000
ln(x2)     -1.3988   0.2327  -6.01  0.000

S = 0.596553   R-Sq = 78.2%   R-Sq(adj) = 76.3%

Analysis of Variance

Source          DF      SS      MS      F      P
Regression       2  30.568  15.284  42.95  0.000
Residual Error  24   8.541   0.356
Total           26  39.109

Predicted Values for New Observations

New Obs    Fit  SE Fit       95% CI           95% PI
      1  3.907   0.118  (3.663, 4.151)  (2.652, 5.162)
```

CHAPTER 14

Section 14.1

1. For each part, we reject H_0 if the *P*-value is $\le \alpha$, which occurs if and only the calculated χ^2 value is <u>greater</u> than or equal to the value $\chi^2_{\alpha,k-1}$ from Table A.7.

- **a.** Since $12.25 \ge \chi^2_{.05,4} = 9.488$, *P*-value $\le$ .05 and we would reject H_0.

- **b.** Since $8.54 < \chi^2_{.01,3} = 11.344$, *P*-value > .01 and we would fail to reject H_0.

- **c.** Since $4.36 < \chi^2_{.10,2} = 4.605$, *P*-value > .10 and we would fail to reject H_0.

- **d.** Since $10.20 < \chi^2_{.01,5} = 15.085$, *P*-value > .01 we would fail to reject H_0.

2. Let p_1, p_2, p_3, p_4 denote the true proportion of all African American, Asian, Caucasian, and Hispanic characters in commercials (broadcast in the Philadelphia area), respectively. The null hypothesis is H_0: p_1 = .177, p_2 = .032, p_3 = .734, p_4 = .057 (that is, the proportions in commercials match census proportions). The alternative hypothesis is that at least one of these proportions is incorrect.

The sample size is n = 404, so the expected counts under H_0 are 404(.177) = 71.508, 404(.032) = 12.928, 404(.734) = 296.536, and 404(.057) = 23.028. The resulting chi-squared goodness-of-fit statistic is

$$\chi^2 = \frac{(57-71.508)^2}{71.508} + \cdots + \frac{(6-23.028)^2}{23.028} = 19.6.$$

At df = 4 – 1 = 3, the *P*-value is less than .001 (since 19.6 > 16.26). Hence, we strongly reject H_0 and conclude that at least one of the racial proportions in commercials is <u>not</u> a match to the census proportions.

3. The uniform hypothesis implies that $p_{i0} = \frac{1}{8} = .125$ for i = 1, …, 8, so the null hypothesis is $H_0 : p_{10} = p_{20} = ... = p_{80} = .125$. Each expected count is np_{i0} = 120(.125) = 15, so

$$\chi^2 = \left[\frac{(12-15)^2}{15} + ... + \frac{(10-15)^2}{15}\right] = 4.80$$

. At df = 8 – 1 = 7, 4.80 < 12.10 ⇒ *P*-value > .10 ⇒ we fail to reject H_0. There is not enough evidence to disprove the claim.

4. Let $p_i = P$(first significant digit is i). We wish to test H_0: $p_i = \log_{10}((i + 1)/i)$ for $i = 1, 2, \ldots, 9$. The observed values, probabilities, expected values, and chi-squared contributions appear in the accompanying table.

Obs	342	180	164	155	86	65	54	47	56
p_{i0}	.3010	.1760	.1249	.0969	.0791	.0669	.0579	.0511	.0457
Exp	345.88	202.32	143.55	111.35	90.979	76.921	66.632	58.774	52.575
$(O–E)^2/E$	.0436	2.4642	2.9119	17.111	.2725	1.8477	2.3950	2.3587	.2231

The χ^2 statistic is .0436 + … + .2231 = 29.6282. With df = 9 – 1 = 8, our χ^2 value of 29.6282 exceeds 26.12, so the P-value < .001 and we strongly reject H_0. There is significant evidence to suggest that the first significant digits deviate from Benford's law. (In particular, the number of observed values with lead digit = 4 is far greater than expected under this law.)

5. The observed values, expected values, and corresponding χ^2 terms are :

Obs	4	15	23	25	38	21	32	14	10	8
Exp	6.67	13.33	20	26.67	33.33	33.33	26.67	20	13.33	6.67
χ^2	1.069	.209	.450	.105	.654	.163	1.065	1.800	.832	.265

χ^2 = 1.069 + … + .265 = 6.612. With df = 10 – 1 = 9, 6.612 < 14.68 ⇒ P-value > .10 ⇒ we cannot reject H_0. There is no significant evidence that the data is not consistent with the previously determined proportions.

6. Under the assumption that each medal pair has probability 1/9, the probabilities of the categories {match, one off, two off} are 3/9, 4/9, and 2/9, respectively. Let p_1, p_2, p_3 denote the probabilities of these three categories, so the hypotheses are H_0: $p_1 = 3/9$, $p_2 = 4/9$, $p_3 = 2/9$ versus H_a: these are not correct.

The sample size is n = 216, so the expected counts are 72, 96, and 48, for a test statistic of
$\chi^2 = \dfrac{(69-72)^2}{72} + \dfrac{(102-96)^2}{96} + \dfrac{(45-48)^2}{48}$ = 0.6875. At df = 3 – 1 = 2, the P-value is much greater than .10 since 0.6875 is much less than 6.25.

Therefore, we fail to reject H_0. The data is consistent with the hypothesis that expert and consumer ratings are independent and equally likely to be Gold, Silver, or Bronze.

7. We test $H_0 : p_1 = p_2 = p_3 = p_4 = .25$ vs. H_a: at least one proportion ≠ .25, and df = 3.

Cell	1	2	3	4
Observed	328	334	372	327
Expected	340.25	340.25	340.25	34.025
χ^2 term	.4410	.1148	2.9627	.5160

χ^2 = 4.0345, and with 3 df, P-value > .10, so we fail to reject H_0. The data fails to indicate a seasonal relationship with incidence of violent crime.

8. $H_0: p_1 = \frac{15}{365}, p_2 = \frac{46}{365}, p_3 = \frac{120}{365}, p_4 = \frac{184}{365},$ versus H_a: at least one proportion is not a stated in H_0.

Cell	1	2	3	4
Observed	11	24	69	96
Expected	8.22	25.21	65.75	100.82
χ^2 term	.9402	.0581	.1606	.2304

$\chi^2 = 1.3893$, df = 4 – 1 = 3 ⇒ P-value > .10 and so H_0 is not rejected. The data does not indicate a relationship between patients' admission date and birthday.

9.

a. Denoting the 5 intervals by $[0, c_1), [c_1, c_2), \ldots, [c_4, \infty)$, we wish c_1 for which $.2 = P(0 \le X \le c_1) = \int_0^{c_1} e^{-x}dx = 1 - e^{-c_1}$, so $c_1 = -\ln(.8) = .2231$. Then $.2 = P(c_1 \le X \le c_2) \Rightarrow .4 = P(0 \le X_1 \le c_2) = 1 - e^{-c_2}$, so $c_2 = -\ln(.6) = .5108$. Similarly, $c_3 = -\ln(.4) =$ .0163 and $c_4 = -\ln(.2) = 1.6094$. The resulting intervals are [0, .2231), [.2231, .5108), [.5108, .9163), [.9163, 1.6094), and $[1.6094, \infty)$.

b. Each expected cell count is 40(.2) = 8, and the observed cell counts are 6, 8, 10, 7, and 9, so $\chi^2 = \left[\frac{(6-8)^2}{8} + \ldots + \frac{(9-8)^2}{8}\right] = 1.25$. Because 1.25 < $\chi^2_{.10,4} = 7.779$, even at level .10 H_0 cannot be rejected; the data is quite consistent with the specified exponential distribution.

10.

a.
$$\chi^2 = \sum_{i=1}^{k}\frac{(n_i - np_{i0})^2}{np_{i0}} = \sum_i \frac{N_i^2 - 2np_{i0}N_i + n^2p_{i0}^2}{np_{i0}} = \sum_i \frac{N_i^2}{np_{i0}} - 2\sum_i N_i + n\sum_i p_{i0}$$
$$= \sum_i \frac{N_i^2}{np_{i0}} - 2n + n(1) = \sum_i \frac{N_i^2}{np_{i0}} - n$$

This formula involves only one subtraction, and that's at the end of the calculation, so it is analogous to the shortcut formula for $s^{2.}$

b. $\chi^2 = \frac{k}{n}\sum_i N_i^2 - n$. For the pigeon data, k = 8, n = 120, and $\Sigma N_i^2 = 1872$, so

$$\chi^2 = \frac{8(1872)}{120} - 120 = 124.8 - 120 = 4.8$$ as before.

11.

a. The six intervals must be symmetric about 0, so denote the 4th, 5th and 6th intervals by $[0, a)$, $[a, b)$, $[b, \infty)$. The constant a must be such that $\Phi(a) = .6667\left(\frac{1}{2} + \frac{1}{6}\right)$, which from Table A.3 gives $a \approx .43$. Similarly, $\Phi(b) = .8333$ implies $b \approx .97$, so the six intervals are $(-\infty, -.97)$, $[-.97, -.43)$, $[-.43, 0)$, $[0, .43)$, $[.43, .97)$, and $[.97, \infty)$.

b. The six intervals are symmetric about the mean of .5. From **a**, the fourth interval should extend from the mean to .43 standard deviations above the mean, i.e., from .5 to .5 + .43(.002), which gives [.5, .50086). Thus the third interval is [.5 – .00086, .5) = [.49914, .5). Similarly, the upper endpoint of the fifth interval is .5 + .97(.002) = .50194, and the lower endpoint of the second interval is .5 – .00194 = .49806. The resulting intervals are (–∞, .49806), [.49806, .49914), [.49914, .5), [.5, .50086), [.50086, .50194), and [.50194, ∞).

c. Each expected count is 45(1/6) = 7.5, and the observed counts are 13, 6, 6, 8, 7, and 5, so $\chi^2 = 5.53$. With 5 df, the *P*-value > .10, so we would fail to reject H_0 at any of the usual levels of significance. There is no significant evidence to suggest that the bolt diameters are not normally distributed with $\mu = .5$ and $\sigma = .002$.

Section 14.2

12.

a. Let θ denote the probability of a male (as opposed to female) birth under the binomial model. The four cell probabilities (corresponding to x = 0, 1, 2, 3) are $\pi_1(\theta) = (1-\theta)^3$, $\pi_2(\theta) = 3\theta(1-\theta)^2$, $\pi_3(\theta) = 3\theta^2(1-\theta)$, and $\pi_4(\theta) = \theta^3$. The likelihood is $3^{n_2+n_3} \cdot (1-\theta)^{3n_1+2n_2+n_3} \cdot \theta^{n_2+2n_3+3n_4}$.
Forming the log likelihood, taking the derivative with respect to θ, equating to 0, and solving yields $\hat{\theta} = \frac{n_2+2n_3+3n_4}{3n} = \frac{66+128+48}{480} = .504$. The estimated expected counts are $160(1-.504)^3 = 19.52$, $480(.504)(.496)^2 = 59.52$, 60.48, and 20.48, so

$$\chi^2 = \left[\frac{(14-19.52)^2}{19.52} + ... + \frac{(16-20.48)^2}{20.48}\right] = 1.56 + .71 + .20 + .98 = 3.45.$$

The number of degrees of freedom for the test is 4 – 1 – 1 = 2. Because 3.45 < 4.60 ⇒ *P*-value > .10 ⇒ H_0 of a binomial distribution is not rejected. The binomial model is judged to be plausible.

b. Now $\hat{\theta} = \frac{53}{150} = .353$ and the estimated expected counts are 13.54, 22.17, 12.09, and 2.20. The last estimated expected count is much less than 5, so the chi-squared test based on 2 df should not be used.

13. According to the stated model, the three cell probabilities are $(1 – p)^2$, $2p(1 – p)$, and p^2, so we wish the value of p which maximizes $(1-p)^{2n_1}\left[2p(1-p)\right]^{n_2} p^{2n_3}$. Proceeding as in Example 14.6 gives $\hat{p} = \frac{n_2+2n_3}{2n} = \frac{234}{2776} = .0843$. The estimated expected cell counts are then $n(1-\hat{p})^2 = 1163.85$, $n\left[2\hat{p}(1-\hat{p})\right]^2 = 214.29$, $n\hat{p}^2 = 9.86$. This gives

$$\chi^2 = \left[\frac{(1212-1163.85)^2}{1163.85} + \frac{(118-214.29)^2}{214.29} + \frac{(58-9.86)^2}{9.86}\right] = 280.3.$$

With df = 4 – 1 – 1 = 2, 280.3 > 13.81 ⇒ *P*-value < .001 ⇒ H_0 is soundly rejected. The stated model is strongly contradicted by the data.

14.

a. We wish to maximize $p^{\Sigma x_i - n}(1-p)^n$, or equivalently $(\Sigma x_i - n)\ln p + n\ln(1-p)$. Equating *d/dp* to 0 yields $\dfrac{(\Sigma x_i - n)}{p} = \dfrac{n}{(1-p)}$, whence $p = \dfrac{(\Sigma x_i - n)}{\Sigma x_i}$. For the given data, $\Sigma x_i = (1)(1) + (2)(31) + ... + (12)(1) = 363$, so $\hat{p} = \dfrac{(363-130)}{363} = .642$, and $\hat{q} = .358$.

b. Each estimated expected cell count is $\hat{p}$ times the previous count, giving $n\hat{q} = 130(.358) = 46.54$, $n\hat{q}\hat{p} = 46.54(.642) = 29.88$, 19.18, 12.31, 17.91, 5.08, 3.26, … . Grouping all values ≥ 7 into a single category gives 7 cells with estimated expected counts 46.54, 29.88, 19.18, 12.31, 7.91, 5.08 (sum = 120.9), and 130 – 120.9 = 9.1. The corresponding observed counts are 48, 31, 20, 9, 6, 5, and 11, giving χ^2 = 1.87. With *k* = 7 and *m* = 1 (*p* was estimated), df = 7 – 1 – 1 = 5 and 1.87 < 9.23 ⇒ the *P*-value > .10 ⇒ we don't reject H_0.

15. The part of the likelihood involving θ is $\left[(1-\theta)^4\right]^{n_1} \cdot \left[\theta(1-\theta)^3\right]^{n_2} \cdot \left[\theta^2(1-\theta)^2\right]^{n_3} \cdot \left[\theta^3(1-\theta)\right]^{n_4} \cdot \left[\theta^4\right]^{n_5} = \theta^{n_2+2n_3+3n_4+4n_5}(1-\theta)^{4n_1+3n_2+2n_3+n_4} = \theta^{233}(1-\theta)^{367}$, so the log-likelihood is $233\ln\theta + 367\ln(1-\theta)$. Differentiating and equating to 0 yields $\hat{\theta} = \dfrac{233}{600} = .3883$, and $(1-\hat{\theta}) = .6117$ [note that the exponent on θ is simply the total # of successes (defectives here) in the *n* = 4(150) = 600 trials]. Substituting this $\hat{\theta}$ into the formula for p_i yields estimated cell probabilities .1400, .3555, .3385, .1433, and .0227. Multiplication by 150 yields the estimated expected cell counts are 21.00, 53.33, 50.78, 21.50, and 3.41. the last estimated expected cell count is less than 5, so we combine the last two categories into a single one (≥ 3 defectives), yielding estimated counts 21.00, 53.33, 50.78, 24.91, observed counts 26, 51, 47, 26, and $\chi^2 = 1.62$. With df = 4 – 1 – 1 = 2, since $1.62 < \chi^2_{.10,2} = 4.605$, the *P*-value > .10, and we do not reject H_0. The data suggests that the stated binomial distribution is plausible.

16.

a. First, we need the maximum likelihood estimate for the unknown mean parameter μ: $\hat{\mu} = \overline{x} = \dfrac{(0)(1627)+(1)(421)+...+(15)(2)}{1627+421+...+2} = \dfrac{2636}{2637} \approx 1$. So, estimated cell probabilities are computed from the Poisson pmf $\hat{p}(x) = \dfrac{e^{-\hat{\mu}}\hat{\mu}^x}{x!} = \dfrac{e^{-1}}{x!}$. In order to keep all expected counts sufficiently large, we collapse the values 5-15 into a "≥ 5" category. For that category, the estimated probability is 1 – [sum of the other probabilities].

x	**0**	**1**	**2**	**3**	**4**	**≥ 5**
Obs.	1627	421	219	130	107	133
Prob.	.367879	.367879	.183940	.061313	.015328	.003661
Exp.	970.098	970.098	485.049	161.683	40.421	9.651

From these, the test statistic is χ^2 = … = 2594, which is extremely large at df = 6 – 1 – 1 = 4 (or any df, really). Hence, we very strongly reject H_0 and conclude the data are not at all consistent with a Poisson distribution.

b. The specified probabilities do not sum to 1, so we'll assume the hypothesized probability for the "12+" category is the remainder (which is .0047). The calculated test statistic from the specified probabilities is $\chi^2 = \ldots = 28.12$; at df = 13 – 1 – 2 = 10 (we lose 2 df estimating the gamma parameters), the corresponding *P*-value is between .005 and .001. Hence, we reject *H*0 again and conclude that this more complicated model also does not fit the data well.

17. $\hat{\mu} = \overline{x} = \frac{(0)(6)+(1)(24)+(2)(42)+...+(8)(6)+(9)(2)}{300} = \frac{1163}{300} = 3.88$, so the estimated cell probabilities are computed from $\hat{p} = e^{-3.88}\frac{(3.88)^x}{x!}$.

x	0	1	2	3	4	5	6	7	≥ 8
***np*(*x*)**	6.2	24.0	46.6	60.3	58.5	45.4	29.4	16.3	13.3
obs	6	24	42	59	62	44	41	14	8

This gives $\chi^2 = 7.789$. At df = 9 – 1 – 1 = 7, 7.789 < 12.01 ⇒ *P*-value > .10 ⇒ we fail to reject H_0. The Poisson model does provide a good fit.

18. $\hat{p}_1 = P(X < .100) = P\left(Z < \frac{.100-.173}{.066}\right) = \Phi(-1.11) = .1335$,
$\hat{p}_2 = P(.100 \le X \le .150) = P(-1.11 \le Z \le -.35) = .2297$, $\hat{p}_3 = P(-.35 \le Z \le .41) = .2959$,
$\hat{p}_4 = P(.41 \le Z \le 1.17) = .2199$, and $\hat{p}_5 = .1210$. The estimated expected counts are then (multiply $\hat{p}_i$ by $n = 83$) 11.08, 19.07, 24.56, 18.25, and 10.04, from which $\chi^2 = 1.67$. With df = 5 – 1 – 2 = 2, the resulting *P*-value is > .10 and the hypothesis of normality cannot be rejected.

19. With $A = 2n_1 + n_4 + n_5$, $B = 2n_2 + n_4 + n_6$, and $C = 2n_3 + n_5 + n_6$, the likelihood is proportional to $\theta_1^A\theta_2^B(1-\theta_1-\theta_2)^C$. Taking the natural log and equating both $\frac{\partial}{\partial\theta_1}$ and $\frac{\partial}{\partial\theta_2}$ to zero gives $\frac{A}{\theta_1} = \frac{C}{1-\theta_1-\theta_2}$ and $\frac{B}{\theta_2} = \frac{C}{1-\theta_1-\theta_2}$, whence $\theta_2 = \frac{B\theta_1}{A}$. Substituting this into the first equation gives $\theta_1 = \frac{A}{A+B+C}$, and then $\theta_2 = \frac{B}{A+B+C}$. Thus $\hat{\theta}_1 = \frac{2n_1+n_4+n_5}{2n}$, $\hat{\theta}_2 = \frac{2n_2+n_4+n_6}{2n}$, and $(1-\hat{\theta}_1-\hat{\theta}_2) = \frac{2n_3+n_5+n_6}{2n}$.
Substituting the observed n_i's yields $\hat{\theta}_1 = \frac{2(49)+20+53}{400} = .4275$, $\hat{\theta}_2 = \frac{110}{400} = .2750$, and $(1-\hat{\theta}_1-\hat{\theta}_2) = .2975$, from which $\hat{p}_1 = (.4275)^2 = .183$, $\hat{p}_2 = .076$, $\hat{p}_3 = .089$, $\hat{p}_4 = 2(.4275)(.275) = .235$, $\hat{p}_5 = .254$, $\hat{p}_6 = .164$.

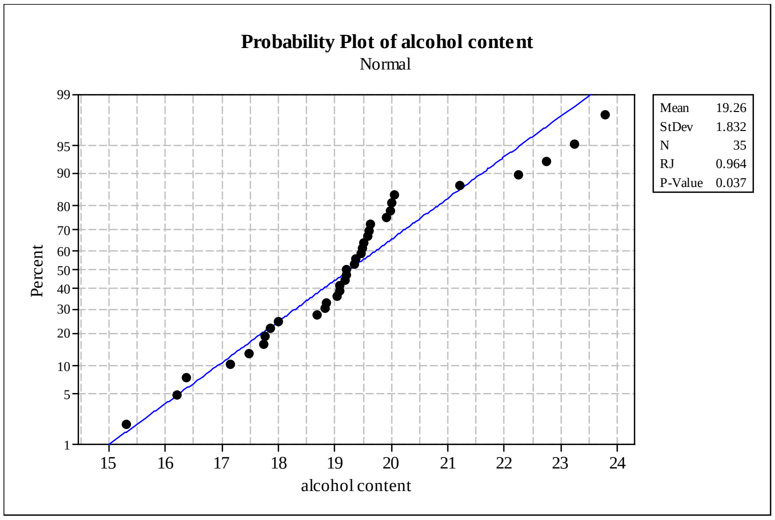
Category	**1**	**2**	**3**	**4**	**5**	**6**
np	36.6	15.2	17.8	47.0	50.8	32.8
observed	49	26	14	20	53	38

This gives $\chi^2 = 29.1$. At df = 6 – 1 – 2 = 3, this gives a *P*-value less than .001. Hence, we reject H_0.

20. The pattern of points in the plot appear to deviate from a straight line, a conclusion that is also supported by the small *P*-value (< .01) of the Ryan-Joiner test. Therefore, it is implausible that this data came from a normal population. In particular, the observation 116.7 is a clear outlier. It would be dangerous to use the one-sample *t* interval as a basis for inference.

21. The Ryan-Joiner test *P*-value is larger than .10, so we conclude that the null hypothesis of normality cannot be rejected. This data could reasonably have come from a normal population. This means that it would be legitimate to use a one-sample *t* test to test hypotheses about the true average ratio.

22. Minitab performs the Ryan-Joiner test automatically, as seen in the accompanying plot. The Ryan-Joiner correlation test statistic is $r = 0.964$ with a *P*-value of 0.037. Thus, we reject the null hypothesis that the alcohol content distribution is normal at the .05 level.



23. Minitab gives $r = .967$, though the hand calculated value may be slightly different because when there are ties among the $x_{(i)}$'s, Minitab uses the same y_i for each $x_{(i)}$ in a group of tied values. $c_{10} = .9707$, and $c_{.05} = 9639$, so .05 < *P*-value < .10. At the 5% significance level, one would have to consider population normality plausible.

Section 14.3

24. H_0: TV watching and physical fitness are independent of each other
H_a: the two variables are not independent
df = (4 – 1)(2 – 1) = 3; with $\alpha = .05$,
Computed $\chi^2 = 6.161 < 7.815 \Rightarrow P\text{-value} > .05$
Fail to reject H_0. The data fail to indicate a significant association between daily TV viewing habits and physical fitness.

25. The hypotheses are H_0: there is no association between extent of binge drinking and age group vs. H_a: there is an association between extent of binge drinking and age group. With the aid of software, the calculated test statistic value is $\chi^2 = 212.907$. With all expected counts well above 5, we can compare this value to a chi-squared distribution with df = (4 – 1)(3 – 1) = 6. The resulting *P*-value is ≈ 0, and so we strongly reject H_0 at any reasonable level (including .01). There is strong evidence of an association between age and binge drinking for college-age males. In particular, comparing the observed and expected counts shows that younger men tend to binge drink more than expected if H_0 were true.

26. Let p_i = the true incidence rate of salmonella for the *i*th type of chicken ($i = 1, 2, 3$). Then the hypotheses are H_0: $p_1 = p_2 = p_3$ vs. H_a: these three true rates are not all equal. To apply the chi-squared test, form a 3x2 table with salmonella contamination classified as yes or no:

	Contaminated	**Not**	*Total*
1.	27	33	60
2.	32	28	60
3.	45	75	120

With the aid of software, the calculated test statistic is $\chi^2 = 4.174$. All expected counts are much larger than 5, so we compare this value to a chi-squared distribution with df = (3 – 1)(2 – 1) = 2. From Table A.11, the *P*-value is > .10, so we fail to reject H_0. We do not have statistically significant evidence sufficient to conclude that the true incidence rates of salmonella differ for these three types of chicken.

27. With $i = 1$ identified with men and $i = 2$ identified with women, and $j = 1, 2, 3$ denoting the 3 categories L>R, L=R, L<R, we wish to test H_0: $p_{1j} = p_{2j}$ for $j = 1, 2, 3$ vs. H_a: $p_{1j} \neq p_{2j}$ for at least one *j*. The estimated cell counts for men are 17.95, 8.82, and 13.23 and for women are 39.05, 19.18, 28.77, resulting in a test statistic of $\chi^2 = 44.98$. With (2 – 1)(3 – 1) = 2 degrees of freedom, the *P*-value is < .001, which strongly suggests that H_0 should be rejected.

28. For the population of Cal Poly students, the hypotheses are H_0: cell phone service provider and email service provider are independent, versus H_a: cell phone service provider and email service provider are not independent.

The accompanying Minitab output shows that all expected counts are ≥ 5, so a chi-squared test is appropriate. The test statistic value is $\chi^2 = 1.507$ at df = (3–1)(3–1) = 4, with an associated *P*-value of .825.

Chi-Square Test: ATT, Verizon, Other

```
Expected counts are printed below observed counts
Chi-Square contributions are printed below expected counts

         ATT  Verizon  Other  Total
    1     28       17      7     52
       25.26    18.42   8.32
       0.298    0.110  0.209

    2     31       26     10     67
       32.54    23.74  10.72
       0.073    0.216  0.048

    3     26       19     11     56
       27.20    19.84   8.96
       0.053    0.036  0.464

Total     85       62     28    175

Chi-Sq = 1.507, DF = 4, P-Value = 0.825
```

At any reasonable significance level, we would fail to reject H_0. There is no evidence to suggest a relationship between cell phone and email providers for Cal Poly students.

29.

a. The null hypothesis is H_0: $p_{1j} = p_{2j} = p_{3j}$ for $j = 1, 2, 3, 4$, where p_{ij} is the proportion of the ith population (natural scientists, social scientists, non-academics with graduate degrees) whose degree of spirituality falls into the jth category (very, moderate, slightly, not at all).

From the accompanying Minitab output, the test statistic value is $\chi^2 = 213.212$ with df = (3–1)(4–1) = 6, with an associated P-value of 0.000. Hence, we strongly reject H_0. These three populations are not homogeneous with respect to their degree of spirituality.

Chi-Square Test: Very, Moderate, Slightly, Not At All

```
Expected counts are printed below observed counts
Chi-Square contributions are printed below expected counts

         Very  Moderate  Slightly  Not At All  Total
    1      56       162       198         211    627
        78.60    195.25    183.16      170.00
        6.497     5.662     1.203       9.889

    2      56       223       243         239    761
        95.39    236.98    222.30      206.33
       16.269     0.824     1.928       5.173

    3     109       164        74          28    375
        47.01    116.78    109.54      101.67
       81.752    19.098    11.533      53.384

Total     221       549       515         478   1763

Chi-Sq = 213.212, DF = 6, P-Value = 0.000
```

b. We're now testing H_0: $p_{1j} = p_{2j}$ for j = 1, 2, 3, 4 under the same notation. The accompanying Minitab output shows $\chi^2 = 3.091$ with df = (2–1)(4–1) = 3 and an associated P-value of 0.378. Since this is larger than any reasonable significance level, we fail to reject H_0. The data provides no statistically significant evidence that the populations of social and natural scientists differ with respect to degree of spirituality.

Chi-Square Test: Very, Moderate, Slightly, Not At All

```
Expected counts are printed below observed counts
Chi-Square contributions are printed below expected counts

        Very  Moderate  Slightly  Not At All  Total
    1     56       162       198         211    627
       50.59    173.92    199.21      203.28
       0.578     0.816     0.007       0.293

    2     56       223       243         239    761
       61.41    211.08    241.79      246.72
       0.476     0.673     0.006       0.242

Total    112       385       441         450   1388

Chi-Sq = 3.091, DF = 3, P-Value = 0.378
```

30. H_0: the design configurations are homogeneous with respect to type of failure vs. H_a: the design configurations are not homogeneous with respect to type of failure.

$\hat{E}_{ij}$	1	2	3	4	
1	16.11	43.58	18.00	12.32	90
2	7.16	19.37	8.00	5.47	40
3	10.74	29.05	12.00	8.21	60
	34	92	38	26	190

$\chi^2 = \frac{(20-16.11)^2}{16.11} + \ldots + \frac{(5-8.21)^2}{8.21} = 13.253$. With 6 df,
$\chi^2_{.05,6} = 12.592 < 13.253 < \chi^2_{.025,6} = 14.440$, so .025 < P-value < .05. Since the P -value is < .05, we reject H_0. (If a smaller significance level were chosen, a different conclusion would be reached.) Configuration appears to have an effect on type of failure.

31.

a. The accompanying table shows the proportions of male and female smokers in the sample who began smoking at the ages specified. (The male proportions were calculated by dividing the counts by the total of 96; for females, we divided by 93.) The patterns of the proportions seems to be different, suggesting there does exist an association between gender and age at first smoking.

		Gender	
		Male	**Female**
Age	**<16**	0.26	0.11
	16–17	0.25	0.34
	18–20	0.29	0.18
	>20	0.20	0.37

b. The hypotheses, in words, are H_0: gender and age at first smoking are independent, versus H_a: gender and age at first smoking are associated. The accompanying Minitab output provides a test statistic value of $\chi^2 = 14.462$ at df = (2–1)(4–1) = 3, with an associated *P*-value of 0.002. Hence, we would reject H_0 at both the .05 and .01 levels. We have evidence to suggest an association between gender and age at first smoking.

Chi-Square Test: Male, Female

```
Expected counts are printed below observed counts
Chi-Square contributions are printed below expected counts

        Male  Female  Total
    1     25      10     35
       17.78   17.22
       2.934   3.029

    2     24      32     56
       28.44   27.56
       0.694   0.717

    3     28      17     45
       22.86   22.14
       1.157   1.194

    4     19      34     53
       26.92   26.08
       2.330   2.406

Total     96      93    189

Chi-Sq = 14.462, DF = 3, P-Value = 0.002
```

32. Let p_i = the true eclosion rate under the *i*th duration ($i = 1$ for 0 days, …, $i = 7$ for 15 days). We wish to test the hypotheses H_0: $p_1 = \ldots = p_7$ vs. H_a: these seven true rates are not all equal. To apply the chi-squared test, form a 2x7 table with eclosion classified as yes or no:

	0	**1**	**2**	**3**	**5**	**10**	**15**
Eclosion	101	38	44	40	38	35	7
No	19	3	3	4	8	7	3
Total	120	41	47	44	46	42	10

The expected count for the bottom right cell is 10(47)/350 < 5, but all other expected counts are ≥ 5. So, we will proceed with the chi-squared test. With the aid of software, the calculated test statistic is $\chi^2 = 7.996$; at df = (7 – 1)(2 – 1) = 6, the *P*-value is > .100 [software gives *P*-value = .238]. Thus, we fail to reject H_0; the evidence suggests that it is at least plausible that eclosion rates do not depend exposure duration.

33. $\chi^2 = \Sigma\Sigma\dfrac{\left(N_{ij} - \hat{E}_{ij}\right)^2}{\hat{E}_{ij}} = \Sigma\Sigma\dfrac{N_{ij}^2 - 2\hat{E}_{ij}N_{ij} + \hat{E}_{ij}^2}{\hat{E}_{ij}} = \Sigma\Sigma\dfrac{N_{ij}^2}{\hat{E}_{ij}} - 2\Sigma\Sigma N_{ij} + \Sigma\Sigma\hat{E}_{ij}$, but $\Sigma\Sigma\hat{E}_{ij} = \Sigma\Sigma N_{ij} = n$, so

$\chi^2 = \Sigma\Sigma\dfrac{N_{ij}^2}{\hat{E}_{ij}} - n$. This formula is computationally efficient because there is only one subtraction to be performed, which can be done as the last step in the calculation.

34. Under the null hypothesis, we compute estimated cell counts by

$$\hat{e}_{ijk} = n\hat{p}_{ijk} = n\hat{p}_{i..}\hat{p}_{.j.}\hat{p}_{..k} = n\frac{n_{i..}}{n}\frac{n_{.j.}}{n}\frac{n_{..k}}{n} = \frac{n_{i..}n_{.j.}n_{..k}}{n^2}$$

This is a 3x3x3 situation, so there are 27 cells. Only the total sample size, *n*, is fixed in advance of the experiment, so there are 26 freely determined cell counts. We must estimate $p_{..1}$, $p_{..2}$, $p_{..3}$, $p_{.1.}$, $p_{.2.}$, $p_{.3.}$, $p_{1..}$, $p_{2..}$, and $p_{3..}$, but $\Sigma p_{i..} = \Sigma p_{.j.} = \Sigma p_{..k} = 1$, so only 6 independent parameters are estimated. The rule for degrees of freedom now gives df = 26 – 6 = 20.
In general, the degrees of freedom for independence in an *I*x*J*x*K* array equals
$(IJK - 1) - [(I - 1) + (J - 1) + (K - 1)] = IJK - (I+J+K) + 2$.

35. With p_{ij} denoting the common value of p_{ij1}, p_{ij2}, p_{ij3}, and p_{ij4} under H_0, $\hat{p}_{ij} = \dfrac{n_{ij.}}{n}$ and $\hat{E}_{ijk} = \dfrac{n_k n_{ij.}}{n}$, where

$n_{ij.} = \displaystyle\sum_{k=1}^{4} n_{ijk}$ and $n = \displaystyle\sum_{k=1}^{4} n_k$. With four different tables (one for each region), there are 4(9 – 1) = 32 freely determined cell counts. Under H_0, the nine parameters $p_{11}, \ldots, p_{33}$ must be estimated, but $\Sigma\Sigma p_{ij} = 1$, so only 8 independent parameters are estimated, giving χ^2 df = 32 – 8 = 24. *Note*: this is really a test of homogeneity for 4 strata, each with 3x3=9 categories. Hence, df = (4 – 1)(9 – 1) = 24.

36.

a.

Observed

13	19	28	60
7	11	22	40
20	30	50	100

Estimated Expected

12	18	30
8	12	20

$\chi^2 = \frac{(13-12)^2}{12} + ... + \frac{(22-20)^2}{20} = .6806$. Because $.6806 < \chi^2_{.10,2} = 4.605$, the P-value is greater than .10 and H_0 is not rejected.

b. Each observation count here is 10 times what it was in **a**, and the same is true of the estimated expected counts, so now $\chi^2 = 6.806$, and H_0 is rejected. With the much larger sample size, the departure from what is expected under H_0, the independence hypothesis, is statistically significant – it cannot be explained just by random variation.

c. The observed counts are .13n, .19n, .28n, .07n, .11n, .22n, whereas the estimated expected $\frac{(.60n)(.20n)}{n} =$ .12n, .18n, .30n, .08n, .12n, .20n, yielding $\chi^2 = .006806n$. H_0 will be rejected at level .10 iff $.006806n \geq 4.605$, i.e., iff $n \geq 676.6$, so the minimum n = 677.

Supplementary Exercises

37. There are 3 categories here – firstborn, middleborn, (2nd or 3rd born), and lastborn. With p_1, p_2, and p_3 denoting the category probabilities, we wish to test H_0: $p_1 = .25$, $p_2 = .50$, $p_3 = .25$ because $p_2 = P$(2nd or 3rd born) = .25 + .25 = .50. The expected counts are (31)(.25) = 7.75, (31)(.50) = 15.5, and 7.75, so $\chi^2 = \frac{(12-7.75)^2}{7.75} + \frac{(11-15.5)^2}{15.5} + \frac{(8-7.75)^2}{7.75} = 3.65$. At df = 3 – 1 = 2, 3.65 < 5.992 $\Rightarrow$ P-value > .05 $\Rightarrow$ H_0 is not rejected. The hypothesis of equiprobable birth order appears plausible.

38. Let p_1 = the true proportion of births under a new moon, p_2 = the true proportion of births under a waxing crescent, and so on through p_8 (there are 8 “phase” categories). If births occur without regard to moon phases, then the proportion of births under a new moon should simply be the proportion of all days that had a new moon; here, that’s 24/699 (since there were 699 days studied). Making the analogous calculations for the other 7 categories, the null hypothesis that births occur without regard to moon phases is

$$H_0: p_1 = 24/699,\ p_2 = 152/699,\ \ldots,\ p_8 = 152/699$$

The alternative hypothesis is that at least one of these proportions is incorrect. The accompanying Minitab output shows a test statistic value of $\chi^2 = 6.31108$ with df = 8 – 1 = 7 and an associated P-value of 0.504. Hence, we fail to reject H_0 at any reasonably significance level. On the basis of the data, there is no reason to believe that births are affected by phases of the moon.

Chi-Square Goodness-of-Fit Test for Observed Counts in Variable: Births

```
                           Test                Contribution
Category  Observed  Proportion  Expected      to Chi-Sq
1             7680    0.034335    7649.2        0.12373
2            48442    0.217454   48445.2        0.00021
3             7579    0.034335    7649.2        0.64491
4            47814    0.213162   47489.0        2.22410
5             7711    0.034335    7649.2        0.49871
6            47595    0.214592   47807.7        0.94654
7             7733    0.034335    7649.2        0.91727
8            48230    0.217454   48445.2        0.95561


     N  DF   Chi-Sq  P-Value
222784   7  6.31108    0.504
```

39.

a. For that top-left cell, the estimated expected count is (row total)(column total)/(grand total) = (189)(406)/(852) = 90.06. Next, the chi-squared contribution is $(O - E)^2/E = (83 - 90.06)^2/90.06 =$ 0.554.

b. No: From the software output, the P-value is .023 > .01. Hence, we fail to reject the null hypothesis of "no association" at the .01 level. We have insufficient evidence to conclude that an association exists between cognitive state and drug status. [*Note*: We would arrive at a different conclusion for $\alpha = .05$.]

40.

a. H_0: The proportion of Late Game Leader Wins is the same for all four sports; H_a: The proportion of Late Game Leader Wins is not the same for all four sports. With 3 df, the computed $\chi^2 = 10.518$, and the P-value < .015 < .05, we reject H_0. There appears to be a relationship between the late-game leader winning and the sport played.

b. Quite possibly: Baseball had many fewer late-game leader losses than expected.

41. The null hypothesis H_0: $p_{ij} = p_{i.}\, p_{.j}$ states that level of parental use and level of student use are independent in the population of interest. The test is based on (3 – 1)(3 – 1) = 4 df.

Estimated expected counts

119.3	57.6	58.1	235
82.8	33.9	40.3	163
23.9	11.5	11.6	47
226	109	110	445

The calculated test statistic value is $\chi^2 = 22.4$; at df = (3 – 1)(3 – 1) = 4, the P-value is < .001, so H_0 should be rejected at any reasonable significance level. Parental and student use level do not appear to be independent.

42. The null hypothesis is H_0: the distribution of the number of concussions is the same for soccer players, non-soccer athletes, and non-athletes; the alternative hypothesis is that H_0 is not true. As the data stands, the conditions for a chi-squared test of homogeneity aren't met: the estimated expected cell count for (non-athletes, ≥ 3 concussions) is $\hat{e} = (53)(15)/240 = 3.31$, which is less than 5. To cope with this, we can collapse the last two columns into one category:

	# of Concussions		
	0	**1**	**≥ 2**
Soccer	45	25	21
N-S Athletes	68	15	13
Non-athletes	45	5	3

The accompanying Minitab output provides a test statistic value of $\chi^2 = 20.604$ with df = (3–1)(3–1) = 4 and an associated *P*-value of 0.000. Hence, we reject H_0 at any reasonable significance level and conclude that the distribution of the number of concussions is not the same for the populations of soccer players, non-soccer athletes, and non-athletes.

Chi-Square Test: 0, 1, 2+

```
Expected counts are printed below observed counts
Chi-Square contributions are printed below expected counts

            0      1     2+  Total
    1      45     25     21     91
        59.91  17.06  14.03
        3.710  3.693  3.464

    2      68     15     13     96
        63.20  18.00  14.80
        0.365  0.500  0.219

    3      45      5      3     53
        34.89   9.94   8.17
        2.928  2.453  3.272

Total     158     45     37    240

Chi-Sq = 20.604, DF = 4, P-Value = 0.000
```

43. This is a test of homogeneity: H_0: $p_{1j} = p_{2j} = p_{3j}$ for $j = 1, 2, 3, 4, 5$. The given SPSS output reports the calculated $\chi^2 = 70.64156$ and accompanying *P*-value (significance) of .0000. We reject H_0 at any significance level. The data strongly supports that there are differences in perception of odors among the three areas.

44. The accompanying table contains both observed and estimated expected counts, the latter in parentheses.

			Age			
Want	127 (131.1)	118 (123.3)	77 (71.7)	61 (55.1)	41 (42.8)	424
Don't	23 (18.9)	23 (17.7)	5 (10.3)	2 (7.9)	8 (6.2)	61
	150	141	82	63	49	485

This gives $\chi^2 = 11.60$; at df = 4, the *P*-value is ~.020. At level .05, the null hypothesis of independence is rejected, though it would not be rejected at the level .01.

45. $(n_1 - np_{10})^2 = (np_{10} - n_1)^2 = (n - n_1 - n(1 - p_{10}))^2 = (n_2 - np_{20})^2$. Therefore

$$\chi^2 = \frac{(n_1 - np_{10})^2}{np_{10}} + \frac{(n_2 - np_{20})^2}{np_{20}} = \frac{(n_1 - np_{10})^2}{n_2}\left(\frac{n}{p_{10}} + \frac{n}{p_{20}}\right)$$

$$= \left(\frac{n_1}{n} - p_{10}\right)^2 \cdot \left(\frac{n}{p_{10}p_{20}}\right) = \frac{(\hat{p}_1 - p_{10})^2}{p_{10}p_{20}/n} = z^2$$

46.

a.

obs	22	10	5	11
exp	13.189	10	7.406	17.405

H_0: probabilities are as specified.
H_a: probabilities are not as specified.

Test Statistic: $\chi^2 = \frac{(22 - 13.189)^2}{13.189} + \frac{(10 - 10)^2}{10} + \frac{(5 - 7.406)^2}{7.406} + \frac{(11 - 17.405)^2}{17.405}$
$= 5.886 + 0 + 0.782 + 2.357 = 9.025$. At df = 4 – 1 = 3, .025 < *P*-value < .030. Therefore, at the .05 level we reject H_0. The model postulated in the exercise is not a good fit.

b.

p_i	0.45883	0.18813	0.11032	0.24272
exp	22.024	9.03	5.295	11.651

$\chi^2 = \frac{(22 - 22.024)^2}{22.024} + \frac{(10 - 9.03)^2}{9.03} + \frac{(5 - 5.295)^2}{5.295} + \frac{(11 - 11.651)^2}{11.651}$
$= .0000262 + .1041971 + .0164353 + .0363746 = .1570332$
With the same rejection region as in a, we do not reject the null hypothesis. This model does provide a good fit.

47.

a. Our hypotheses are H_0: no difference in proportion of concussions among the three groups v. H_a: there is a difference in proportion of concussions among the three groups.

Observed	Concussion	No Concussion	Total
Soccer	45	46	91
Non Soccer	28	68	96
Control	8	45	53
Total	81	159	240

Expected	Concussion	No Concussion	Total
Soccer	30.7125	60.2875	91
Non Soccer	32.4	63.6	96
Control	17.8875	37.1125	53
Total	81	159	240

$$\chi^2 = \frac{(45-30.7125)^2}{30.7125} + \frac{(46-60.2875)^2}{60.2875} + \frac{(28-32.4)^2}{32.4} + \frac{(68-63.6)^2}{63.6}$$

$+\dfrac{(8-17.8875)^2}{17.8875}+\dfrac{(45-37.1125)^2}{37.1125}$ = 19.1842. The df for this test is $(I-1)(J-1) = 2$, so the P-value is less than .001 and we reject H_0. There is a difference in the proportion of concussions based on whether a person plays soccer.

b. The sample correlation of $r = -.220$ indicates a weak negative association between "soccer exposure" and immediate memory recall. We can formally test the hypotheses H_0: $\rho = 0$ vs H_a: $\rho < 0$. The test statistic is $t = \dfrac{r\sqrt{n-2}}{\sqrt{1-r^2}} = \dfrac{-.22\sqrt{89}}{\sqrt{1-.22^2}} = -2.13$. At significance level $\alpha = .01$, we would fail to reject H_0 and conclude that there is no significant evidence of negative association in the population.

c. We will test to see if the average score on a controlled word association test is the same for soccer and non-soccer athletes. H_0: $\mu_1 = \mu_2$ vs H_a: $\mu_1 \neq \mu_2$. Since the two sample standard deviations are very close, we will use a pooled-variance two-sample t test. From Minitab, the test statistic is $t = -0.91$, with an associated P-value of 0.366 at 80 df. We clearly fail to reject H_0 and conclude that there is no statistically significant difference in the average score on the test for the two groups of athletes.

d. Our hypotheses for ANOVA are H_0: all means are equal vs H_a: not all means are equal. The test statistic is $f = \dfrac{MSTr}{MSE}$.

$SSTr = 91(.30-.35)^2 + 96(.49-.35)^2 + 53(.19-.35)^2 = 3.4659$ $MSTr = \dfrac{3.4659}{2} = 1.73295$

$SSE = 90(.67)^2 + 95(.87)^2 + 52(.48)^2 = 124.2873$ and $MSE = \dfrac{124.2873}{237} = .5244$.

Now, $f = \dfrac{1.73295}{.5244} = 3.30$. Using df = (2,200) from Table A.9, the P-value is between .01 and .05. At significance level .05, we reject the null hypothesis. There is sufficient evidence to conclude that there is a difference in the average number of prior non-soccer concussions between the three groups.

48.

a. H_0: $p_0 = p_1 = \ldots = p_9 = .10$ vs H_a: at least one $p_i \neq .10$, with df = 9.

b. H_0: $p_{ij} = .01$ for i and $j = 0,1,2,\ldots,9$ vs H_a: at least one $p_{ij} \neq .01$, with df = 99.

c. For this test, the number of p's in the hypothesis would be 10^5 = 100,000 (the number of possible combinations of 5 digits). Using only the first 100,000 digits in the expansion, the number of non-overlapping groups of 5 is only 20,000. We need a much larger sample size!

d. Based on these P-values, we could conclude that the digits of π behave as though they were randomly generated.

49. According to Benford's law, the probability a lead digit equals x is given by $\log_{10}(1 + 1/x)$ for $x = 1, \ldots, 9$. Let p_i = the proportion of Fibonacci numbers whose lead digit is i ($i = 1, \ldots, 9$). We wish to perform a goodness-of-fit test H_0: $p_i = \log_{10}(1 + 1/i)$ for $i = 1, \ldots, 9$. (The alternative hypothesis is that Benford's formula is incorrect for at least one category.) The table below summarizes the results of the test.

Digit	1	2	3	4	5	6	7	8	9
Obs. #	25	16	11	7	7	5	4	6	4
Exp. #	25.59	14.97	10.62	8.24	6.73	5.69	4.93	4.35	3.89

Expected counts are calculated by $np_i = 85 \log_{10}(1 + 1/i)$. Some of the expected counts are too small, so combine 6 and 7 into one category (obs = 9, exp = 10.62); do the same to 8 and 9 (obs = 10, exp = 8.24).

The resulting chi-squared statistic is $\chi^2 = \dfrac{(25-25.59)^2}{25.59} + \cdots + \dfrac{(10-8.24)^2}{8.24} = 0.92$ at df = 7 – 1 = 6 (since there are 7 categories after the earlier combining). Software provides a *P*-value of .988!

We certainly do not reject H_0 — the lead digits of the Fibonacci sequence are highly consistent with Benford's law.

CHAPTER 15

Section 15.1

1. Refer to Table A.13.

a. With $n = 12$, $P_0(S_+ \geq 56) = .102$.

b. With $n = 12$, $61 < 62 < 64 \Rightarrow P_0(S_+ \geq 62)$ is between .046 and .026.

c. With $n = 12$ and a lower-tailed test, P-value $= P_0(S_+ \geq n(n + 1)/2 - s_+) = P_0(S_+ \geq 12(13)/2 - 20) = P_0(S_+ \geq 58)$. Since $56 < 58 < 60$, the P-value is between .055 and .102.

d. With $n = 14$ and a two-tailed test, P-value $= 2P_0(S_+ \geq \max\{21, 14(15)/2 - 21\}) = 2P_0(S_+ \geq 84) = .025$.

e. With $n = 25$ being "off the chart," use the large-sample approximation:

$$z = \frac{s_+ - n(n+1)/4}{\sqrt{n(n+1)(2n+1)/24}} = \frac{300 - 25(26)/4}{\sqrt{25(26)(51)/24}} = 3.7 \Rightarrow \text{two-tailed } P\text{-value} = 2P(Z \geq 3.7) \approx 0.$$

2. With μ = the population mean expense ratio, our hypotheses are H_0: $\mu = 1$ versus H_a: $\mu > 1$. For each of the 20 values, calculate $x_i - 1$, and then replace each value by the rank of $|x_i - 1|$. For example, the first observation converts to $1.03 - 1 = .03$; since $|.03| = .03$ turns out to be the smallest of the absolute values, its rank is 1.
The test statistic value is $s_+ = 1 + 12 + 2 + 7 + 19 + 20 + 15 + 17 + 16 + 2 + 13 = 124$. (Depending on how you deal with ties, you could also get $s_+ = 125$.) With $n = 20$, since this test statistic value is less than the critical value $c = 140$ associated with the .101 level, P-value $= P_0(S_+ \geq 124) > .101$. Therefore, we fail to reject H_0 at the .10 level (or .05 or .01 level). The data do not provide statistically significant evidence that the population mean expense ratio exceeds 1%.

3. We test H_0: $\mu = 7.39$ vs. H_a: $\mu \neq 7.39$, so a two tailed test is appropriate. The $(x_i - 7.39)$'s are –.37, –.04, –.05, –.22, –.11, .38, –.30, –.17, .06, –.44, .01, –.29, –.07, and –.25, from which the ranks of the three positive differences are 1, 4, and 13. Thus $s_+ = 1 + 4 + 13 = 18$, and the two-tailed P-value is given by $2P_0(S_+ \geq \max\{18, 14(15)/2 - 18\}) = 2P_0(S_+ \geq 87)$, which is between 2(.025) and 2(.010) or .05 and .02. In particular, since P-value < .05, H_0 is rejected at level .05.

4. The appropriate test is H_0: $\mu = 30$ vs. H_a: $\mu < 30$. The $(x_i - 30)$'s are 0.6, 0.1, –14.4, –3.3, –2.9, –4.6, 5, 0.8, 1.9, 23.2, –17.5, –6.8, –21.2, –5.1, 0.2. From these $n = 15$ values, $s_+ = 3 + 1 + 9 + 4 + 5 + 15 + 2 = 39$. The lower-tailed P-value is $P_0(S_+ \geq 15(16)/2 - 39) = P_0(S_+ \geq 81) > .104$. Therefore, H_0 cannot be rejected. There is not enough evidence to prove that the mean diagnostic time is less than 30 minutes at the 10% significance level.

5. The data are paired, and we wish to test H_0: $\mu_D = 0$ vs. H_a: $\mu_D \neq 0$.

d_i	–.3	2.8	3.9	.6	1.2	–1.1	2.9	1.8	.5	2.3	.9	2.5
rank	1	10*	12*	3*	6*	5	11*	7*	2*	8*	4*	9*

$s_+ = 10 + 12 + \ldots + 9 = 72$, so the 2-tailed P-value is $2P_0(S_+ \geq \max\{72, 12(13)/2 - 72\}) = 2P_0(S_+ \geq 72) <$ $2(.005) = .01$. Therefore, H_0 is rejected at level .05.

6. The data in Ch. 9 Exercise 39 are paired, and we wish to test H_0: $\mu_D = 0$ vs. H_a: $\mu_D \neq 0$. Looking at the differences provided, the 11 positive values have a rank sum of $s_+ = 91$ (the three negative values have ranks 1, 10, and 3, and the rank total with $n = 14$ is 105). The P-value is $2P_0(S_+ \geq \max\{91, 14(15)/2 - 91\})$ $= 2P_0(S_+ \geq 91)$, which is between $2(.010) = .02$ and $2(.005) = .01$. Hence, at $\alpha = .05$ we reject H_0 and conclude that the true average difference between intake values is something other than zero.

7. The data are paired, and we wish to test H_0: $\mu_D = .20$ vs. H_a: $\mu_D > .20$ where $\mu_D = \mu_{\text{outdoor}} - \mu_{\text{indoor}}$. Because $n = 33$, we'll use the large-sample test.

d_i	$d_i - .2$	rank	d_i	$d_i - .2$	rank	d_i	$d_i - .2$	rank
0.22	0.02	2	0.15	–0.05	5.5	0.63	0.43	23
0.01	–0.19	17	1.37	1.17	32	0.23	0.03	4
0.38	0.18	16	0.48	0.28	21	0.96	0.76	31
0.42	0.22	19	0.11	–0.09	8	0.2	0	1
0.85	0.65	29	0.03	–0.17	15	–0.02	–0.22	18
0.23	0.03	3	0.83	0.63	28	0.03	–0.17	14
0.36	0.16	13	1.39	1.19	33	0.87	0.67	30
0.7	0.5	26	0.68	0.48	25	0.3	0.1	9.5
0.71	0.51	27	0.3	0.1	9.5	0.31	0.11	11
0.13	–0.07	7	-0.11	–0.31	22	0.45	0.25	20
0.15	–0.05	5.5	0.31	0.11	12	–0.26	–0.46	24

From the table, $s_+ = 424$, so $z = \dfrac{s_+ - n(n+1)/4}{\sqrt{n(n+1)(2n+1)/24}} = \dfrac{424 - 280.5}{\sqrt{3132.25}} = \dfrac{143.5}{55.9665} = 2.56$. The upper-tailed P-value is $P(Z \geq 2.56) = .0052 < .05$, so we reject $H0$. There is statistically significant evidence that the true mean difference between outdoor and indoor concentrations exceeds .20 nanograms/m^3.

8. With μ = the true average alcohol content for port wine, our hypotheses are H_0: $\mu = 18.5$ vs. H_a: $\mu > 18.5$. Using software, the sum of the ranks for the positive differences among the values $(x_i - 18.5)$ is $s_+ = 463$. Using the large-sample approximation, the expected value and variance of S_+ under H_0 are $n(n + 1)/4 =$ $35(36)/4 = 315$ and $n(n + 1)(2n + 1)/24 = 3727.5$. This corresponds to a P-value of

$$P\left(Z \geq \frac{463 - 315}{\sqrt{3727.5}}\right) = P(Z \geq 2.42) = 1 - \Phi(2.42) = .0078$$

Since .0078 < .01, we reject H_0 at the .01 level and conclude that the true average alcohol content for port wine exceeds 18.5.

9.

R_1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	4	4	4	4	4	4
R_2	2	2	3	3	4	4	1	1	3	3	4	4	1	1	2	2	4	4	1	1	2	2	3	3
R_3	3	4	2	4	2	3	3	4	1	4	1	3	2	4	1	4	1	2	2	3	1	3	1	2
R_4	4	3	4	2	3	2	4	3	4	1	3	1	4	2	4	1	2	1	3	2	3	1	2	1
D	0	2	2	6	6	8	2	4	6	12	10	14	6	10	8	14	16	18	12	14	14	18	18	20

When H_0 is true, each of the above 24 rank sequences is equally likely, which yields the distribution of D:

d	0	2	4	6	8	10	12	14	16	18	20
$p(d)$	1/24	3/24	1/24	4/24	2/24	2/24	2/24	4/24	1/24	3/24	1/24

Then $c = 0$ yields $\alpha = 1/24 = .042$ (too small) while $c = 2$ implies $\alpha = 1/24 + 3/24 = .167$, and this is the closest we can come to achieving a .10 significance level.

Section 15.2

10. Refer to Table A.14.

a. With $m = 5$, $n = 6$, upper-tailed P-value = $P_0(W \geq 41) = .026$.

b. With $m = 5$, $n = 6$, lower-tailed P-value = $P_0(W \geq 5(5 + 6 + 1) - 22) = P_0(W \geq 38) > .041$.

c. With $m = 5$, $n = 6$, two-tailed P-value = $2P_0(W \geq \max\{45, 5(5 + 6 + 1) - 45\}) = 2P_0(W \geq 45) < 2(.004) = .008$.

d. First, $w = 4 + 7 + \ldots + 24 = 182$. Next, since $m = n = 12$ is "off the chart," use the large-sample test:

$$z = \frac{w - m(m+n+1)/2}{\sqrt{mn(m+n+1)/12}} = \frac{182 - 12(25)/2}{\sqrt{12(12)(25)/12}} = 1.85,\ P\text{-value} = P(Z \geq 1.85) = .0322.$$

11. The ordered combined sample is 163(y), 179(y), 213(y), 225(y), 229(x), 245(x), 247(y), 250(x), 286(x), and 299(x), so $w = 5 + 6 + 8 + 9 + 10 = 38$. With $m = n = 5$, Table A.14 gives P-value = $P_0(W \geq 38)$, which is between .008 and .028. In particular, P-value < .05, so H_0 is rejected in favor of H_a.

12. Identifying x with pine (corresponding to the smaller sample size) and y with oak, we wish to test H_0: $\mu_1 - \mu_2 = 0$ vs. H_a: $\mu_1 - \mu_2 \neq 0$. The x ranks are 3 (for .73), 4 (for .98), 5 (for 1.20), 7 (for 1.33), 8 (for 1.40), and 10 (for 1.52), so $w = 37$. With $m = 6$ and $n = 8$, the two-tailed P-value is $2P_0(W \geq 37) > 2(.054)$ .108. Hence, we fail to reject H_0.

13. Identifying x with unpolluted region ($m = 5$) and y with polluted region ($n = 7$), we wish to test the hypotheses H_0: $\mu_1 - \mu_2 = 0$ vs. H_a: $\mu_1 - \mu_2 < 0$. The x ranks are 1, 5, 4, 6, 9, so $w = 25$. In this particular order, the test is lower-tailed, so P-value $= P_0(W \geq 5(5 + 7 + 1) - 25) = P_0(W \geq 40) > .053$. So, we fail to reject H_0 at the .05 level: there is insufficient evidence to conclude that the true average fluoride level is higher in polluted areas.

14. Let μ_1 and μ_2 denote the true average scores using these two methods. The competing hypotheses are H_0: $\mu_1 - \mu_2 = 0$ versus H_a: $\mu_1 - \mu_2 \neq 0$. With $m = n = 18$, we'll use the large-sample version of the rank-sum test. The sum of the ranks for the first sample (Method 1) is $w = 4 + 5 + 6 + 13 + 15 + 17 + 18 + 19 + 22 + 23 + 27 + 29 + 30 + 32 + 33 + 34 + 35 + 36 = 398$. The expected value and variance of W under H_0 are $m(m + n + 1)/2 = 18(18 + 18 + 1)/2 = 333$ and $mn(m + n + 1)/12 = 999$, respectively. The resulting z-value and two-tailed P-value are

$$z = \frac{398 - 333}{\sqrt{999}} = 2.06 \text{ ; } P\text{-value} = 2P(Z \geq 2.06) = 2[1 - \Phi(2.06)] = .0394$$

This is a reasonably low P-value; in particular, we would reject H_0 at the traditional .05 significance level. Thus, on the basis of the rank-sum test, we conclude the true average scores using these two methods are <u>not</u> the same.

15. Let μ_1 and μ_2 denote true average cotanine levels in unexposed and exposed infants, respectively. The hypotheses of interest are H_0: $\mu_1 - \mu_2 = -25$ vs. H_0: $\mu_1 - \mu_2 < -25$. Before ranking, –25 is subtracted from each x_i (i.e. 25 is added to each), giving 33, 36, 37, 39, 45, 68, and 136. The corresponding x ranks in the combined set of 15 observations are 1, 3, 4, 5, 6, 8, and 12, from which $w = 1 + 3 + \ldots + 12 = 39$. With $m = 7$ and $n = 8$, P-value $= P_0(W \geq 7(7 + 8 + 1) - 39) = P_0(W \geq 73) = .027$. Therefore, H_0 is rejected at the .05 level. The true average level for exposed infants appears to exceed that for unexposed infants by more than 25 (note that H_0 would not be rejected using level .01).

16.

a.

x	rank	y	rank
0.43	2	1.47	9
1.17	8	0.80	7
0.37	1	1.58	11
0.47	3	1.53	10
0.68	6	4.33	16
0.58	5	4.23	15
0.50	4	3.25	14
2.75	12	3.22	13

We verify that w = sum of the ranks of the x's = 41.

b. We are testing H_0: $\mu_1 - \mu_2 = 0$ versus H_a: $\mu_1 - \mu_2 < 0$. The reported P-value (significance) is .0027, which is < .01 so we reject H_0. There is evidence that the distribution of good visibility response time is to the left (or lower than) that response time with poor visibility.

Section 15.3

17. $n = 8$, so from Table A.15, a 95% CI (actually 94.5%) has the form $\left(\overline{x}_{(36-32+1)}, \overline{x}_{(32)}\right) = \left(\overline{x}_{(5)}, \overline{x}_{(32)}\right)$. It is easily verified that the 5 smallest pairwise averages are $\frac{5.0+5.0}{2} = 5.00$, $\frac{5.0+11.8}{2} = 8.40$, $\frac{5.0+12.2}{2} = 8.60$, $\frac{5.0+17.0}{2} = 11.00$, and $\frac{5.0+17.3}{2} = 11.15$ (the smallest average not involving 5.0 is $\overline{x}_{(6)} = \frac{11.8+11.8}{2} = 11.8$), and the 5 largest averages are 30.6, 26.0, 24.7, 23.95, and 23.80, so the confidence interval is (11.15, 23.80).

18. With $n = 14$ and $\frac{n(n+1)}{2} = 105$, from Table A.15 we see that $c = 93$ and the 99% interval is $\left(\overline{x}_{(13)}, \overline{x}_{(93)}\right)$. Subtracting 7 from each x_i and multiplying by 100 (to simplify the arithmetic) yields the ordered values –5, 2, 9, 10, 14, 17, 22, 28, 32, 34, 35, 40, 45, and 77. The 13 smallest *sums* are –10, –3, 4, 4, 5, 9, 11, 12, 12, 16, 17, 18, and 19, so $\overline{x}_{(13)} = \frac{14.19}{2} = 7.095$, while the 13 largest sums are 154, 122, 117, 112, 111, 109, 99, 91, 87, and 86, so $\overline{x}_{(93)} = \frac{14.86}{2} = 7.430$. The desired CI is thus (7.095, 7.430).

19. First, we must recognize this as a paired design; the eight <u>differences</u> (Method 1 minus Method 2) are –0.33, –0.41, –0.71, 0.19, –0.52, 0.20, –0.65, and –0.14. With $n = 8$, Table A.15 gives $c = 32$, and a 95% CI for μ_D is $(\overline{x}_{(8(8+1)/2-32+1)}, \overline{x}_{(32)}) = (\overline{x}_{(5)}, \overline{x}_{(32)})$.
Of the 36 pairwise averages created from these 8 differences, the 5th smallest is $\overline{x}_{(5)} = -0.585$, and the 5th-largest (aka the 32nd-smallest) is $\overline{x}_{(32)} = 0.025$. Therefore, we are 94.5% confident the true mean difference in extracted creosote between the two solvents, μ_D, lies in the interval (–.585, .025).

20. For $n = 4$ Table A.13 shows that a two tailed test can be carried out at level .124 or at level .250 (or, of course even higher levels), so we can obtain either an 87.6% CI or a 75% CI. With $\frac{n(n+1)}{2} = 10$, the 87.6% interval is $\left(\overline{x}_{(1)}, \overline{x}_{(10)}\right) = (.045, .177)$.

21. $m = n = 5$ and from Table A.16, $c = 21$ and the 90% (actually 90.5%) interval is $\left(d_{ij(5)}, d_{ij(21)}\right)$. The five smallest $x_i - y_j$ differences are –18, –2, 3, 4, 16 while the five largest differences are 136, 123, 120, 107, 87 (construct a table like Table 15.5), so the desired interval is (16, 87).

22. $m = 6$, $n = 8$, $mn = 48$, and from Table A.16 a 99% interval (actually 99.2%) requires $c = 44$ and the interval is $\left(d_{ij(5)}, d_{ij(44)}\right)$. The five largest $x_i - y_j$ differences are 1.52 – .48 = 1.04, 1.40 – .48 = .92, 1.52 – .67 = .85, 1.33 – .48 = .85, and 1.40 – .67 = .73, while the five smallest are –1.04, –.99, –.83, –.82, and –.79, so the confidence interval for $\mu_1 - \mu_2$ (where 1 = pine and 2 = oak) is (–.79, .73).

Section 15.4

23. Below we record in parentheses beside each observation the rank of that observation in the combined sample.

1:	5.8(3)	6.1(5)	6.4(6)	6.5(7)	7.7(10)	$r_{1.} = 31$
2:	7.1(9)	8.8(12)	9.9(14)	10.5(16)	11.2(17)	$r_{2.} = 68$
3:	5.1(1)	5.7(2)	5.9(4)	6.6(8)	8.2(11)	$r_{3.} = 26$
4:	9.5(13)	1.0.3(15)	11.7(18)	12.1(19)	12.4(20)	$r_{4.} = 85$

The computed value of k is $k = \frac{12}{20(21)}\left[\frac{31^2 + 68^2 + 26^2 + 85^2}{5}\right] - 3(21) = 14.06$. At 3 df, the P-value is < .005, so we reject H_0.

24. After ordering the 9 observation within each sample, the ranks in the combined sample are

1:	1	2	3	7	8	16	18	22	27	$r_{1.} = 104$
2:	4	5	6	11	12	21	31	34	36	$r_{2.} = 160$
3:	9	10	13	14	15	19	28	33	35	$r_{3.} = 176$
4:	17	20	23	24	25	26	29	30	32	$r_{4.} = 226$

The computed k is $k = \frac{12}{36(37)}\left[\frac{104^2 + 160^2 + 176^2 + 226^2}{5}\right] - 3(37) = 7.587$; at 3 df, the corresponding P-value is slightly more than .05 (since 7.587 > 7.815). Therefore, $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$ cannot be rejected at the .05 level (but only barely).

25. The ranks are 1, 3, 4, 5, 6, 7, 8, 9, 12, 14 for the first sample; 11, 13, 15, 16, 17, 18 for the second; 2, 10, 19, 20, 21, 22 for the third; so the rank totals are 69, 90, and 94.

$k = \frac{12}{22(23)}\left[\frac{69^2}{10} + \frac{90^2}{6} + \frac{94^2}{5}\right] - 3(23) = 9.23$; at 2 df, the P-value is roughly .01. Therefore, we reject $H_0 : \mu_1 = \mu_2 = \mu_3$ at the .05 level.

26.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	r_i	r_i^2
A	2	2	2	2	2	2	2	2	2	2	20	400
B	1	1	1	1	1	1	1	1	1	1	10	100
C	4	4	4	4	3	4	4	4	4	4	39	1521
D	3	3	3	3	4	3	3	3	3	3	31	961
												2982

The computed value of F_r is $\frac{12}{4(10)(5)}(2982)-3(10)(5)=28.92$. At 3 df, P-value < .005, and so H_0 is rejected.

27.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	r_i	r_i^2
I	1	2	3	3	2	1	1	3	1	2	19	361
H	2	1	1	2	1	2	2	1	2	3	17	289
C	3	3	2	1	3	3	3	2	3	1	24	576
												1226

The computed value of F_r is $\frac{12}{10(3)(4)}(1226)-3(10)(4)=2.60$. At 2 df, P-value > .10, and so we don't reject H_0 at the .05 level.

Supplementary Exercises

28. The Wilcoxon signed-rank test will be used to test H_0: $\mu_D = 0$ vs. H_a: $\mu_D \neq 0$, where μ_D = the true mean difference between expected rate for a potato diet and a rice diet. The d_i's are (in order of magnitude) .16, .18, .25, –.56, .60, .96, 1.01, and –1.24, so $s_+ = 1 + 2 + 3 + 5 + 6 + 7 = 24$. With $n = 8$, the two tailed P-value is > 2(.098) = .196 from Table A.13. Therefore, H_0 is not rejected.

29. Friedman's test is appropriate here. It is easily verified that $r_{1.} = 28$, $r_{2.} = 29$, $r_{3.} = 16$, $r_{4.} = 17$, from which the defining formula gives $f_r = 9.62$ and the computing formula gives $f_r = 9.67$. Either way, at 3 df the P-value is < .025, and so we reject $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ at the .05 level. We conclude that there are effects due to different years.

30. The Kruskal-Wallis test is appropriate for testing $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$.

Treatment			**ranks**			r_i
I	4	1	2	3	5	15
II	8	7	10	6	9	40
III	11	15	14	12	13	65
IV	16	20	19	17	18	90

$k = \frac{12}{420}\left[\frac{225+1600+4225+8100}{5}\right] - 63 = 17.86.$ At 3 df, P-value < .005, so we reject H_0.

31. From Table A.16, $m = n = 5$ implies that $c = 22$ for a confidence level of 95%, so $mn - c + 1 = 25 - 22 = 1 = 4$. Thus the confidence interval extends from the 4th smallest difference to the 4th largest difference. The 4 smallest differences are –7.1, –6.5, –6.1, –5.9, and the 4 largest are –3.8, –3.7, –3.4, –3.2, so the CI is (–5.9, –3.8).

32.

a. We're testing $H_0 : \mu_1 - \mu_2 = 0$ vs $H_a : \mu_1 - \mu_2 \neq 0$.

Gait	D	L	L	D	D	L	L
Obs	.85	.86	1.09	1.24	1.27	1.31	1.39
Gait	D	L	L	L	D	D	
obs	1.45	1.51	1.53	1.64	1.66	1.82	

$w = 1 + 4 + 5 + 8 + 12 + 13 = 43$, and the two-tailed P-value is $2P_0(W \geq 43) > 2(.051) = .102$. Because the P-value is > .05, we fail to reject H_0 and conclude there is no statistically significant difference between the two means.

b.

Differences

		Lateral Gait						
		.86	1.09	1.31	1.39	1.51	1.53	1.64
	.85	.01	.24	.46	.54	.66	.68	.79
Diagonal gait	1.24	–.38	–.15	.07	.15	.27	.29	.40
	1.27	–.41	–.18	.04	.12	.24	.26	.37
	1.45	–.59	–.36	–.14	–.06	.06	.08	.19
	1.66	–.80	–.57	–.35	–.27	–.15	–.13	–.02
	1.82	–.96	–.73	–.51	–.43	–.31	–.29	–.18

From Table A.16, $c = 35$ and $mn - c + 1 = 8$, giving (–.41, .29) as the CI.

33.

a. With “success” as defined, then Y is binomial with $n = 20$. To determine the binomial proportion p, we realize that since 25 is the hypothesized median, 50% of the distribution should be above 25, thus when H_0 is true $p = .50$. The upper-tailed P-value is $P(Y \geq 15$ when $Y \sim \text{Bin}(20, .5)) = 1 - B(14; 20, .5) = .021$.

b. For the given data, y = (# of sample observations that exceed 25) = 12. Analogous to **a**, the P-value is then $P(Y \geq 12$ when $Y \sim \text{Bin}(20, .5)) = 1 - B(11; 20, .5) = .252$. Since the P-value is large, we fail to reject H_0 — we have insufficient evidence to conclude that the population median exceeds 25.

34.

a. Using the same logic as in Exercise 33, $P(Y \leq 5) = .021$, and $P(Y \geq 15) = .021$, so the significance level is $\alpha = .042$.

b. The null hypothesis will not be rejected if the median is between the 6th smallest observation in the data set and the 6th largest, exclusive. (If the median is less than or equal to 14.4, then there are at least 15 observations above, and we reject H_0. Similarly, if any value at least 41.5 is chosen, we have 5 or less observations above.) Thus with a confidence level of 95.8%, the population median lies between 14.4 and 41.5.

35.

Sample:	y	x	y	y	x	x	x	y	y
Observations:	3.7	4.0	4.1	4.3	4.4	4.8	4.9	5.1	5.6
Rank:	1	3	5	7	9	8	6	4	2

The value of W' for this data is $w' = 3 + 6 + 8 + 9 = 26$. With $m = 4$ and $n = 5$, he upper-tailed P-value is $P_0(W \geq 26) > .056$. Thus, H_0 cannot be rejected at level .05.

36. The only possible ranks now are 1, 2, 3, and 4. Each rank triple is obtained from the corresponding *x* ordering by the "code" 1 = 1, 2 = 2, 3 = 3, 4 = 4, 5 = 3, 6 = 2, 7 = 1 (so e.g. the *x* ordering 256 corresponds to ranks 2, 3, 2).

x ordering	ranks	w''	*x* ordering	ranks	w''	*x* ordering	ranks	w''
123	123	6	156	132	6	267	221	5
124	124	7	157	131	5	345	343	10
125	123	6	167	121	4	346	342	9
126	122	5	234	234	9	347	341	8
127	121	4	235	233	8	356	332	8
134	134	8	236	232	7	357	331	7
135	133	7	237	231	6	367	321	6
136	132	6	245	243	9	456	432	9
137	131	5	246	242	8	457	431	8
145	143	8	247	241	7	467	421	7
146	142	7	256	232	7	567	321	6
147	141	6	257	231	6			

Since when H_0 is true the probability of any particular ordering is 1/35, we can easily obtain the null distribution of W''.

w''	4	5	6	7	8	9	10
$p(w'')$	2/35	4/35	9/35	8/35	7/35	4/35	1/35

In particular, $P(W'' \geq 9) = 4/35 + 1/35 = 1/7 \approx .14$.

CHAPTER 16

Section 16.1

1. All ten values of the quality statistic are between the two control limits, so no out-of-control signal is generated.

2. All ten values are between the two control limits. However, it is readily verified that all but one plotted point fall below the center line (at height .04975). Thus even though no single point generates an out-of-control signal, taken together, the observed values do suggest that there may be a decrease in the average value of the quality statistic. Such a "small" change is more easily detected by a CUSUM procedure (see section 16.5) than by an ordinary chart.

3. P(10 successive points inside the limits) = P(1st inside) x P(2nd inside) x…x P(10th inside) = $(.998)^{10}$ = .9802. P(25 successive points inside the limits) = $(.998)^{25}$ = .9512. $(.998)^{52}$ = .9011, but $(.998)^{53}$ = .8993, so for 53 successive points the probability that at least one will fall outside the control limits when the process is in control is 1 - .8993 = .1007 > .10.

4.

a. With $X \sim N(3.04, .02)$, the probability X stays within the spec limits is $P(2.9 \le X \le 3.1)$ = $\Phi\left(\frac{3.1-3.04}{.02}\right)-\Phi\left(\frac{2.9-3.04}{.02}\right)$ = $\Phi(3) - \Phi(-7) \approx .9987 - 0 = .9987$.

b. With $X \sim N(3.00, .05)$, the probability X stays within the spec limits is $P(2.9 \le X \le 3.1)$ = $\Phi\left(\frac{3.1-3.00}{.05}\right)-\Phi\left(\frac{2.9-3.00}{.05}\right)$ = $\Phi(2) - \Phi(-2) = .9772 - .0228 = .9544$. This is smaller than the probability in part (a): even though the mean is now exactly halfway between the spec limits (which is often desirable), the greatly increased variability makes it less likely for a cork's diameter to breach the spec limits.

5.

a. For the case of 4(a), with $\sigma = .02$, C_p = $\frac{\text{USL}-\text{LSL}}{6\sigma}=\frac{3.1-2.9}{6(.02)}$ = 1.67. This is indeed a very good capability index. In contrast, the case of 4(b) with $\sigma = .05$ has a capability index of C_p = $\frac{3.1-2.9}{6(.05)}$ = 0.67. This is quite a bit less than 1, the dividing line for "marginal capability."

b. For the case of 4(a), with $\mu = 3.04$ and $\sigma = .02$, $\frac{\text{USL}-\mu}{3\sigma}=\frac{3.1-3.04}{3(.02)}$ = 1 and $\frac{\mu-\text{LSL}}{3\sigma}=\frac{3.04-2.9}{3(.02)}$ = 2.33, so $C_{pk} = \min\{1, 2.33\} = 1$.

For the case of 4(b), with $\mu = 3.00$ and $\sigma = .05$, $\frac{\text{USL}-\mu}{3\sigma}=\frac{3.1-3.00}{3(.05)}$ = .67 and $\frac{\mu-\text{LSL}}{3\sigma}=\frac{3.00-2.9}{3(.05)}$ = .67, so $C_{pk} = \min\{.67, .67\} = .67$. Even using this mean-adjusted capability index, process (a) is more "capable" than process (b), though C_{pk} for process (a) is now right at the "marginal capability" threshold.

c. In general, $C_{pk} \le C_p$, and they are equal iff $\mu = \frac{\text{LSL}+\text{USL}}{2}$, i.e. the process mean is the midpoint of the spec limits. To demonstrate this, suppose first that $\mu = \frac{\text{LSL}+\text{USL}}{2}$. Then $\frac{\text{USL}-\mu}{3\sigma} = \frac{\text{USL}-(\text{LSL}+\text{USL})/2}{3\sigma} = \frac{2\text{USL}-(\text{LSL}+\text{USL})}{6\sigma} = \frac{\text{USL}-\text{LSL}}{6\sigma} = C_p$, and similarly $\frac{\mu-\text{LSL}}{3\sigma} = C_p$. In that case, $C_{pk} = \min\{C_p, C_p\} = C_p$.

Otherwise, suppose μ is closer to the lower spec limit than to the upper spec limit (but between the two), so that $\mu - \text{LSL} < \text{USL} - \mu$. In such a case, $C_{pk} = \frac{\mu-\text{LSL}}{3\sigma}$. However, in this same case $\mu < \frac{\text{LSL}+\text{USL}}{2}$, from which $\frac{\mu-\text{LSL}}{3\sigma} < \frac{(\text{LSL}+\text{USL})/2-\text{LSL}}{3\sigma} = \frac{\text{USL}-\text{LSL}}{6\sigma} = C_p$. That is, $C_{pk} < C_p$. Analogous arguments for all other possible values of μ also yield $C_{pk} < C_p$.

Section 16.2

6. For Z, a standard normal random variable, $P(-c \le Z \le c) = .995$ implies that $\Phi(c) = P(Z \le c) = .995 + \frac{.005}{2} = .9975$. Table A.3 then gives c = 2.81. The appropriate control limits are therefore $\mu \pm 2.81\sigma$.

7.

a. P(point falls outside the limits when $\mu = \mu_0 + .5\sigma$) $= 1 - P\left(\mu_0 - \frac{3\sigma}{\sqrt{n}} < \bar{X} < \mu_0 + \frac{3\sigma}{\sqrt{n}} \text{ when } \mu = \mu_0 + .5\sigma\right)$
$= 1 - P\left(-3 - .5\sqrt{n} < Z < 3 - .5\sqrt{n}\right) \quad = 1 - P(-4.12 < Z < 1.882) = 1 - .9699 = .0301$.

b. $1 - P\left(\mu_0 - \frac{3\sigma}{\sqrt{n}} < \bar{X} < \mu_0 + \frac{3\sigma}{\sqrt{n}} \text{ when } \mu = \mu_0 - \sigma\right) = 1 - P\left(-3 + \sqrt{n} < Z < 3 + \sqrt{n}\right)$
$= 1 - P(-.76 < Z < 5.24) = .2236$

c. $1 - P\left(-3 - 2\sqrt{n} < Z < 3 - 2\sqrt{n}\right) = 1 - P(-7.47 < Z < -1.47) = .9292$

8. The limits are $13.00 \pm \frac{(3)(.6)}{\sqrt{5}} = 13.00 \pm .80$, from which LCL = 12.20 and UCL = 13.80. Every one of the 22 $\bar{x}$ values is well within these limits, so the process appears to be in control with respect to location.

9. $\bar{\bar{x}} = 12.95$ and $\bar{s} = .526$, so with $a_5 = .940$, the control limits are $12.95 \pm 3\frac{.526}{.940\sqrt{5}} = 12.95 \pm .75 = 12.20, 13.70$. Again, every point $(\bar{x})$ is between these limits, so there is no evidence of an out-of-control process.

10. From the data, $\bar{\bar{x}}$ = 12.9888 and $\bar{r}$ = 0.1032. With n = 4 the control limits are $\bar{\bar{x}} \pm 3 \cdot \frac{\bar{r}}{b_4\sqrt{4}} = 12.9888 \pm 3\frac{0.1032}{2.058(2)}$ = 12.9136 and 13.0640. Based on these limits, the process is not in control: $\bar{x}_3$ = 13.0675 > UCL, $\bar{x}_{18}$ = 13.0725 > UCL, and $\bar{x}_{25}$ = 12.905 < LCL.

11. $\bar{\bar{x}} = \frac{2317.07}{24} = 96.54$, $\bar{s} = 1.264$, and $a_6 = .952$, giving the control limits $96.54 \pm 3\frac{1.264}{.952\sqrt{6}} = 96.54 \pm 1.63 = 94.91, 98.17$. The value of $\bar{x}$ on the 22nd day lies above the UCL, so the process appears to be out of control at that time.

12. Now $\bar{\bar{x}} = \frac{2317.07 - 98.34}{23} = 96.47$ and $\bar{s} = \frac{30.34 - 1.60}{23} = 1.250$, giving the limits $96.47 \pm 3\frac{1.250}{.952\sqrt{6}} = 96.47 \pm 1.61 = 94.86, 98.08$. All 23 remaining $\bar{x}$ values are between these limits, so no further out-of-control signals are generated.

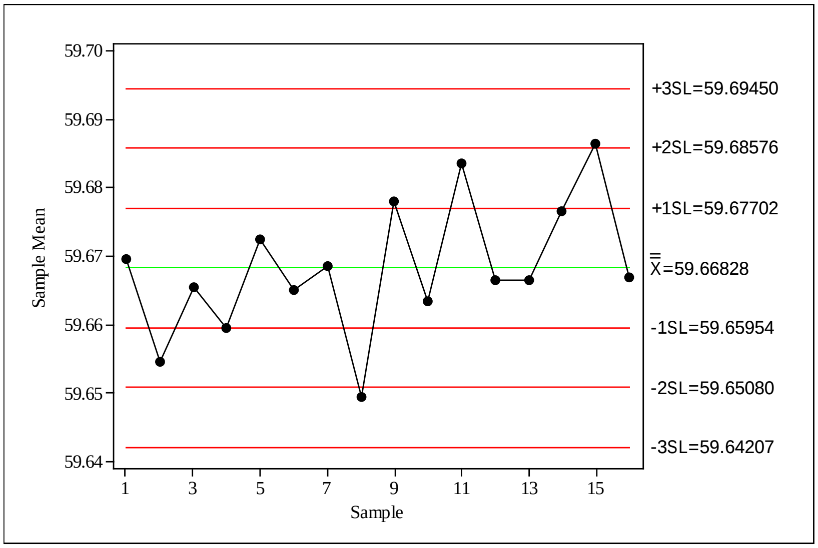
13.

a. $P\left(\mu_0 - \frac{2.81\sigma}{\sqrt{n}} < \bar{X} < \mu_0 + \frac{2.81\sigma}{\sqrt{n}} \text{ when } \mu = \mu_0\right) = P(-2.81 < Z < 2.81) = .995$, so the probability that a point falls outside the limits is .005 and $\text{ARL} = \frac{1}{.005} = 200$.

b. P(a point is inside the limits) = $P\left(\mu_0 - \frac{2.81\sigma}{\sqrt{n}} < \bar{X} < \mu_0 + \frac{2.81\sigma}{\sqrt{n}} \text{ when } \mu = \mu_0 + \sigma\right) = \ldots =$ $P\left(-2.81 - \sqrt{n} < Z < 2.81 - \sqrt{n}\right) = P(-4.81 < Z < .81)$ [when n = 4] $\approx \Phi(.81) = .7910 \Rightarrow$ $p = P$(a point is outside the limits) $= 1 - .7910 = .209 \Rightarrow \text{ARL} = \frac{1}{.2090} = 4.78$.

c. Replace 2.81 with 3 above. For **a**, $P(-3 < Z < 3) = .9974$, so $p = 1 - .9974 = .0026$ and $\text{ARL} = \frac{1}{.0026} = 385$ for an in-control process. When $\mu = \mu_0 + \sigma$ as in **b**, the probability of an out-of-control point is $1 - P(-3 - \sqrt{n} < Z < 3 - \sqrt{n}) = 1 - P(-5 < Z < 1) \approx 1 - \Phi(1) = .1587$, so $\text{ARL} = \frac{1}{.1587} = 6.30$.

14. An $\overline{x}$ control chart from Minitab appears below. Since the process never breaches the ± 3σ limits, the process is in control. Applying the supplemental rules, there are no alerts: it is never the case that (1) two out of three successive points fall outside 2σ limits on the same side, (2) four out of five successive points fall outside 1σ limits on the same side, or (3) eight successive points fall on the same side of the center line.



15. $\overline{\overline{x}} = 12.95$, IQR = .4273, $k_5 = .990$. The control limits are $12.95 \pm 3\dfrac{.4273}{.990\sqrt{5}} = 12.37, 13.53$.

Section 16.3

16. $\Sigma s_i = 4.895$ and $\overline{s} = \dfrac{4.895}{24} = .2040$. With a_5 = .940, the lower control limit is zero and the upper limit is $.2040 + \dfrac{3(.2040)\sqrt{1-(.940)^2}}{.940} = .2040 + .2221 = .4261$. Every s_i is between these limits, so the process appears to be in control with respect to variability.

17.

a. $\overline{r} = \dfrac{85.2}{30} = 2.84$, b_4 = 2.058, and c_4 = .880. Since n = 4, LCL = 0 and UCL $= 2.84 + \dfrac{3(.880)(2.84)}{2.058} = 2.84 + 3.64 = 6.48$.

b. $\overline{r} = 3.54$, $b_8 = 2.844$, and $c_8 = .820$, and the control limits are $3.54 \pm \dfrac{3(.820)(3.54)}{2.844} = 3.54 \pm 3.06 = .48, 6.60$.

18. Let $y = \sqrt{x}$. For example, the transformed data at des dim = 200 are 3.74166, 5.56776, and 3.46410, from which $\overline{y}_1$ = 4.25784, r_1 = 2.10366, and s_1 = 1.14288. Continuing these calculations for all k = 17 rows provides $\overline{\overline{y}}$ = 3.9165, $\overline{r}$ = 2.272, and $\overline{s}$ = 1.160.

Start with an S chart. Since $n = 3 \le 5$, LCL = 0. UCL = $\overline{s} + 3\overline{s}\sqrt{1-a_3^2} / a_3$ = $1.16 + 3(1.16)\sqrt{1-.886^2} / .886$ = 2.981. Since all 17 s_i values are between 0 and 2.981, the process variation (as measured by standard deviation) is in control.

Next, make an R chart. Since $n = 3 \le 6$, LCL = 0. UCL = $\overline{r} + 3c_3\overline{r} / b_3$ = 2.272 + 3(.888)(2.272)/1.693 = 5.847. Since all 17 r_i values are between 0 and 5.847, the process variation (as measured by range) is in control

Making an $\overline{X}$ -chart (really, a $\overline{Y}$ -chart) with control limits based on the sample ranges, we have CL = $\overline{\overline{y}} \pm 3 \cdot \dfrac{\overline{r}}{b_n\sqrt{n}} = 3.9165 \pm 3 \cdot \dfrac{2.272}{1.693\sqrt{3}}$ = 3.9165 ± 2.3244 = 1.59, 6.24. That is, we have LCL = 1.59 and UCL = 6.24. Since all 17 $\overline{y}_i$ values are between these control limits, the process is in statistical control.

19. $\overline{s} = 1.2642$, $a_6 = .952$, and the control limits are

$$1.2642 \pm \frac{3(1.2642)\sqrt{1-(.952)^2}}{.952} = 1.2642 \pm 1.2194 = .045, 2.484.$$

The smallest s_I is s_{20} = .75, and the largest is s_{12} = 1.65, so every value is between .045 and 2.434. The process appears to be in control with respect to variability.

20. $\Sigma s_i^2 = 39.9944$ and $\overline{s}^2 = \dfrac{39.9944}{24} = 1.6664$, so LCL = $\dfrac{(1.6664)(.210)}{5} = .070$, and UCL = $\dfrac{(1.6664)(20.515)}{5} = 6.837$. The smallest s^2 value is $s_{20}^2 = (.75)^2 = .5625$ and the largest is $s_{12}^2 = (1.65)^2 = 2.723$, so all s_i^2 's are between the control limits.

Section 16.4

21. $\bar{p} = \Sigma \frac{\hat{p}_i}{k}$ where $\Sigma \hat{p}_i = \frac{x_1}{n} + ... + \frac{x_k}{n} = \frac{x_1 + ... + x_k}{n} = \frac{578}{100} = 5.78$. Thus $\bar{p} = \frac{5.78}{25} = .231$.

a. The control limits are $.231 \pm 3\sqrt{\frac{(.231)(.769)}{100}} = .231 \pm .126 = .105, .357$.

b. $\frac{13}{100} = .130$, which is between the limits, but $\frac{39}{100} = .390$, which exceeds the upper control limit and therefore generates an out-of-control signal.

22. $\Sigma x_i = 567$, from which $\bar{p} = \frac{\Sigma x_i}{nk} = \frac{567}{(200)(30)} = .0945$. The control limits are

$.0945 \pm 3\sqrt{\frac{(.0945)(.9055)}{200}} = .0945 \pm .0621 = .0324, .1566$. The smallest x_i is $x_7 = 7$, with $\hat{p}_7 = \frac{7}{200} = .0350$. This (barely) exceeds the LCL. The largest x_i is $x_5 = 37$, with $\hat{p}_5 = \frac{37}{200} = .185$. Thus $\hat{p}_5 > UCL = .1566$, so an out-of-control signal is generated. This is the only such signal, since the next largest x_i is $x_{25} = 30$, with $\hat{p}_{25} = \frac{30}{200} = .1500 < UCL$.

23. LCL > 0 when $\bar{p} > 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$, i.e. (after squaring both sides) $50\bar{p}^2 > 9\bar{p}(1-\bar{p})$, i.e. $50\bar{p} > 3(1-\bar{p})$, i.e. $53\bar{p} > 3 \Rightarrow \bar{p} = \frac{3}{53} = .0566$.

24. The suggested transformation is $Y = h(X) = \sin^{-1}\left(\sqrt{X/n}\right)$, with approximate mean value $\sin^{-1}\left(\sqrt{p}\right)$ and approximate variance $\frac{1}{4n}$. $\sin^{-1}\left(\sqrt{x_1/n}\right) = \sin^{-1}\left(\sqrt{.050}\right) = .2255$ (in radians), and the values of $y_i = \sin^{-1}\left(\sqrt{x_i/n}\right)$ for i = 1, 2, 3, …, 30 are

0.2255	0.2367	0.2774	0.3977	0.3047	0.3537	0.3381	0.2868
0.3537	0.3906	0.2475	0.2367	0.2958	0.2774	0.3218	0.3218
0.4446	0.2868	0.2958	0.2678	0.3133	0.3300	0.3047	0.3835
0.1882	0.3047	0.2475	0.3614	0.2958	0.3537		

These give $\Sigma y_i = 9.2437$ and $\bar{y} = .3081$. The control limits are $\bar{y} \pm 3\sqrt{\frac{1}{4n}} = .3081 \pm 3\sqrt{\frac{1}{800}} = .3081 \pm .1091 = .2020, .4142$. In contrast to the result of exercise 20, there is now one point below the LCL (.1882 < .2020) as well as one point above the UCL.

25. $\Sigma x_i = 102$, $\bar{x} = 4.08$, and $\bar{x} \pm 3\sqrt{\bar{x}} = 4.08 \pm 6.06 \approx (-2.0, 10.1)$. Thus LCL = 0 and UCL = 10.1. Because no x_i exceeds 10.1, the process is judged to be in control.

26. $\bar{x} - 3\sqrt{\bar{x}} < 0$ is equivalent to $\sqrt{\bar{x}} < 3$, i.e. $\bar{x} < 9$.

27. With $u_i = \frac{x_i}{g_i}$, the u_i 's are 3.75, 3.33, 3.75, 2.50, 5.00, 5.00, 12.50, 12.00, 6.67, 3.33, 1.67, 3.75, 6.25, 4.00, 6.00, 12.00, 3.75, 5.00, 8.33, and 1.67 for i = 1, …, 20, giving $\bar{u} = 5.5125$. For $g_i = .6$, $\bar{u} \pm 3\sqrt{\frac{\bar{u}}{g_i}} = 5.5125 \pm 9.0933$, LCL = 0, UCL = 14.6. For $g_i = .8$, $\bar{u} \pm 3\sqrt{\frac{\bar{u}}{g_i}} = 5.5125 \pm 7.857$, LCL = 0, UCL = 13.4. For $g_i = 1.0$, $\bar{u} \pm 3\sqrt{\frac{\bar{u}}{g_i}} = 5.5125 \pm 7.0436$, LCL = 0, UCL = 12.6. Several u_i's are close to the corresponding UCL's but none exceed them, so the process is judged to be in control.

28. $y_i = 2\sqrt{x_i}$ and the y_i's are 3.46, 5.29, 4.47, 4.00, 2.83, 5.66, 4.00, 3.46, 3.46, 4.90, 5.29, 2.83, 3.46, 2.83, 4.00, 5.29, 3.46, 2.83, 4.00, 4.00, 2.00, 4.47, 4.00, and 4.90 for i = 1, …, 25, from which $\Sigma y_i = 98.35$ and $\bar{y} = 3.934$. Thus $\bar{y} \pm 3 = 3.934 \pm 3 = .934, 6.934$. Since every y_i is well within these limits it appears that the process is in control.

Section 16.5

29. $\mu_0 = 16$, $k = \frac{\Delta}{2} = 0.05$, $h = .20$, $d_i = \max\left(0, d_{i-1} + \left(\bar{x}_i - 16.05\right)\right)$, $e_i = \max\left(0, e_{i-1} + \left(\bar{x}_i - 15.95\right)\right)$.

i	$\bar{x}_i - 16.05$	d_i	$\bar{x}_i - 15.95$	e_i
1	–0.058	0	0.024	0
2	0.001	0.001	0.101	0
3	0.016	0.017	0.116	0
4	–0.138	0	–0.038	0.038
5	–0.020	0	0.080	0
6	0.010	0.010	0.110	0
7	–0.068	0	0.032	0
8	–0.151	0	–0.054	0.054
9	–0.012	0	0.088	0
10	0.024	0.024	0.124	0
11	–0.021	0.003	0.079	0
12	–0.115	0	–0.015	0.015
13	–0.018	0	0.082	0
14	–0.090	0	0.010	0
15	0.005	0.005	0.105	0

For no time r is it the case that $d_r > .20$ or that $e_r > .20$, so no out-of-control signals are generated.

30. $\mu_0 = .75$, $k = \frac{\Delta}{2} = 0.001$, h = .003, $d_i = \max\left(0, d_{i-1} + \left(\bar{x}_i - .751\right)\right)$, $e_i = \max\left(0, e_{i-1} + \left(\bar{x}_i - .749\right)\right)$.

i	$\bar{x}_i - .751$	d_i	$\bar{x}_i - .749$	e_i
1	–.0003	0	.0017	0
2	–.0006	0	.0014	0
3	–.0018	0	.0002	0
4	–.0009	0	.0011	0
5	–.0007	0	.0013	0
6	.0000	0	.0020	0
7	–.0020	0	.0000	0
8	–.0013	0	.0007	0
9	–.0022	0	–.0002	.0002
10	–.0006	0	.0014	0
11	.0006	.0006	.0026	0
12	–.0038	0	–.0018	.0018
13	–.0021	0	–.0001	.0019
14	–.0027	0	–.0007	.0026
15	–.0039	0	–.0019	.0045*
16	–.0012	0	.0008	.0037
17	–.0050	0	–.0030	.0067
18	–.0028	0	–.0008	.0075
19	–.0040	0	–.0020	.0095
20	–.0017	0	.0003	.0092
21	–.0048	0	–.0028	.0120
22	–.0029	0	–.0009	.0129

Clearly $e_{15} = .0045 > .003 = h$, suggesting that the process mean has shifted to a value smaller than the target of .75.

31. Connecting 600 on the in-control ARL scale to 4 on the out-of-control scale and extending to the *k*′ scale gives *k*′ = .87. Thus $k' = \frac{\Delta/2}{\sigma/\sqrt{n}} = \frac{.002}{.005/\sqrt{n}}$ from which $\sqrt{n} = 2.175 \Rightarrow n = 4.73 = s$. Then connecting .87 on the *k*′ scale to 600 on the out-of-control ARL scale and extending to *h*′ gives *h*′ = 2.8, so $h = \left(\frac{\sigma}{\sqrt{n}}\right)(2.8) = \left(\frac{.005}{\sqrt{5}}\right)(2.8) = .00626$.

32. In control ARL = 250, out-of-control ARL = 4.8, from which $k' = .7 = \frac{\Delta/2}{\sigma/\sqrt{n}} = \frac{\sigma/2}{\sigma/\sqrt{n}} = \frac{\sqrt{n}}{2}$. So $\sqrt{n} = 1.4 \Rightarrow n = 1.96 \approx 2$. Then *h*′ = 2.85, giving $h = \left(\frac{\sigma}{\sqrt{n}}\right)(2.85) = 2.0153\sigma$.

Section 16.6

33. For the binomial calculation, $n = 50$ and we wish $P(X \le 2) = \binom{50}{0} p^0 (1-p)^{50} + \binom{50}{1} p^1 (1-p)^{49} + \binom{50}{2} p^2 (1-p)^{48}$ when p = .01, .02, …, .10. For the hypergeometric calculation, $P(X \le 2) = \frac{\binom{M}{0}\binom{500-M}{50}}{\binom{500}{50}} + \frac{\binom{M}{1}\binom{500-M}{49}}{\binom{500}{50}} + \frac{\binom{M}{2}\binom{500-M}{48}}{\binom{500}{50}}$, to be calculated for M = 5, 10, 15, …, 50. The resulting probabilities appear below.

p	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
Hypg.	.9919	.9317	.8182	.6775	.5343	.4047	.2964	.2110	.1464	.0994
Bin.	.9862	.9216	.8108	.6767	.5405	.4162	.3108	.2260	.1605	.1117

34. $P(X \le 1) = \binom{50}{0} p^0 (1-p)^{50} + \binom{50}{1} p^1 (1-p)^{49} = (1-p)^{50} + 50p(1-p)^{49}$

p	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
$P(X \le 1)$	.9106	.7358	.5553	.4005	.2794	.1900	.1265	.0827	.0532	.0338

35. $P(X \le 2) = \binom{100}{0} p^0 (1-p)^{100} + \binom{100}{1} p^1 (1-p)^{99} + \binom{100}{2} p^2 (1-p)^{98}$

p	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
$P(X \le 2)$	.9206	.6767	.4198	.2321	.1183	.0566	.0258	.0113	.0048	.0019

For values of p quite close to 0, the probability of lot acceptance using this plan is larger than that for the previous plan, whereas for larger p this plan is less likely to result in an “accept the lot” decision (the dividing point between “close to zero” and “larger p” is someplace between .01 and .02). In this sense, the current plan is better.

36. $\frac{\text{LTPD}}{\text{AQL}} = \frac{.07}{.02} = 3.5 \approx 3.55$, which appears in the $\frac{p_1}{p_2}$ column in the c = 5 row. Then $n = \frac{np_1}{p_1} = \frac{2.613}{.02} = 130.65 \approx 131$.

$P(X > 5$ when $p = .02) = 1 - \sum_{x=0}^{5} \binom{131}{x} (.02)^x (.98)^{131-x} = .0487 \approx .05$

$P(X \le 5$ when $p = .07) = \sum_{x=0}^{5} \binom{131}{x} (.07)^x (.93)^{131-x} = .0974 \approx .10$

37. P(accepting the lot) = $P(X_1 = 0 \text{ or } 1) + P(X_1 = 2, X_2 = 0, 1, 2, \text{ or } 3) + P(X_1 = 3, X_2 = 0, 1, \text{ or } 2)$
$= P(X_1 = 0 \text{ or } 1) + P(X_1 = 2)P(X_2 = 0, 1, 2, \text{ or } 3) + P(X_1 = 3)P(X_2 = 0, 1, \text{ or } 2)$.

$p = .01$: $= .9106 + (.0756)(.9984) + (.0122)(.9862) = .9981$

$p = .05$: $= .2794 + (.2611)(.7604) + (.2199)(.5405) = .5968$

$p = .10$: $= .0338 + (.0779)(.2503) + (.1386)(.1117) = .0688$

38. P(accepting the lot) = $P(X_1 = 0 \text{ or } 1) + P(X_1 = 2, X_2 = 0 \text{ or } 1) + P(X_1 = 3, X_2 = 0)$ [since $c_2 = r_1 - 1 = 3$] = $P(X_1 = 0 \text{ or } 1) + P(X_1 = 2)P(X_2 = 0 \text{ or } 1) + P(X_1 = 3)P(X_2 = 0)$

$$= \sum_{x=0}^{1}\binom{50}{x}p^x(1-p)^{50-x} + \binom{50}{2}p^2(1-p)^{48}\cdot\sum_{x=0}^{1}\binom{100}{x}p^x(1-p)^{100-x} = \binom{50}{3}p^3(1-p)^{47}\cdot\binom{100}{0}p^0(1-p)^{100}.$$

$p = .02$: $= .7358 + (.1858)(.4033) + (.0607)(.1326) = .8188$

$p = .05$: $= .2794 + (.2611)(.0371) + (.2199)(.0059) = .2904$

$p = .10$: $= .0338 + (.0779)(.0003) + (.1386)(.0000) = .0038$

39.

a. $\text{AOQ} = pP(A) = p[(1-p)^{50} + 50p(1-p)^{49} + 1225p^2(1-p)^{48}]$

p	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
AOQ	.010	.018	.024	.027	.027	.025	.022	.018	.014	.011

b. $p = .0447$, AOQL = $.0447P(A) = .0274$

c. ATI = $50P(A) + 2000(1 - P(A))$

p	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
ATI	77.3	202.1	418.6	679.9	945.1	1188.8	1393.6	1559.3	1686.1	1781.6

40. $\text{AOQ} = pP(A) = p[(1-p)^{50} + 50p(1-p)^{49}]$. Exercise 32 gives $P(A)$, so multiplying each entry in the second row by the corresponding entry in the first row gives AOQ:

p	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
AOQ	.0091	.0147	.0167	.0160	.0140	.0114	.0089	.0066	.0048	.0034

ATI = $50P(A) + 2000(1 - P(A))$

p	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
ATI	224.3	565.2	917.2	1219.0	1455.2	1629.5	1753.3	1838.7	1896.3	1934.1

$\frac{d}{dp}\text{AOQ} = \frac{d}{dp}\left[\, pP(A) = p[(1-p)^{50} + 50p(1-p)^{49}]\right] = 0$ gives the quadratic equation $2499p^2 - 48p - 1 = 0$, from which $p = .0318$, and $AOQL = .0318P(A) \approx .0167$.

Supplementary Exercises

41. $n = 6$, $k = 26$, $\Sigma\bar{x}_i = 10{,}980$, $\bar{\bar{x}} = 422.31$, $\Sigma s_i = 402$, $\bar{s} = 15.4615$, $\Sigma r_i = 1074$, $\bar{r} = 41.3077$

S chart: $15.4615 \pm \frac{3(15.4615)\sqrt{1-(.952)^2}}{.952} = 15.4615 \pm 14.9141 \approx .55, 30.37$

R chart: $41.31 \pm \frac{3(.848)(41.31)}{2.536} = 41.31 \pm 41.44$, so LCL = 0, UCL = 82.75

$\bar{X}$ chart based on $\bar{s}$: $422.31 \pm \frac{3(15.4615)}{.952\sqrt{6}} = 402.42, 442.20$

$\bar{X}$ chart based on $\bar{r}$: $422.31 \pm \frac{3(41.3077)}{2.536\sqrt{6}} = 402.36, 442.26$

42. A *c* chart is appropriate here. $\Sigma\bar{x}_i = 92$ so $\bar{x} = \frac{92}{24} = 3.833$, and $\bar{x} \pm 3\sqrt{\bar{x}} = 3.833 \pm 5.874$, giving LCL = 0 and UCL = 9.7. Because $x_{22} = 10 >$ UCL, the process appears to have been out of control at the time that the 22nd plate was obtained.

43.

i	$\bar{x}_i$	s_i	r_i
1	50.83	1.172	2.2
2	50.10	.854	1.7
3	50.30	1.136	2.1
4	50.23	1.097	2.1
5	50.33	.666	1.3
6	51.20	.854	1.7
7	50.17	.416	.8
8	50.70	.964	1.8
9	49.93	1.159	2.1
10	49.97	.473	.9
11	50.13	.698	.9
12	49.33	.833	1.6
13	50.23	.839	1.5
14	50.33	.404	.8
15	49.30	.265	.5
16	49.90	.854	1.7
17	50.40	.781	1.4
18	49.37	.902	1.8
19	49.87	.643	1.2
20	50.00	.794	1.5
21	50.80	2.931	5.6
22	50.43	.971	1.9

$\Sigma s_i = 19.706$, $\overline{s} = .8957$, $\Sigma \overline{x}_i = 1103.85$, $\overline{\overline{x}} = 50.175$, $a_3 = .886$, from which an *s* chart has LCL = 0 and UCL = $.8957 + \dfrac{3(.8957)\sqrt{1-(.886)^2}}{.886} = 2.3020$, and $s_{21} = 2.931 > UCL$. Since an assignable cause is assumed to have been identified we eliminate the 21st group. Then $\Sigma s_i = 16.775$, $\overline{s} = .7998$, $\overline{\overline{x}} = 50.145$. The resulting UCL for an *s* chart is 2.0529, and $s_i < 2.0529$ for every remaining *i*. The $\overline{x}$ chart based on $\overline{s}$ has limits $50.145 \pm \dfrac{3(.7988)}{.886\sqrt{3}} = 48.58, 51.71$. All $\overline{x}_i$ values are between these limits.

44. $\overline{p} = .0608$, *n* = 100, so $\text{UCL} = n\overline{p} + 3\sqrt{n\overline{p}(1-\overline{p})} = 6.08 + 3\sqrt{6.08(.9392)}$ = 6.08 + 7.17 = 13.25 and LCL = 0. All points are between these limits, as was the case for the *p*-chart. The *p*-chart and *np*-chart will always give identical results since $\overline{p} - 3\sqrt{\dfrac{\overline{p}(1-\overline{p})}{n}} < \hat{p}_i < \overline{p} + 3\sqrt{\dfrac{\overline{p}(1-\overline{p})}{n}}$ iff $n\overline{p} - 3\sqrt{n\overline{p}(1-\overline{p})} < n\hat{p}_i = x_i < n\overline{p} + 3\sqrt{n\overline{p}(1-\overline{p})}$.

45. $\Sigma n_i = 4(16) + (3)(4) = 76$, $\Sigma n_i \overline{x}_i = 32{,}729.4$, $\overline{\overline{x}} = 430.65$,

$s^2 = \dfrac{\Sigma(n_i - 1)s_i^2}{\Sigma(n_i - 1)} = \dfrac{27{,}380.16 - 5661.4}{76 - 20} = 590.0279$, so *s* = 24.2905. For variation: when *n* = 3,

$\text{UCL} = 24.2905 + \dfrac{3(24.2905)\sqrt{1-(.886)^2}}{.886} = 24.29 + 38.14 = 62.43$; when *n* = 4,

$\text{UCL} = 24.2905 + \dfrac{3(24.2905)\sqrt{1-(.921)^2}}{.921} = 24.29 + 30.82 = 55.11$. For location: when *n* = 3, $430.65 \pm 47.49 = 383.16, 478.14$; when *n* = 4, $430.65 \pm 39.56 = 391.09, 470.21$.

46.

a. Provided the $E(\overline{X}_i) = \mu$ for each *i*,

$$\begin{aligned} E(W_t) &= \alpha E(\overline{X}_t) + \alpha(1-\alpha)E(\overline{X}_{t-1}) + \ldots + \alpha(1-\alpha)^{t-1}E(\overline{X}_1) + (1-\alpha)^t \mu \\ &= \mu\left[\alpha + \alpha(1-\alpha) + \ldots + \alpha(1-\alpha)^{t-1} + (1-\alpha)^t\right] \\ &= \mu\left[\alpha\left(1 + (1-\alpha) + \ldots + (1-\alpha)^{t-1}\right) + (1-\alpha)^t\right] = \mu\left[\alpha\sum_{i=0}^{\infty}(1-\alpha)^i - \alpha\sum_{i=t}^{\infty}(1-\alpha)^i + (1-\alpha)^t\right] \\ &= \mu\left[\frac{\alpha}{1-(1-\alpha)} - \alpha(1-\alpha)^t \cdot \frac{1}{1-(1-\alpha)} + (1-\alpha)^t\right] = \mu \end{aligned}$$

b.

$$V\left(W_t\right)=\alpha^2 V\left(\bar{X}_t\right)+\alpha^2\left(1-\alpha\right)^2 V\left(\bar{X}_{t-1}\right)+\ldots+\alpha^2\left(1-\alpha\right)^{2(t-1)}V\left(\bar{X}_1\right)$$

$$=\alpha^2\left[1+\left(1-\alpha\right)^2+\ldots+\left(1-\alpha\right)^{2(t-1)}\right]\cdot V\left(\bar{X}_1\right)=\alpha^2\left[1+C+\ldots+C^{t-1}\right]\cdot\frac{\sigma^2}{n} \quad \text{where } C=(1-\alpha)^2$$

$$=\alpha^2\frac{1-C^t}{1-C}\cdot\frac{\sigma^2}{n}$$

which gives the desired expression.

c. From Example 16.8, σ = .5 (or $\bar{s}$ can be used instead). Suppose that we use α = .6 (not specified in the problem). Then

$w_0=\mu_0=40$

$w_1=.6\bar{x}_1+.4\mu_0=.6\left(40.20\right)+.4\left(40\right)=40.12$

$w_2=.6\bar{x}_2+.4w_1=.6\left(39.72\right)+.4\left(40.12\right)=39.88$

$w_3=.6\bar{x}_3+.4w_2=.6\left(40.42\right)+.4\left(39.88\right)=40.20$

w_4 = 40.07, w_5 = 40.06, w_6 = 39.88, w_7 = 39.74, w_8 = 40.14, w_9 = 40.25, w_{10} = 40.00, w_{11} = 40.29, w_{12} = 40.36, w_{13} = 40.51, w_{14} = 40.19, w_{15} = 40.21, w_{16} = 40.29

$$\sigma_1^2=\frac{.6\left[1-\left(1-.6\right)^2\right]}{2-.6}\cdot\frac{.25}{4}=.0225,\ \sigma_1=.1500,\ \sigma_2^2=\frac{.6\left[1-\left(1-.6\right)^4\right]}{2-.6}\cdot\frac{.25}{4}=.0261,\ \sigma_2=.1616,$$
$$\sigma_3=.1633,\ \sigma_4=.1636,\ \sigma_5=.1637=\sigma_6,\ldots,\sigma_{16}$$

Control limits are:

For t = 1, $40\pm3\left(.1500\right)=39.55, 40.45$

For t = 2, $40\pm3\left(.1616\right)=39.52, 40.48$

For t = 3, $40\pm3\left(.1633\right)=39.51, 40.49$

These last limits are also the limits for t = 4, …, 16.

Because w_{13} = 40.51 > 40.49 = UCL, an out-of-control signal is generated.